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NEWTON'S METHOD, CIRCLE MAPS, AND CHAOTIC MOTION

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1. Introduction. For most of us, Newton's method for determining the zeros of a polynomial was one of the first iterative systems we encountered. As for its properties, it is known that should the initial point be "sufficiently close" to a zero of a polynomial f, then the iterates,

$$x_{i+1} = N(x_i) = x_i - (f(x_i)/f'(x_i)),$$

will converge to the zero set of f, $Z(f)$. But what happens if the initial point isn't sufficiently close to $Z(f)$? In other words, what about those nonconvergent points; what is their set theoretic nature and what is the behavior of the Newton sequences they define? Some reasons for nonconvergence are obvious; the sequence may "blow-up" because some iterate hits set $Z(f')$ so the next iterate will not be defined, or there may exist some two points where the iterates bounce forever from one to the other. But this doesn't capture all of the nonconvergent behavior. For example, for a polynomial of degree greater than three, it turns out that the nonconvergent points define sequences which are highly erratic. Indeed, suppose all the roots of a polynomial f are real and distinct, and consider the bounded intervals determined by the zeros of f'. Perhaps the most erratic behavior would be to specify an arbitrary, infinite listing of these bounded intervals, and then assert that there is an initial point p so that its ith iterate under Newton's method will lie in the ith selected interval, $i = 1, 2, 3 \ldots$! This does occur, and we will indicate why.

Newton's method clearly is a deterministic system. Yet, we are claiming that this system contains subsystems which have highly random behavior in the sense described above. This is not an isolated behavior peculiar to Newton's method; it is beginning to emerge that a considerable number of deterministic dynamical systems contain subsystems with such random motion. For example, in the 1920's, J. Chazy [6] classified the long term asymptotic behavior of the three-body problem of Newtonian mechanics. He found he could not preclude the possibility of one type of motion whereby the system vacillates between remaining bounded or expanding to infinity. More precisely, let $R(t)$ represent the maximum spacing between particles at time t. Chazy wasn't able to eliminate the possibility of motion where limsup $R = \infty$, but liminf $R < \infty$ as $t \to \infty$. (Similar problems arise in the asymptotic classification of n-body systems [21].) It wasn't until 1960 that K. Sitnikov [23] established the existence of this motion. Eventually, others built upon his framework to show that the deterministic dynamics of the three-body problem admits "random" or "chaotic" motion in that all sorts of "near escape" or "near capture" orbits are possible [17]. More recently, it has been shown that certain models coming from biology [14] exhibit this chaotic behavior, and there have been some persuasive arguments [20] that turbulent behavior in fluids may be caused by mechanisms which are somewhat similar to this random motion. It turns out that even the study of the transient instabilities of an electric power system may admit such an explanation [13]!

Although the study of chaos is recent, its origins go back at least to the work of Julia [12] and of Fatou [9] (also, see [5]) concerning iterations in the complex plane. Here the iterates are given

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by rational functions, $z_{i+1} = h(z_i)/g(z_i)$. In this beautiful theory, the convergence, the nonconvergence, and the other properties of the sequences are related to regions in the plane where certain function theoretic properties of the rational functions are satisfied. This work is related to the current paper in that if $f(z)$ is a polynomial (or even a rational function) over the complex plane, then the corresponding Newton's map is a rational function. Consequently, the analysis of Newton's method for polynomials with complex valued roots can be subsumed by the Julia-Fatou theory.

A major contribution and impetus to this area of "random motion" was given by S. Smale with his famous "horseshoe" example [**24**]. Here, some local, geometric properties of an iteration mapping were considered. Essentially, the mapping is from the plane to the plane. Locally this mapping is characterized by starting with some narrow strip, stretching the length, shrinking the width, bending it into the shape of an horseshoe, and placing it back upon the original strip. (See, for example, [**18**].) It turns out that this simple property, which arises in several dynamical systems, carries with it some complicated and rich consequences concerning the iterates of this mapping. (The extensions of Sitnikov's result concerning oscillatory motion in the three-body problem are proved in this way [**17**].) To describe the consequences of this horseshoe property, Smale introduced a "coding" procedure. This coding permeates the modern dynamical systems literature, and a version of it will be used here.

The onset of chaos is not restricted to complicated dynamical systems. Readers of this MONTHLY have encountered it in the Li and Yorke paper "Period three implies chaos" [**14**]. This paper, which stimulated considerable discussion and research, described the behavior of the iterates of a simple quadratic mapping from the unit interval back into itself. The potential complexity of iterates resulting from interval maps and the role of "period three" was highlighted in an earlier paper by Sarkovskii [**22**] which analyzed the periods of the iterates of continuous maps of the interval back into itself. (A sequence $\{p_i\}_{i=1}^{\infty}$ has period n if $p_{i+n} = p_i$ for all i, and n is the smallest integer for which this is true.) Sarkovskii proved the surprising result that should f have a sequence of iterates with period n, then it has another sequence of iterates of period k for any k which follows n in the following listing (which includes all of the positive integers):

$$\{3,5,6,9,\ldots;\ 2\cdot 3,2\cdot 5,2\cdot 7,\ldots;\ 2^2\cdot 3,2^2\cdot 5,\ldots;\ 2^3,2^2,2^1,1\}.$$

(A nice proof of this is given in [**4**].) L. Block [**3**] was able to show that a similar conclusion holds for continuous maps from the circle back into the circle. In Section 3, we will extend Block's result to the Newton iteration process.

By now fairly sophisticated machinery has been developed to detect when random motion occurs. Some of it concerns interval maps, others consider different types of dynamical systems, etc. (There is a large, growing literature, but a starting point might be [**3–5**], [**7–14**], [**16–20**], [**22**], [**24**].) Nevertheless, when stripped of some of their technical difficulties and sophisticated trappings, the central idea for several of these arguments is compatible with that presented here for Newton's method. Technically, the existence of such motion is based upon the fact that a nested, decreasing sequence of bounded, closed sets has a nonempty intersection. In order to underscore this theme, the proofs which are based on other arguments are deferred to Section 4.

We conclude this section by defining some terms and by providing a partial description of some of B. Barna's pioneering work concerning the Newton iteration process. Let f be a smooth function, and consider the iterates formed by the Newton's method N. For any initial point on R, the real line, only one of two possible events can occur—either the forward iterates of this point under N converge to an element in the zero set of f, or they do not. Let

$$S_1 = \{x \in R: N^n(x) \to Z(f) \quad \text{as } n \to \infty\}$$

be the set of points of the first type where $N^n(x) = N(N^{n-1}(x))$ is the nth Newton iterate of point x.

One possible reason a point x is not in S_1 is that some forward iterate lands in $Z(f')$, and at

this iterate Newton's method is no longer defined. Let S_2 be the set of these points; that is,

$$S_2 = \{x: \text{there exists a positive integer } m = m(x) \text{ such that the } m\text{th Newton iterate of } x \text{ is in } Z(f')\}.$$

Let $S_3 = R/\{S_1 \cup S_2\}$ be the remaining points. Clearly S_3 is an invariant set; N: $S_3 \to S_3$. (Indeed, so are sets S_1 and S_2.) Our goal is to determine the set theoretic structure of S_3 and to find a characterization of N when it is restricted to S_3. An informal characterization of the dynamics of N on S_3 was given in the introductory paragraph; a precise representation will be given in the next section.

Before the set theoretic nature of S_3 is described, we digress a bit. One of the delightful aspects of that first course concerning the properties of the real line was being introduced to the "counter-intuitive" cantor sets. On the one hand, these sets are "large and abundant" in the sense that every point serves as an accumulation point for the set and the set is closed. (Hence the set has cardinality c.) On the other hand, the set is sparse and thin because it is nowhere-dense; it contains no interior points. While these sets may seem to be somewhat esoteric, Barna [1] established that S_3 is a cantor set should f be a polynomial where all $n > 3$ of its roots are real and distinct! Indeed, it turns out that cantor sets often are associated with chaotic behavior.

THEOREM (Barna [1]). *Let f be an nth degree polynomial with $n > 3$ distinct, real roots. Then, set S_3 is a cantor set; namely, it is a closed, nowhere dense set where each point is an accumulation point of the set.*

(The above represents only part of Barna's results. For example, he was able to show with the type of conditions above that the theorem holds for polynomials with real roots, that set S_3 has Lebesque measure zero, etc. Also, he constructed examples of polynomials with complex roots for which S_3 contains open intervals.)

2. The Dynamics. The properties of polynomials which we need are that

(1) the zeros of f separate the zeros of f', and the zeros of f' separate the zeros of f''. For convenience of exposition, assume that the separation is strict (i.e., f and f' have simple zeros) and, of course, that the zero sets consist of isolated points. Next, the derivative of a polynomial has fewer zeros. In fact,

(2) the set $Z(f')$ is contained within the interval defined by the extreme points of $Z(f)$. A similar constraint holds for $Z(f'')$ with respect to the zeros of f'. Call any function which satisfies these two properties "polynomial-like." The function $\cos(x)$ is an example of such a function with an infinite number of zeros.

For such a function f, we start by describing a symbolic representation for the dynamics of the corresponding Newton map N on S_3. Toward this end, label the bounded intervals of $R/Z(f')$ by $a_1, a_2, \ldots$, and let $\mathscr{A}$ be the set of these distinguishing labels. If function f has n roots, then $\mathscr{A}$ has precisely $(n-2)$ elements; on the other hand, for the sine function, the set has a countable infinity of elements.

To specify the interval in which the ith Newton iterate should lie, $i = 0, 1, 2, \ldots$, is the same as specifying an infinite sequence. So, form infinite sequences where the ith entry, b_i, is some element of the "alphabet" $\mathscr{A}$. Any such sequence $b = (b_0, b_1, \ldots)$ is a listing of the suggested future for some Newton sequence. That is, for a given sequence b we seek some point p which lies in the interval corresponding to symbol b_0 such that the ith iterate of p lies in the interval b_i, $i = 1, 2, \ldots$. The set of all possible sequences of this type is given by $D = \mathscr{A}^I$, where I is the set of nonnegative integers.

This dynamic requiring $N^i(p)$ to lie in interval b_i admits a more compact representation. Toward this end, adopt the convention that for a given sequence it is the very first entry which determines the interval of immediate interest; after all, this identifies the interval in which the point should lie. For example, the sequence $(a_6, a_4, a_1, \ldots)$ is identified with the interval a_6. Since

this sequence defines the "future" of some Newton sequence, we need a mapping which will transfer attention from the interval corresponding to the first symbol, a_6, to that corresponding to the second symbol, a_4. This is accomplished by taking a sequence, dropping the first entry, and then shifting all remaining entries one slot to the left. So, define the shift map T: $D \to D$ as $T((b_0, b_1, b_2, \ldots)) = (b_1, b_2, \ldots)$. The ith iterate of T yields sequence $(b_i, b_{i+1}, \ldots)$ which is associated with the interval b_i. With the shift map, the issue is to find a point p so that $N^i(p)$ will lie in the interval associated with $T^i(b)$, $i = 0, 1, 2, \ldots$.

If there is some mapping h: $S_3 \to D$, then a compact form of the above is obtained should Diagram 2.1 commute. This means that p in the interval b_0 is mapped to $h(p) = (b_0, b_1, \ldots)$. The point $N(p)$ is mapped to the sequence $h(N(p)) = T(b_0, b_0, \ldots) = (b_1, \ldots)$; thus, $N(p)$ is in the interval b_1. Repeated application of this diagram describes the dynamic above because it identifies the ith iterate of N with the ith iterate of T. The assertion that all possible types of random behavior can occur is equivalent to the statement that h is surjective:

$$\begin{array}{ccc} S_3 & \xrightarrow{N} & S_3 \\ h\downarrow & & \downarrow h \\ D & \xrightarrow[T]{} & D \end{array} \tag{2.1}$$

If h is continuous, then some of the topological structure of S_3 will be inherited from D through the inverse images of h. (Although this could be used to prove Barna's result, we will provide a direct proof.) The appropriate topology on D to make h continuous is the natural one—the usual product topology where we assume that $\mathscr{A}$ has the discrete topology. An example of an open set in D would be the set of all sequences where a finite number of designated components have specified entries from $\mathscr{A}$. Namely, for any nonnegative integers m and k and for any finite sequence $(b_m, \ldots, b_{m+k})$, let

$$B(b_m, \ldots, b_{m+k}) = \{(c_0, c_1, \ldots) \in D: c_i = b_i, i = m, \ldots, m + k\}.$$

All sets of this type form a basis for the topology of D.

With this notation the behavior of Newton's method can be characterized on the set S_3.

THEOREM 1. *Let f be a smooth, "polynomial like" function where f' has at least 3 distinct zeros. Let the set S_3 have the subspace topology inherited from R. Then there exists a continuous, surjective map h: $S_3 \to D$ such that Diagram 2.1 commutes. Furthermore, with the exception of the constant sequences, corresponding to any periodic sequence in D (i.e., a sequence which is determined by a continually repeating, finite subsequence), there is a periodic orbit of N with the same period. For every constant sequence in D, there is a pair of period two points in the indicated interval.*

The theorem means that on S_3, $N(x)$ is identified with a shift map on D; thus, N defines a "chaotic dynamic." This result is meaningful only for those functions f where f' has at least three zeros; otherwise $\mathscr{A}$ would contain at most one symbol. The surjectivity of h implies that any type of imagined behavior of the nature described in the introductory paragraph does occur. Furthermore, not only does this erratic dynamic take place, but it characterizes N on S_3! The last sentence in the theorem asserts that the period two points are but a special case of what can occur—it is possible to find a periodic orbit of any order which bounces through any prescribed ordering of the bounded intervals defined by $Z(f')$! In fact, it will turn out that there exist sequences which assume any type of prescribed behavior before they settle down to a prescribed periodic orbit.

We now turn to the structure of the set which accompanies this errant behavior. Here, an interesting phenomenon occurs. Should $Z(f)$ be finite (e.g., f is a polynomial with real roots), then S_3 is a perfect set and S_2 is a closed set which is disjoint from S_3. But, should f have an infinite number of zeros (e.g., $f = \sin(x)$), then the two sets S_2 and S_3 become intertwined and it is their union which forms the perfect set! By imposing appropriate additional conditions on f, these perfect sets are nowhere dense, hence they are cantor sets. Part of the conditions imposed here is the Turan inequality which, for function g, requires that $gg'' - (g')^2$ is negative.

THEOREM 2. *Define* $CS = S_3$ *if* $Z(f)$ *is bounded and* $S_2 \cup S_3$ *otherwise. With the assumptions of Theorem* 1, *CS is a perfect set; it is closed and every point is an accumulation point of the set.*

Suppose that both f and f' satisfy the Turan inequality. Then the inverse image of a constant sequence under h is a unique set of period 2 points. If there exist constants $A, B > 1$ such that each bounded interval of $R/Z(f')$ has length less than B, and $|N'| \geqslant A$ at the period two points, then CS is a cantor set. Furthermore, the periodic points of S_3 are dense in CS.

Polynomials with real roots satisfy these conditions, so Barna's result follows. An example of a function with an infinite zero set which satisfies these conditions is $\sin(x)$. This theorem will be used in a technical statement which appears at the end of Section 4. The new statement shows that under certain conditions on f, N satisfies a group of properties collectively known as "Axiom A."

We now turn to the proof of Theorem 1. The essential property which allows Newton's method to admit this chaotic behavior is that the graph of $N(x)$ approaches infinity as $x \to Z(f')$. In fact, with the assumed properties of function f, the graph of N is similar to that given in Figure 1. (This will be verified in Section 4.) The vertical asymptotes are located at $Z(f')$; Newton's method isn't defined at these points. In any unbounded interval of $R/Z(f')$, the graph of N intersects the line $y = x$ in precisely one point, and then it approaches infinity in the indicated manner. The points of intersection of the graph of N and the line $y = x$ correspond to the zeros of f. At these points the slope of N is zero. In each interval, N can have at most one other critical point, and this is given by the zero of f''.

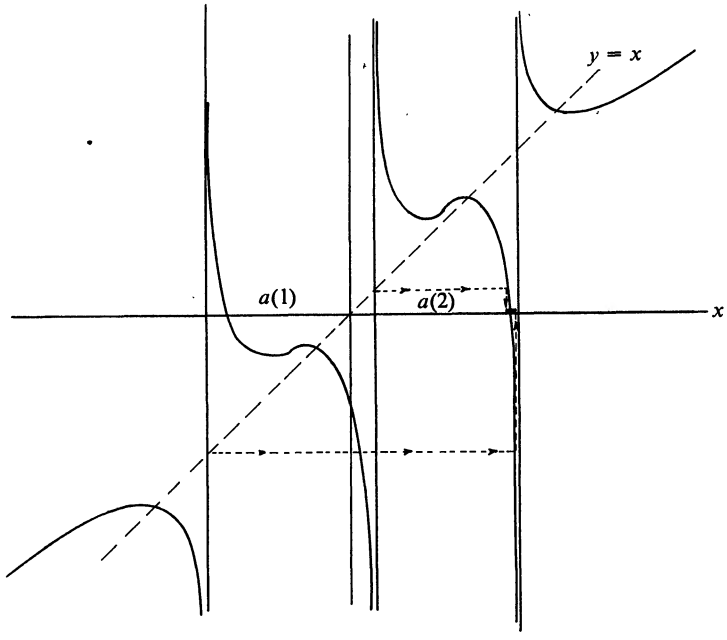


FIG. 1. Graph of $N(f(x))$, $n = 4$.

The key point of the proof is to use these properties of N to define a surjective function h so that the diagram commutes. (Recall, the surjectivity ensures that all types of chaotic behavior exist!)

Proof of Theorem 1. We start with the assertion that any unbounded interval of $R/Z(f')$ lies in S_1. This follows immediately from Figure 1. Assume that the unbounded interval lies to the right of the set $Z(f')$ and let z' denote the zero of f in this interval. If the initial point p is such that $z' < p$, then it follows from the graph that $z' < N(p) < p$. From this it is clear that the resulting

sequence is strictly decreasing, and it converges to z'. If the initial iterate p is such that $p < z'$, then $z' < N(p)$. Thus, the next iterate lies to the right of z'. The argument above now applies. A similar argument holds should the unbounded interval be to the left of $Z(f')$.

Let $p \in S_3$. The definition of $h(p)$ is determined by the future of p as given by the sequence $\{N^n(p)\}_{n=0}^{\infty}$. For each integer n, $N^n(p)$ lies in some bounded component of $R/Z(f')$. So, let b_n be the symbol from $\mathscr{A}$ which corresponds to this interval, and define $h(p) = (b_0, b_1, \ldots)$. It follows from the above that should an iterate of N land in an unbounded interval of $R/Z(f')$, then that sequence will converge to $Z(f)$. This cannot occur for $p \in S_3$. Consequently, $h(p)$ is well defined.

To establish the surjectivity of h, let $b = (b_0, b_1, \ldots) \in D$. The goal is to show the existence of a point p in the interval b_0 so that the nth iterate of p lies in the interval b_n. Toward this end, let N_b denote the restriction of N to the interval b, and let $\bar{b}$ denote the closure of this interval. N is continuous when restricted to $R/Z(f')$, and N_b maps the interval b onto R. So, it follows that $N_{b_0}^{-1}(\bar{b}_1)$ is a closed subset of $\bar{b}_0$. (This is represented by the dotted lines in Figure 1. This is where the unboundedness of N is used.) In fact, the set

$$C(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_n) = N_{b_0}^{-1}\left(N_{b_1}^{-1}\left(\ldots\left(N_{b_{n-1}}^{-1}(\bar{b}_n)\ldots\right)\right)\right)$$

is the closed subset of points which N^i maps into the interval $\bar{b}_i$, for $i = 1, 2, \ldots, n-1$, and onto $\bar{b}_n$ for $i = n$. By construction these sets are nested, and they lie in the bounded interval $\bar{b}_0$. Thus, it follows from the nested set theorem that as $n \to \infty$, the intersection of the C sets is nonempty. Any point p in this intersection will define a Newton sequence which has the described properties. We must show that this intersection contains points which are not in $S_1 \cup S_2$.

It follows from the definition of set S_2 that the intersection of the C sets must be disjoint from S_2. (The points in S_2 have only a finite future.) Similarly, if $p \in S_1$, the Newton iterates converge to a zero of f, so eventually they remain in one interval of $R/Z(f')$. Thus, unless sequence b is "eventually constant" (it consists of only one symbol after some index n), this intersection cannot contain points of S_1. So, to complete the proof of the surjectivity of h, it remains to show that if b is eventually constant, the corresponding intersection of the C sets does contain points which are not in S_1. This will be done at the same time that the periodicity statement is verified.

The continuity of h follows if $h^{-1}(B(b_0, b_1, \ldots, b_n))$ is an open set of S_3 for any choice of the symbols $b_0, b_1, \ldots, b_n$. This is the set $C(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_n) \cap S_3$. However, should $p \in S_3$, then the forward iterates of p cannot hit the boundaries of these intervals because they are in the set $Z(f')$. This requires $N^i(p)$ to lie in the interior of set $\bar{b}_i$. Consequently, the set $C(b_0, \ldots, b_n) \cap S_3$ is equal to the intersection of the iterated inverse images of the *interior* of the intervals b_i. This is an open set of S_3, so the continuity of h follows.

We conclude by indicating why there exist periodic points of any order. Let $b_0 \in \mathscr{A}$. We wish to show that in the interval b_0 there exist period n points. First consider $n = 1$. Any such point corresponds to the intersection of the graph of $y = N(x)$ with the line $y = x$. However, since N_{b_0} maps b_0 onto R, the boundedness of the interval b_0 requires the graph of N_{b_0} to cross that of $y = x$. This verifies the conclusion for $n = 1$.

The proof of the existence of periodic orbits for other values of n is much the same. It relies upon the idea that the graph of the higher iterates of n is, in some sense, similar to the graph of N, except now the whole graph is squeezed on subintervals of the bounded components a_i. To see that this is so, consider the graph of N^2. The graph of $N \circ N_b$ approaches infinity at $N_b^{-1}(Z(f'))$. It follows from the asymptotic and the continuity properties of N_{b_0} that there are subintervals of b_0 which are the inverse images of the intervals of $R/Z(f')$. When $N \circ N_{b_0}$ is restricted to any such subinterval, its range is R; thus the graph of N^2 must intersect the line $y = x$. These points of intersection correspond to the period two points, and the choice of the subintervals determines the sequence from D which is associated with them. (For example, if an intersection point is on the subinterval $N_{b_0}^{-1}(b_1)$, then the period two point has the associated sequence $(b_0, b_1, b_0, \ldots)$.

It remains to show that there are points of intersection on the subinterval $N_{b_0}^{-1}(b_0)$ which differ

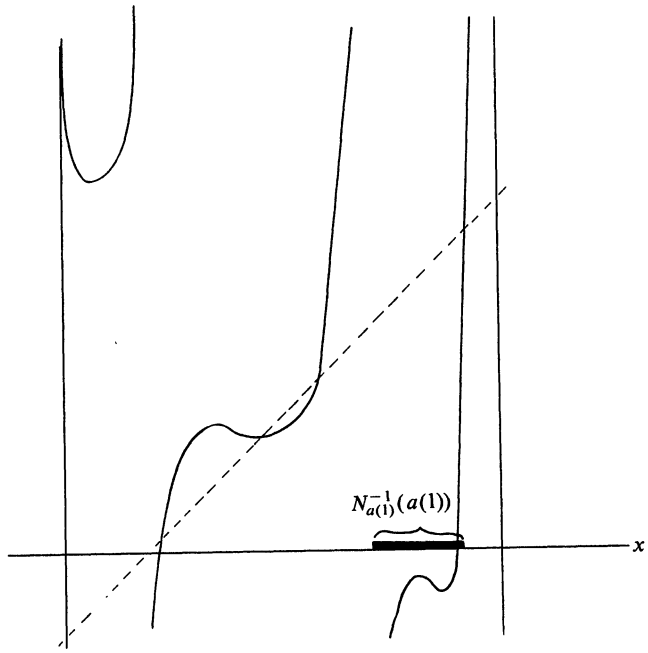


FIG. 1a

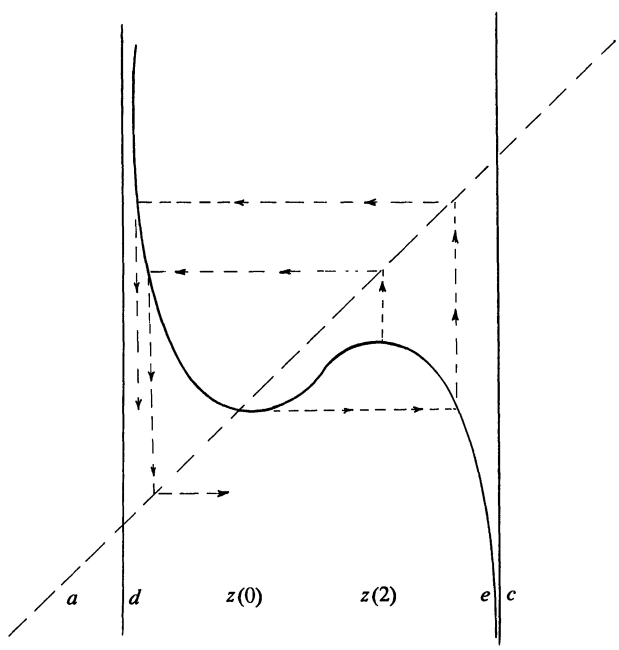


FIG. 2

from the fixed point of N. That is, it must be shown that there are period 2 points associated with the constant sequence. This follows from a more precise representation of N^2 on the interval given by b_0. Assume this interval is (a, c) and that z_0 is the fixed point for N. (See Figures 1a and 2.) On at least one of the intervals (a, z_0) or (z_0, c), the mapping N is orientation reversing. Assume

it is the first. Then N maps $(a, z_0]$ to $[z_0, \infty)$ and it takes z_0 to itself. Therefore, the inverse image of $[z_0, c)$ lies in $(a, z_0]$. This means that somewhere to the left of z_0, the graph of N^2 approaches $-\infty$. But, according to the chain rule,

$$\left(N^2(z_0)\right)' = N'(N(z_0))N'(z_0) = 0.$$

Consequently, the graph of N^2 passes through z_0, and immediately to the left of z_0 it lies above the line $y = x$. This means there is another point of intersection to the left of z_0; this is a period two point associated with the constant sequence $(b_0, b_0, \ldots)$. A similar argument shows that there is another period two point to the right of z_0.

The above shows that there exist periodic points of period 2. The only complication was to show that there is a period two point associated with a constant sequence from D. This difficulty doesn't arise for $n > 2$; consequently, the induction step is immediate—on subintervals of b the image of N^n is R, so the graph of N^n must intersect the line $y = x$. The identification of a periodic point of N with a periodic sequence from D is given by the construction. This completes the proof.

From this proof, it follows that there exist periodic points of order n where the first $(n-1)$ iterates can be made to lie in any specified bounded components of $R/Z(f')$ and in any specified ordering. Furthermore, by taking the inverse image of points, it is easy to show there exist sequences which have any specified ordering for the first m iterates before the iterates settle down into some specified n-periodic orbit.

The most important element of the proof above was the asymptotic properties of N. Thus, most of these assertions extend to other iterative processes which possess a similar graph; e.g., the iterations given by $x_{i+1} = \tan(x_i)$. Another example would be the Newton's function associated with polynomials with real, but not necessarily distinct roots. (The main difference is describing the behavior of N' on $Z(f)$.) Finally, any C^1 small perturbation of N will preserve the important characteristics. Thus, any C^2 small perturbation of an f with distinct, real roots will result in a Newtonian iteration sequence which is similar to that of $N(f)$.

3. What Else Can Occur? After h was defined, the proof of Theorem 1 relied upon the fact that N maps the bounded components of $R/Z(f')$ onto R. Now suppose we have a function f for which the corresponding Newton map does not map some bounded component of $R/Z(f')$ onto R. This arises should the zeros of f not separate the zeros of f'. (An example of this is a polynomial with complex roots.) Here, the graph of $N \to \pm\infty$ as $x \to Z(f')$, but it cannot cross the line $y = x$ (as this would correspond to a zero of f). To illustrate the problem, suppose a_k is an interval of $R/Z(f')$ such that the image set $N(a_k)$ is (e, ∞). It is apparent that N cannot map any point in the interval a_k to any interval of $R/Z(f')$ which lies strictly to the left of the point e. Accordingly, suppose for the sequence $d \in D$ the symbol a_k is immediately followed by a symbol corresponding to an interval which lies to the left of the point e. Clearly, this sequence d cannot be in the image of h. Thus, to characterize the chaotic behavior of N, the issue is to characterize which sequences remain in the image set of h.

One way to handle this problem is to augment the nested set argument with a bookkeeping which keeps track of which symbols can follow others. It turns out that this bookkeeping involves examining the orbit of the point e under all future iterates of N, checking the bifurcation values of e (when the orbit of e enters the set S_2), etc. All of this can be done should there be only one interval of this type. But even simple choices of functions f define a Newton map where this "one-sided boundedness" behavior occurs on several intervals of $R/Z(f')$. (This crops up for polynomials with complex roots.) Whenever this occurs, the analysis becomes increasingly more difficult. A measure of the added complexity can be gleaned by examining what types of periodic orbits the iterative dynamic admits, and then comparing them with the earlier statements. (Recall that the conditions of Theorem 1 allow for all possible types of periodic orbits.) The surprising fact is that an ordering is imposed on the types of periodic orbits which can be admitted!

In the introductory section, the Sarkovskii sequence was described. L. Block [3] was able to show that a similar type of conclusion holds for continuous circle maps. By using Block's result,

we will show that the conclusion extends to the Newton iteration scheme when it is applied to a wide class of functions which includes the polynomials.

DEFINITION. Let G consist of those smooth functions which satisfy the following conditions:

(1) For f in G, both f and f' have a finite number of zeros.
(2) Let N be the Newton map for an f in G. Then as $|x| \to \infty$, either
 (a) $|N| \to \infty$ or
 (b) N approaches a fixed limit.

Polynomials satisfy Condition 2a.

THEOREM 3. *Let f be a smooth function in the set G and let N be the associated Newton map. Assume that $|N| \to \infty$ as $|x| \to \infty$. If N has a point of period $n > 1$, then at least one of the following holds.*

(a) *For every integer m where $n < m$, N has a period m point.*
(b) *For every integer m where m follows n in the Sarkovskii sequence, N has a point of period m.*

If $|N|$ does not approach ∞ as $|x| \to \infty$, but f has a real zero, then the above holds with one possible exceptional value for m.

One way to appreciate what the theorem does and does not state about these iterative sequences is to use an analogy with the decimal representations of numbers in the unit interval. Since the periodic orbits are characterized by a block of symbols which are repeated infinitely often, they correspond to the "rational number" sequences. All this theorem states about such sequences is the size of the "blocks" which can be admitted; it does not state anything about the entries. Furthermore, this theorem does not provide any information about the "irrational" sequences, and it does not describe the image of h. Most of these questions remain unanswered for an arbitrarily chosen f. However, the sharp restriction imposed upon the size and the ordering of the allowable "blocks" demonstrates that the complexity of the analysis has increased.

Proof. Block showed that the conclusion of this theorem holds, with no exceptional values allowed for m, for a continuous mapping from the circle to the circle which has a fixed point. Thus, except for explaining the possible exceptional value for m, the theorem holds once the Newton's map can be identified with a continuous map from S^1 to S^1 where S^1 denotes the circle. To do this, use the stereographic projection of S^1 to R. Namely, let NP represent the North Pole of S^1 and let H represent the stereographic mapping from S^1/NP to R. Define K to be $H^{-1} \circ N \circ H$. The mapping K is defined and continuous everywhere except possibly at NP and at those points of S^1 which are mapped to $Z(f')$. A standard argument shows that N is defined and smooth at a multiple zero of f, so K is defined at those points corresponding to $Z(f) \cap Z(f')$. It is easily seen from the graphical representation of Newton's method that N must approach $\pm\infty$ as $x \to Z(f')/Z(f)$, the exact sign depends upon whether x approaches this set from the right or the left and whether the value of f at this point is positive or negative. (See Section 4.) But $\pm\infty$ on R corresponds to the NP on S^1, so K can be extended continuously on $H^{-1}(Z(f'))$. Finally, the conditions imposed upon the tail behavior of N allow K to be extended continuously to NP. Thus K is a continuous map from S^1 to S^1.

If $|N| \to \infty$ as $|x| \to \infty$, then $K(NP) = NP$, so K has a fixed point. If f has a real zero, then N has a fixed point. In either case, K has a fixed point, so Block's theorem applies. Finally, the only way some one period from the specified sequence may be missing for a N map is if its associated circle map K has NP as a periodic point of that period. But if $|N| \to \infty$ as $|x| \to \infty$, then this can't happen because NP is a fixed point for K.

In conclusion, note that not all Newton maps can be identified with a circle map; if $Z(f')$ has an infinite number of isolated zeros, then the associated circle map K could not be continuous at NP.

4. Proof of Statements. In this concluding section, the proofs of Theorem 2 and the assertion that the graph of N behaves as indicated in Figure 1 will be given. The paper ends with a technical statement showing that N satisfies certain "Axiom A" type conditions.

The Asymptotic Properties of N: Let f be as described in the introductory paragraph of Section 2, and let N be the associated Newton map. It is well known that $N(x)$ is undefined only at those zeros of f' which are not zeros of f; that is, N is not defined at set $A = Z(f')/Z(f)$. The easiest way to determine the asymptotic properties of N as $x \to A$ seems to be to use the usual geometric representation for Newton's method on the graph of f. From this it follows immediately that $|N(x)| \to \infty$ as $x \to A$. Indeed, it follows from the geometry that as x approaches some point in A,

$$N(x) \to (\text{sign}(-f(x)f'(x)))\infty.$$

With our assumption that f has no multiple zeros ($A = Z(f')$), N must map bounded components of $R/Z(f')$ onto R. This is because f' retains the same sign between elements of A, but f must change sign once. Consequently, the asymptotic properties of the graph of N are as indicated in Figure 1.

It remains to verify the assertions concerning the critical points of N; that is, function N has at most two critical points in each component of R/A. Because

$$N'(x) = f(x)f''(x)/(f'(x))^2,$$

the critical points of N are given by the zeros of f and f''. It follows from the assumption concerning the separation of these zeros that f and f'' each can have at most one zero in any interval of R/A.

In an unbounded region of R/A, N has exactly one critical point. (This is because the zeros of f'' are contained in the interval defined by the extreme points of $Z(f')$, so in the unbounded regions f'' is never zero.) This critical point of N corresponds to a zero of f. Because f has no multiple zeros, f changes sign as it passes through this point; thus the sign of N' changes. Since N has no additional critical points in this interval, N' keeps the same sign. Thus N is either monotonically increasing or monotonically decreasing. That N' is positive in the left-hand unbounded intervals of $R/Z(f)$ follows because as f approaches its first zero, it must differ in sign from f'. A similar argument shows that $N' > 0$ as $x \to \infty$. Finally, as $|x| \to \infty$, N must lie on one side of the line $y = x$; otherwise, the intersection would correspond to a nonexistent zero of f. (Should f be a polynomial, it is easy to show that $N(x)$ is asymptotic to $(n-1)x/n$ as $|x| \to \infty$.) In any case, it follows that in the left-hand unbounded region, N has a maximum at the zero of f while in the right-hand region, N has a minimum at the zero of f.

Finally, the following fairly obvious assertion will be used in what follows.

Claim. In any bounded component of $R/Z(f')$, the closed interval defined by the zero of f and the zero of f'' is in S_1. This interval is characterized by N' being nonnegative.

It follows from the analysis above concerning the critical points of N that N' is positive on the interior of this closed interval and negative outside of it. The rest of the proof follows by using the graph of N and an argument similar to that in the first paragraph of the proof of Theorem 1. If the zero of f is greater than the zero of f'', then the graph of N lies above $y = x$ on this interval; otherwise, it lies below.

Now we turn to the proof of Theorem 2. For a first reading, some readers may be interested only in the case where f is a polynomial with real roots. Thus, those parts of the proof which assume that $Z(f)$ is unbounded can be omitted. Other appropriate guideposts are provided.

Proof of Theorem 2. Assume that f is a mapping with properties specified in the statement of the theorem. First we verify that the set CS is a perfect set.

Set CS Is Closed. To show that the set CS is closed, it suffices to show that S_1 is an open set and, if $Z(f)$ is bounded, that no point of S_2 is a boundary point for S_3. Let $p \in S_1$; some forward

iterate of p will be "sufficiently close" to a point in $Z(f)$. This means that all the points in a small neighborhood, V, of this iterate also lie in S_1. Now, from the continuity of N, the inverse iterates of the open set V define open sets of points in S_1. By construction, one of these open sets must contain p. Thus, S_1 is an open set.

Next, suppose that the set $Z(f)$ consists of only a finite number of points. Let point $p \in S_2$. We must show that there is some neighborhood of p which is disjoint from S_3. Because $p \in S_2$, there is some iterate of p, say $N^n(p)$, which is in $Z(f')$. But, by the asymptotic properties of N near $Z(f')$, there is some neighborhood of $N^n(p)$, say U, such that N maps $U/\{N^n(p)\}$ to the unbounded components of $R/Z(f)$. These components are in S_1. (See the proof of Theorem 1.) This means that $U/\{N^n(p)\}$ is in S_1. It follows from the continuity of N and by taking backward iterates of this open set that there is a neighborhood of p where all of the points (except p) are in S_1. This completes the proof that CS is a closed set.

Set CS Is a Perfect Set. Next, we show that every point in the set CS is an accumulation point of S_3. Let $p \in S_3$; then $h(p)$ is some sequence $(b_0, b_1, \ldots)$. This means that for all values of n, p is contained inside a component of the set $C(\bar{b}_0, \ldots, \bar{b}_n)$. (See the proof of Theorem 1 for the definition of this set.) Because these components form a nested set, their lengths form a decreasing sequence which converges to nonnegative number a. If $a = 0$, then for any neighborhood U of p, there is some value of n so that this component of set $C(\bar{b}_0, \ldots, \bar{b}_n)$ is so small that it is contained in U. This means that in U there is an open set of points which are mapped by N^n onto b_n. Because b_n contains points of S_3, so does U.

Next, assume that $a > 0$. This means that the intersection of these components of the C sets is an interval. If this interval is in S_3, then, trivially, p is an accumulation point. Furthermore, unless $h(p)$ eventually becomes a constant sequence, this interval must be in S_3.

So, the difficulties again correspond to the eventually constant sequences. As before, the problem is that the intersection of these C sets includes points converging to $Z(f)$. It has to be shown that it can't include a subset of isolated points of S_3. The argument is technical, and probably should be omitted in a first reading.

Assume that $h(p)$ is a constant sequence corresponding to interval b. (If it eventually becomes one, then focus attention on the appropriate iterate of p so that the above holds.) Assume without loss of generality that p lies to the left of the zero of f in this interval. Suppose N^2 intersects the line $y = x$ in more than one point in that part of the interval b which lies to the left of the zero of f. Denote by P the interval defined by the extreme intersection points. If N^2 maps P into P, then, because P contains no zeros of f, it must lie in S_3. None of these points are isolated.

The case which remains is if N^2 doesn't map P into itself. So, suppose N^2 maps some part of P to the right of P. This means that the graph (restricted to P) extends above p^*, where p^* is the value of the right hand endpoint of P. Since p^* is a fixed point for N^2, at some point the graph has to decrease to assume the value of p^*. That is, there is some $x \leqslant p^*$ such that $N^2(x) = p^*$ and $(N^2(x))' \leqslant 0$. The first condition forces x to be in S_3 as its image is a period two point. By use of the chain rule and the "Claim," it follows from the second condition that either x or $N(x)$ lies between the zeros of f and f''. (Either $N'(N(x))$ or $N'(x)$ must be nonnegative.) This forces x to be in S_1, which is a contradiction. A similar argument using the left-hand end-point of P shows that P is an invariant set for N. Hence, P lies in S_3.

If $p \in P$ and P is an interval, then p is an accumulation point. There remains the possibility that p lies to the left or to the right of P. (Here the argument holds whether P is an interval or a point.) If it lies to the right of P, then N^2 lies above the line $y = x$. An argument similar to the above and in Theorem 1 demonstrates that p is in S_1, not S_3. If p lies to the left of P, then this component of N^2 lies below the line $y = x$ for the rest of this interval. This forces the iterates of p to decrease and eventually to leave the interval b. This contradicts the fact that $h(p)$ is a constant sequence.

The only case which remains is if $P = \{p\}$ and p is one of the pair of period two points in the

interval b. This means that to the left of p, the graph of N^2 lies below that of $y = x$. An argument similar to the concluding one in the preceding paragraph demonstrates that arbitrarily close to the left of p is an open interval V which is mapped, by some iterate of N, onto some other interval b'. Because the interval b' contains a point of S_3, so does V. This completes the proof when $p \in S_3$.

Now, assume that $Z(f)$ is an unbounded set and that $p \in S_2$. It must be shown that p is an accumulation point of S_3. The idea is that eventually the iterated image of a neighborhood of p contains an interval with a point of $Z(f')$ as an interior point. In turn, because of the asymptotic behavior of N near a zero of f', N maps this interval to a neighborhood of ∞. But, $Z(f)$ is unbounded, so some bounded interval of $R/Z(f')$ is in this neighborhood of ∞. Because any bounded neighborhood of $R/Z(f')$ contains points of S_3 (this follows from the surjectivity of h), this neighborhood of ∞ contains points of S_3. The invariance of S_3 provides that the inverse images of points in S_3 are again in S_3. This means that the original neighborhood of p contained points of S_3, and this completes the proof.

The only technical difficulty in carrying out the details of the description above is to verify that the iterated image eventually contains a zero of f' as an interior point. But this will follow from the inverse function theorem if it can be shown that the iterate of N which maps p to $Z(f')$, say, $N^n(p)$, has a nonzero derivative. According to the chain rule,

$$(N^n(p))' = \sum_{i=0}^{n-1} N'(N^i(p)).$$

If this derivative were equal to zero, then some iterate of p would land on a critical point of N. That is, this iterate would correspond to a zero of f or f''. So, by the "Claim," this means that this iterate is in S_1; hence, so is p. This contradiction shows that $(N^n(p))' \neq 0$. The remaining details of the proof follow immediately.

Cantor Set. The purpose of the additional assumption in Theorem 2 is to force the length of any component of the $C(b_0,\ldots,b_n)$ sets to zero as $n \to \infty$. We now show that whenever this occurs, the set CS is a cantor set. To demonstrate this, it remains to show that CS is nowhere dense. It suffices to show that if U is some neighborhood of $p \in CS$, then there is some open set contained in U which is disjoint from CS.

Assume that $p \in S_3$, and let $h(p)$ be the sequence $(b_0, b_1,\ldots)$. Because the size of the component of $C(\bar{b}_0, \bar{b}_1,\ldots,\bar{b}_n)$ which contains p goes to zero as $n \to \infty$, for some value of n the C set is contained in the open set U. Recall that the points in this C set get mapped by N^n onto the interval $\bar{b}_n$. In this interval, there is an open set of points which converge to $Z(f)$ (for example, those points which are "sufficiently close" to a zero of f); so they are in S_1. Since inverse iterates of points in S_1 are again in S_1, it follows from the continuity properties of N that there is an open set of points in U which are in S_1. This completes the proof for bounded $Z(f)$.

If $Z(f)$ is unbounded, then we must also consider $p \in S_2$. The argument above showing that $p \in S_2$ is an accumulation point also demonstrates that some iterate of any neighborhood of p eventually contains a bounded component of $R/Z(f')$. In this component, there is an open set of points of S_1. Again, the inverse image argument shows that U contains an open subset of S_1. This completes the proof.

The Turan Inequality Assumption Gives a Unique Period Two Point. Here we are to show that for each constant sequence in D, there is a unique pair of period two points. Most of the proof concentrates on the properties of N in the neighborhood of these period two points as determined in the proof of Theorem 1. The basic role of the Turan inequality is that it forces N to have the appropriate concavity in the region outside of the period two points.

Let b be a symbol corresponding to a bounded interval of R/A. Without loss of generality, assume that this interval is (a, c), that z_0 and z_2 are, respectively, the zeros of f and f'' in this

interval, and that $z_0 < z_2$. (See Fig. 2.) It is easy to show that the same conclusions hold should $z_0 \geqslant z_2$. The first statement outlines the properties of f near a and c.

LEMMA 1. *Let f be a function as defined in the statement of Theorem 2. Then N is decreasing in the intervals (a, z_0) and (z_2, c); N is concave up in the former interval and concave down in the latter. N is increasing in the interval (z_0, z_2).*

The only new statement in this Lemma concerns the concavity of N. By differentiating the terms involved,

$$N'' = \left\{ f\left[f'f''' - (f'')^2 \right] - f''\left[ff'' - (f')^2 \right] \right\} / (f')^3.$$

The Turan inequality assumption forces the expressions in the brackets to be negative. In the intervals of concern, f and f'' are of different signs; thus in these regions N'' is of one sign. That N'' is of the indicated sign follows directly either from the sign of f' or from the behavior of N as it approaches either $\pm\infty$ on this bounded interval.

It follows from the graph of N that backward iterates of point z_2 under the mapping N alternate from one side to the other side of z_0 forming two monotonic sequences within set b. One is to the left of z_0, and it is monotonically decreasing to a limit point d. The other is to the right of z_0, and it is monotonically increasing to a limit point e. By this construction, for any x from the interval (d, e), $N^n(x)$ converges to the zero of f. Furthermore, by the continuity of N, $N(d) = e$ and $N(e) = d$. Lemma 1 will be used to establish the known fact that this is the only set of period two points in interval b.

The way we prove this is by describing the graph of the component of N^2 which passes through z_0. Since N maps the region (a, z_0) in a monotonic, orientation reversing fashion to (z_0, ∞), the graph of N^2 on (a, z_0) is, essentially, a compression and reversal of the graph of N on (z_0, ∞). In particular, z_0 is a fixed point for N^2, and, according to the chain rule,

$$\left(N^2(z_0)\right)' = N'\left(N(z_0)\right)N'(z_0) = 0.$$

Furthermore, using the chain rule we have that N^2 is decreasing on the interval $(N^{-1}(z_2), z_0)$ and increasing on the interval $[d, N^{-1}(z_2))$; that is, N^2 is increasing at d. This means that on the interval (d, z_0), the graph of N^2 lies above the line $y = x$.

A direct computation which uses the signs of terms determined in Lemma 1 and the construction above demonstrates that to the left of d, the second derivative of N^2 is always negative. Now, suppose N^2 has fixed points to the left of d, and let p be the one with the smallest value. The properties of N^2 require its graph to meet the line $y = x$ from below. Consequently the slope must be at least unity at this point. If it equals unity, then by the fact that the second derivative of N^2 is strictly negative, the graph of N^2 meets the line and then drops strictly below it until at least after point d. But this contradicts the fact that d is a fixed point of N^2. If the slope of N^2 at p is greater than unity, then the graph crosses the line $y = x$, but the slope of N^2 is decreasing. Thus at the next fixed point of N^2, the graph of N^2 must meet the line $y = x$ from above. At this point the slope of N^2 is bounded above by unity. If this second fixed point is not d, then the contradiction above obtains. If it is d, then the graph crosses $y = x$ and lies below this straight line. This is a contradiction to what was established above. A similar argument (with slight complications occurring in the region (z_0, z_2)) shows that e is the only fixed point of N^2 to the right of z_0, but a simpler argument uses the graph of N to establish that for any pair of period two points in b where one of the iterates is to the right of z_0, its mate must be to the left of z_0. The conclusion follows from what has been shown above.

The Assumption on N'. The argument above demonstrates that the slope of N^2 at a period two point is bounded below by some constant greater than unity. If it is known that $|N'| \geqslant A > 1$ at all such period two points, then it follows that this same inequality holds outside of the intervals defined by the period two points. This is because at these points $N' \leqslant -A < -1$. The concavity

property of N means that N' becomes even more negative outside of these intervals.

The above means that in each of the nested sets corresponding to a symbol, each closed interval is shrinking in size. Indeed, $A > 1$ is the minimum of $|N'|$ on S_3 and B is a bound on the length of the longest bounded interval of $R/Z(f')$. Thus, the length of the nth nested interval is no more than $B/(A)^{n-1}$. From this it follows that the size of the components of the C sets goes to zero as $n \to \infty$.

Periodicity. Let $p \in CS$. It was shown above that in any neighborhood of p there is an open set containing an S_3 point. Because the size of the C sets approach zero as $n \to \infty$, there is some n such that $C(b_0,\ldots,b_n)$ is in this neighborhood of the S_3 point. It now follows either from the proof of Theorem 1, or the above, that this neighborhood contains a periodic point corresponding to the periodic sequence determined by $\{b_0,\ldots,b_n\}$.

The final theorem addresses a question raised in S. Smale's paper [25]. It shows the type of conditions on f for which N satisfies a collection of properties known as "Axiom A." (For a definition and a discussion of Axiom A, see [18].) As Smale points out, these conditions can be used to determine the measure of S_3. For more information, see [25].

THEOREM 4. *Let f satisfy the conditions of Theorem 2. Let $\Omega(N) = \{S_3 \cup Z(f)\}$. Then*

(a) *any $p \in R/\{\Omega(N) \cup S_2\}$ has a neighborhood V so that $N^n(V)$ is disjoint from V for all choices of n.*

(b) *$|(N^n(p))'| > A^n$ for $p \in S_3$ and any positive integer n.*

(c) *$|N^n(p)'| < (1/2)^n$ for $p \in Z(f)$.*

If $Z(f)$ is bounded, then the above holds for $p \in R/\Omega(N)$.

Suppose in addition that N is asymptotic to ax as $|x| \to \infty$, where a is a positive constant less than unity. Let K denote the associated circle map. Let

$$\Omega(K) = \{H^{-1}(S_3) \cup H^{-1}(Z(f)) \cup \{NP\}\}.$$

Then, for any $p \in S^1/\Omega(K)$, (a) *holds for K.*

(b′) *For $p \in \{H^{-1}(S_3) \cup NP\}$, $|DK^n(p)(v)| > CA^n|v|$, where v is a tangent vector of S^1 at p, and C is some constant.*

(c′) *For $p \in H^{-1}(Z(f))$, $|DK^n(p)(v)| < (1/2)^n|v|$.*

Conditions (b) and (c) follow from the chain rule and the fact that $|N'| \geqslant A > 1$ on S_3, while $N' = 0$ on $Z(f)$. Conditions (b′) and (c′) are almost as immediate. We use the fact that $K^n = H^{-1} \circ N^n \circ H$, the chain rule, and the conditions above on N. Also, we use the fact that S_3 lies in a compact set of R, and C denotes the maximum of the derivatives of H related to this set. Finally, $|K'(NP)| = (1/a)$. What remains is the wandering set condition (a). This is fairly straightforward. The idea is that if p were sufficiently close to a zero of f, then it would be easy to construct such a set V in terms of the value of N'; this is because each iterate of p satisfies this condition. But then backward iterates must also satisfy this separability condition. Thus, such sets can be constructed.

Incidentally, Barna [2] provides an elegant, elementary argument that a polynomial with real (not necessarily distinct) roots satisfies the condition of $|N'| \geqslant A > 1$ at the period two points. From this and from obvious modifications of the proof of Theorem 1, it follows that Theorems 1, 2, 4 can be extended to any polynomial with real roots where $Z(f')$ consists of at least three different points.

Acknowledgements. The first-named author presented these results at the 1979 TIMS/ORSA meeting in New Orleans and at one of his invited lectures during an MAA summer conference held at Michigan Tech in Houghton, Michigan, during July, 1979. These results are published now in response to renewed interest in these issues stimulated by S. Smale's paper [25], as well as their connection to the "complexity" of information. Recently, a

related study of the Newton iteration map, which emphasized different features, was undertaken by C. Martin and R. Hurley [15]. This work was supported in part by an NSF grant.

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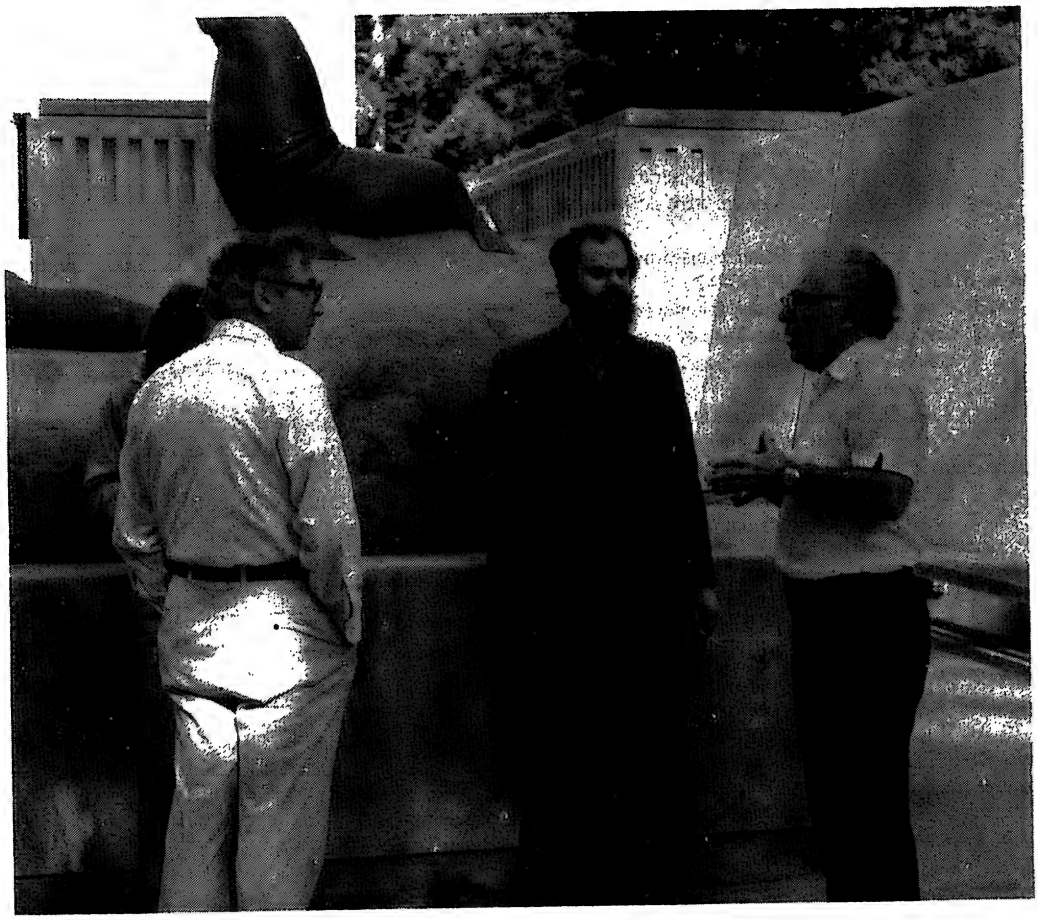
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Who is this group? The answer is simple. (See p. 51.)

COROLLARY. *Every geometric* 1*-series converges in* A *if and only if every* f *with* $\|f - 1\| < 1$ *has an inverse.*

The proof follows from that of the Proposition.

REMARK. The reader may check easily that $G(\mathcal{P})$ is not open for the normed algebra $\mathcal{P}$ of the Example.

We now present examples of incomplete normed algebras satisfying the condition that every geometric 1-series is convergent.

Let K be an infinite Hausdorff compact space. We recall that K is said to be zero-dimensional if the topology of K has a base of open-closed subsets of K. Let us suppose that K is *zero-dimensional.* Let $C(K)$ be the Banach algebra of all real or complex continuous functions defined on K. A function in $C(K)$ is called *simple* if the range of f is finite. Let $C_s(K)$ be the uniformly dense subalgebra of all simple functions of $C(K)$. We now have the following result:

THEOREM. $C_s(K)$ *is an incomplete normed algebra such that every geometric* 1*-series is convergent. In particular,* $G(C_s(K))$ *is open.*

Proof. We shall show that if f is in $C_s(K)$ with $\|f - 1\|_\infty < 1$, then f is invertible, and then invoke the Corollary. There is a finite partition of K, consisting of open-closed sets, A_j, $j = 1,\ldots,n$, i.e., pairwise disjoint finite open cover of K, such that

$$f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

where $\alpha_j, j = 1,\ldots,n$ are scalars and where χ_{A_j} is the characteristic function of A_j. Since

$$\|f - 1\|_\infty = \max_{1 \leqslant j \leqslant n} |\alpha_j - 1| < 1$$

for every j we have that $\alpha_j \neq 0$. Thus, we obtain

$$f^{-1} = \sum_{j=1}^{n} (1/\alpha_j)\chi_{A_j},$$

and, therefore, f^{-1} is a simple function.

From the Theorem we obtain the following consequence:

COROLLARY. *The normed algebra* m_0 *of all scalar sequences with finite range, provided with the supremum norm, satisfies the invertibility criterion.*

Proof. Apply the Theorem to the Stone-Čech compactification of the positive integers with the discrete topology. Alternatively, note that the proof of the theorem applies directly to the present case.

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ANSWER TO PHOTO ON PAGE 18

Reading from left to right: Walter Feit, John Thompson, Daniel Gorenstein. Photo by Joseph Gallian.

RELATIONS AMONG SOME CLASSES OF SUBSETS OF R

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Mathematicians have studied many classes of subsets of the real line R. These subsets are defined using various mathematical criteria. Most broadly, these criteria include cardinality, topological properties, and properties such as Lebesgue measure which depend rather heavily on the metric structure of R. In this paper, we study the interrelationships among several of these classes of sets. Included in the analysis are two new classes of sets which we call M-shrets and C-shrets; examples of these sets have appeared in the literature but their class-defining properties have not been previously distinguished. Finally, a Venn diagram is provided showing the inclusions, disjointness, and nonempty intersections existing among the classes of sets.

1. Definitions and Terminology. We define below the classes that are the concern of this paper and we introduce the relevant terminology. Though we will deal solely with subsets of R, some of the definitions below apply in more general settings.

(a) *Bernstein Sets.* A set B is a Bernstein set if B and its complement $\tilde{B}$ each contain at least one point from every uncountable closed set [7, p.23].

(b) *Borel Sets.* The members of the smallest σ-algebra containing the open sets. (A collection of sets is called a σ-algebra if it is closed under the formation of complements and countable unions [4, p. 28].)

(c) *Nowhere Dense Sets.* A set A is nowhere dense if for x, y in A and $x \neq y$, there is an interval between x and y that is disjoint from A.

(d) *First Category Sets.* A set A is of the first category if it is the countable union of nowhere dense sets. A set that is not of the *first category* is said to be of the *second category*. R is a second category set.

(e) *Measurable Sets.* These are the Lebesgue measurable subsets of R. A fact used repeatedly in the paper is that the properties of first category, nowhere dense, and measure zero persist under the formation of subsets.

(f) *M-shrets*: A set A is an M-shret if for every interval I,

$$\mu^*(A \cap I) = \mu^*(\tilde{A} \cap I) = \mu^*(I), \tag{1}$$

where μ^* denotes Lebesgue outer measure. An obvious consequence of this definition is that every M-shret is nonmeasurable. The term "shret" was introduced in the unpublished paper [3] and is a contraction of the words "shredding set," the "M" here referring to measure. The defining property of M-shrets has appeared previously in the literature; see for instance [1] and [4, p. 70]. We will say that a set A is an *M-shret on an interval* if (1) holds for all subintervals of some given interval. If A is an M-shret, one can show that in fact for any measurable set E,

$$\mu^*(A \cap E) = \mu^*(\tilde{A} \cap E) = \mu(E)$$

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so that in particular $A \cap E$ is not measurable if $\mu(E) > 0$, where μ denotes Lebesgue measure. (The proof relies on the approximation of bounded measurable sets by finite unions of intervals.) Finally, we note that the union of an M-shret and a set of measure zero is again an M-shret.

(g) *C-shrets.* A set A is a C-shret if for every nondegenerate interval I, $A \cap I$ and $\tilde{A} \cap I$ are both of the second category. A *C-shret on an interval* is defined, mutatis mutandis, as in (f). The "C" here refers to category. The union of a C-shret and a set of the first category is again a C-shret.

(h) *Property of Baire* (POB). A set A has the property of Baire if it can be represented in the form $A = (G \cap \tilde{P}) \cup (P \cap \tilde{G})$, where G is open and P is of the first category. Thus A is the symmetric difference of an open set and a set of the first category. Alternatively, the class of sets having POB is the smallest σ-algebra containing the open sets and the sets of first category. It can be shown that a set does not have POB if and only if it is a C-shret on some interval. (See [7, p. 19ff] and [5, p. 87ff].)

2. Some Useful Classic Examples. Our analysis makes extensive use of the examples below.

(a) C, the Cantor ternary set on [0,1]. This is an example of a measure zero, nowhere dense, uncountable closed set.

(b) FC, "Fat Cantor" sets. These are nowhere dense Cantor-like sets of positive measure on [0,1] as in [2].

(c) N, the set of normal numbers, and its complement NN the set of nonnormal numbers. A normal number, roughly speaking, is one in whose decimal expansion all blocks of digits of equal length occur with the same frequency [10, p. 94]. The set of normal numbers is of the first category. The set of nonnormal numbers is of the second category and has measure zero. See [8] for a proof of these facts.

(d) B will denote a generic Bernstein set. These can be constructed using the well-ordering principle, as in [7].

(e) NB denotes a non-Borel measure-zero subset of the Cantor set C. Such a set may be obtained either by a cardinality argument (there are 2^R measure-zero subsets of C and the cardinality of the class of all Borel sets is the same as R) or by the technique used in [6].

(f) Q will denote the set of rational numbers.

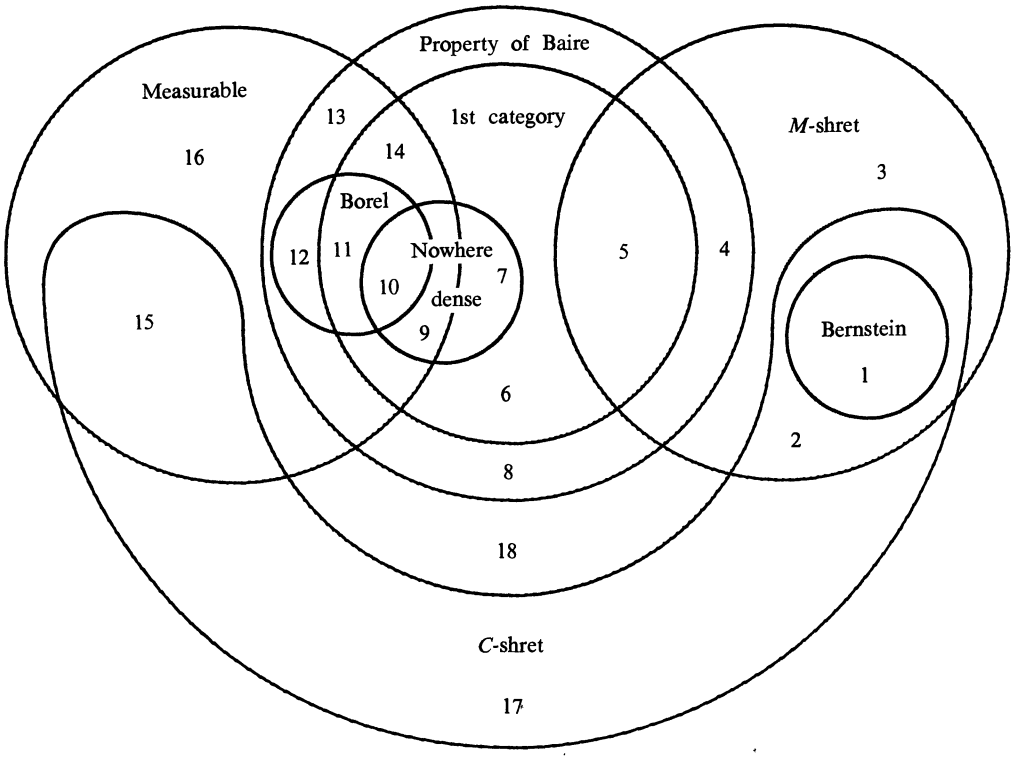
3. The Diagram with Explanations. The Venn diagram on p. 21 illustrates the relationships among the various classes of sets described in Section 1. The classes of sets identified by words or names are represented by complete disks (except that the C-shrets are represented by a distorted disk) whereas the numbers refer to disjoint portions of these regions (except for 18 which is a complement). Classes represented by disjoint disks will be shown to be disjoint and for each numbered subregion we will construct an appropriate example to establish that it is nonempty.

(1) B, a Bernstein set. It is shown in [7] that B is not measurable and does not possess POB. To be consistent with the diagram, we must show the stronger properties that B is an M-shret and a C-shret. It is shown in [1] that a set is an M-shret if and only if it and its complement each contain at least one point from every closed set of positive measure. It then follows that B is an M-shret. We now prove that B is a C-shret.

PROPOSITION. *B is a C-shret.*

Proof. Suppose B is not a C-shret. Then $\tilde{B}$ is not a C-shret and by symmetry we assume without loss of generality that there is an interval I such that $B \cap I$ is second category and $\tilde{B} \cap I$ is first category. Now $\tilde{B} \cap I$ has POB since it is first category and it follows from $B \cap I = \widetilde{(\tilde{B} \cap I)} \cap I$ that $B \cap I$ also has POB, since the sets in POB contain the intervals and form a σ-algebra. It is proved in [7, p. 24] that any subset of B with POB must be of the first category, which contradicts the assumption that $B \cap I$ is of the second category.

(2) $B \cup C$. To obtain a non-Bernstein set that is both an M-shret and a C-shret, we destroy the Bernstein property by uniting B with C (a closed, uncountable, measure-zero, first category set.)



$B \cup C$ is not a Bernstein set since its complement has no point from C. However, as explained previously the M-shret property is preserved because C is of measure zero and the C-shret property is preserved since C is of the first category.

(3) $B \cup (NN \cap [0,1])$. The set of nonnormal numbers has measure zero so $B \cup (NN \cap [0,1])$ is still an M-shret. But since the set of normal numbers is first category, the complement of $B \cup (NN \cap [0,1])$ is of the first category on $[0,1]$. Thus the set in question is not a C-shret. However, this set still does not have POB since it is a C-shret on $[1,2]$.

(4) $B \cup NN$. This set is still an M-shret since NN has measure zero, but its complement, a subset of N, is of the first category and thus POB. Therefore, $B \cup NN$ also has POB.

(5) $\widetilde{B \cup NN}$. As explained above, this set is of the first category and an M-shret.

(6) $(B \cup [0,1]) \cap N$. This set of the first category is not an M-shret on $[0,1]$ but is still an M-shret on $[1,2]$ and hence is nonmeasurable. The set contains all the normal numbers in $[0,1]$ and thus is not nowhere dense.

(7) $B \cap FC$. This is clearly nowhere dense and also nonmeasurable, since it is the intersection of an M-shret and a set of positive measure. Clearly the class of nowhere dense sets is disjoint from the M-shrets and the C-shrets.

(8) $B \cup NN \cup [0,1]$. This set is the union of the interval $[0,1]$ and the set given in (4). POB is retained but the set is not an M-shret on $[0,1]$. Also the set is of the second category on $[1,2]$ and nonmeasurable on $[1,2]$.

(9) NB. This is the non-Borel subset of the Cantor set described in Section 2.

(10) C. The Cantor set is closed and so it is Borel.

(11) Q. The set of rational numbers.

(12) $[0,1]$.

(13) $NB \cup [1,2]$. This set is the union of $[1,2]$ and the set of NB of example (9). It is of the second category, has POB and is measurable. It is clearly not Borel.

(14) $NB \cup Q$. This set is the union of the examples in (9) and (11). It is measurable, of the first category, and neither a Borel set nor a nowhere dense set.

(15) $B \cap NN$. This set is a measurable C-shret. If I is an interval, then $(B \cap I \cap N) \cup (B \cap I \cap NN) = B \cap I$ which is of the second category. Since $B \cap I \cap N$ is of the first category, $B \cap I \cap NN$ must be of the second category. $B \cap NN$ has measure zero.

(16) $(B \cap NN) \cup [0,1]$. This set is measurable and does not have POB since it is a C-shret on $[1,2]$. It is not a C-shret on $[0,1]$.

(17) $B \cup ([0,1] \cap N)$. This set is not an M-shret on $[0,1]$ (its relative complement there is of measure zero). However, it is a C-shret since the union of a set of the first category and a C-shret is still a C-shret.

(18) $B \cup [0,1]$. This set is a C-shret on $[1,2]$ and an M-shret on $[1,2]$ and hence does not have POB and is nonmeasurable. However it is neither an M-shret nor a C-shret as it contains the entire interval $[0,1]$.

4. Final Remarks. In this paper we have developed a visual model of the interrelationships among several classes of subsets of R. An interesting and salient aspect of our construction techniques is the use of the "classic sets" which incorporate the dual properties of being large with respect to one class and insignificant with respect to another. For example, the nonnormal numbers are large in a topological sense (being of the second category) but small in the sense of measure. This duality allows us to discriminate among the various classes of sets.

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BISHOP'S STONE-WEIERSTRASS THEOREM

To the Memory of Errett Bishop [1928–1983] and Silvio Machado [1932–1981]

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Introduction. The Stone-Weierstrass theorem is one of the principal pillars of modern abstract analysis and nowadays is part of every mathematician's graduate education. Many proofs of this important theorem have evolved since its discovery by Marshall Harvey Stone in 1937. Some of

A minibiography of the author appeared in this MONTHLY in vol. 88, p. 396. The only thing new that the author now reports is that he is working on the second volume of *An Introduction to Classical Complex Analysis.*

these are very elegant and even "elementary"; see, e.g., Brosowski and Deutsch (1981). On the other hand, the theorem has experienced a torrent of generalizations, including noncommutative extensions (to von Neumann algebras). One direction of generalization which is important in the study of function algebras concerns the hypothesis that the algebra be conjugate-closed. Here there is a well-known dichotomy illustrated by the closed unit disc D in the complex plane $\mathbb{C}$, the algebra A of all continuous complex-valued functions on D, and the proper subalgebra H of A comprising those functions which are also holomorphic in the interior of D. H is the uniform closure of the algebra of all polynomials in the complex variable z, while A is the uniform closure of the algebra of all polynomials in z and $\bar{z}$. The algebra H is as far from being conjugate-closed as possible: any time both a function f and its conjugate $\bar{f}$ belong to H, f is a constant. This led Errett Bishop, in the general case of a compact Hausdorff space X and an algebra A of continuous functions on it, to isolate the subsets of X which behaved like D in this respect; i.e., on which a function in A must be constant if it is real-valued. If on each of these islands of "analyticity" a continuous function f matches something in A, then the other hypotheses of the Stone-Weierstrass theorem without conjugation-closure suffice for the conclusion $f \in A$. Earlier Šilov (1951) had reached the corresponding conclusion concerning the islands of constancy when the point-separation hypothesis is dropped from the Stone-Weierstrass theorem.

Proofs of Bishop's result make substantial use of the Riesz representation theorem, the Hahn-Banach theorem, the Krein-Milman theorem and other nontrivial tools from functional analysis. [See deBranges (1959), Bishop (1961), Glicksberg (1962).] In 1971 João Prolla adapted these methods to the case of vector-valued functions and in 1977 Silvio Machado formulated a quantitative version of the theorem in the vector-valued case and gave an awesomely beautiful proof which was completely elementary. The object of this exposition is to present his proof together with the background necessary to make it completely self-contained for the reader's convenience.

For interesting short histories and references to the literature on Stone-Weierstrass theorems, see Semadeni (1971) and Bierstedt (1975, 1977). The author's monograph (1972) is devoted to related themes.

Definitions and Statements of Results. Throughout X is a nonvoid compact Hausdorff space, and E is a real or complex Banach space. $C(X, E)$ is the set of continuous maps of X into E. Each such function f is necessarily bounded and $\sup_x \|f(x)\|_E$ is a realized maximum (thanks to the compactness of X). Here $\| \|_E$ denotes the norm in E. We write $\|f\|_E$ for this supremum; it makes $C(X, E)$ into a Banach space. We write $C_{\mathbb{R}}(X)$ and $C_{\mathbb{C}}(X)$ for $C(X, \mathbb{R})$ and $C(X, \mathbb{C})$, respectively, and in these two cases write $\|f\|_\infty$ for $\|f\|_E$. For $S \subset X$ and f a function on X write $f|S$ for *the restriction of f to S*.

Notice that $C(X, E)$ is always a module, with respect to pointwise multiplication of functions, over $C_{\mathbb{R}}(X)$ and is also a module over $C_{\mathbb{C}}(X)$ when E is a complex space. Note that the norm in $C_{\mathbb{R}}(X)$ and $C_{\mathbb{C}}(X)$ is submultiplicative and these two spaces are actually Banach algebras. For $f \in C(X, E)$, $\phi \in C_{\mathbb{R}}(X)$ (respectively, $C_{\mathbb{C}}(X)$) we have $\|\phi f\|_E \leqslant \|\phi\|_\infty \|f\|_E$, so $C(X, E)$ is a Banach module.

A subalgebra A of $C_{\mathbb{R}}(X)$ or $C_{\mathbb{C}}(X)$ is called *unital* if 1 is an element of A. We say that A is *point-separating* if for every pair of distinct elements of X there is some element of A which has distinct nonzero evaluations at these points.

If A is a subalgebra of $C_{\mathbb{R}}(X)$ or $C_{\mathbb{C}}(X)$, a nonvoid subset S of X is called an *A-antisymmetric set* if whenever f is an element of A and the restriction of f to S is real-valued, then the restriction of f to S is constant. Thus for real algebras A an antisymmetric set is just the same thing as a *set of constancy* for the algebra.

If A is a subalgebra of $C_{\mathbb{R}}(X)$ or $C_{\mathbb{C}}(X)$, a nonvoid subset S of X is called an *A-peak set* if there exists an element f of A such that $S = \{x \in X: f(x) = 1\}$ and $|f(x)| < 1$ for all $x \in X \setminus S$. We say that such a function f *peaks on S*.

REMARKS 1. Let A be an algebra of continuous functions on X.

(i) If S and T are A-antisymmetric sets that have nonvoid intersection, then the union of S and T is A-antisymmetric. For if x is an element of that intersection and if f is an element of A that is real-valued on that union, then $f(S) = \{f(x)\} = f(T)$ and so $f(S \cup T) = \{f(x)\}$.
(ii) If Y is an A-antisymmetric set and $\mathscr{C}$ is a nonempty collection of A-antisymmetric sets each of which contains Y, then $\cup \mathscr{C}$ is an A-antisymmetric set. Thus the union of *all* the A-antisymmetric sets which contain Y is a (the) *maximal* A-antisymmetric set containing Y.
(iii) Each singleton $\{x\}$ is trivially an A-antisymmetric set and so by (ii) lies in a unique maximal A-antisymmetric set, which we may denote by $A(x)$.
(iv) The closure of an A-antisymmetric set is again A-antisymmetric. For if S is an A-antisymmetric set and if f is an element of A whose restriction to the closure of S is real-valued, then the restriction of f to S is constant, and so, by the continuity of f, is the restriction of f to the closure of S.
(v) Maximal A-antisymmetric sets are closed.
(vi) By (i) maximal A-antisymmetric sets are disjoint. Thus, associated to A is a unique decomposition of X into pairwise disjoint, non-void, closed, maximal A-antisymmetric sets. Denote this family by $A(X)$.

After these preliminaries we can state

ERRETT BISHOP'S THEOREM (1961). *Let X be a compact Hausdorff space, A a uniformly closed unital subalgebra of $C_{\mathbf{C}}(X)$. Then $f \in C_{\mathbf{C}}(X)$ belongs to A if and only if $f|S \in \{F|S : F \in A\}$ for each $S \in A(X)$.*

REMARKS 2. If A is *conjugate-closed* (that is, $f \in A$ implies $\bar{f} \in A$) and if A separates the points of X, then each S in $A(X)$ is a singleton. For if s and t are distinct elements of X, then there is a function $f \in A$ with $f(s) \neq f(t)$. Further, if $s \in S$, $f_1 = \operatorname{Re} f = (f + \bar{f})/2$ and $f_2 = \operatorname{Im} f = (f - \bar{f})/2i$ lie in A, are real-valued and $f_j(s) \neq f_j(t)$ for at least one $j \in \{1,2\}$. The function f_j is real-valued on S and hence is constant on S, so $t \notin S$.

With every S being a single point, every $f \in C_{\mathbf{C}}(X)$ trivially satisfies $f|S \in \{F|S : F \in A\}$, for every $S \in A(X)$, so $f \in A$; that is, $A = C_{\mathbf{C}}(X)$.

This is Marshall Harvey Stone's generalization (1937) of the Weierstrass polynomial approximation theorem (1885).

Our goal is the following quantitative and qualitative generalization of the above.

PRINCIPAL THEOREM [Silvio Machado (1977)]. *Let G be a vector subspace of $C(X, E)$ which is a module over a unital subalgebra A of $C_{\mathbf{R}}(X)$ or $C_{\mathbf{C}}(X)$. Then for each $f \in C(X, E)$ there exists an A-antisymmetric set S_f such that*

$$\inf\{\|f - g\|_E : g \in G\} = \inf\{\|f|S_f - g|S_f\|_E : g \in G\}.$$

REMARKS 3. Let A be a unital subalgebra of $C_{\mathbf{R}}(X)$.

(i) A nonempty intersection of finitely many A-peak sets is an A-peak set. If A is uniformly closed, then it is even true that a nonempty intersection of countably many A-peak sets is an A-peak set. Indeed, if S_n is an A-peak set and f_n peaks on S_n, then $\|f_n\|_\infty = 1$ and $f = \sum_{n=1}^{\infty} 2^{-n} f_n \in A$. If $\cap_{n=1}^{\infty} S_n$ is not empty, then f evidently peaks there.

(ii) Each Y in $A(X)$ is an intersection of A-peak sets. For, as noted before, A being real, A-antisymmetric sets are just sets of constancy for A and so if $y \in Y$ is fixed, then $Y = \{x \in X : f(x) = f(y)$ for each $f \in A\}$, by maximality. Thus, setting $Y_f = \{x \in X : f(x) = f(y)\}$, we have $Y = \cap_{f \in A} Y_f$, so it suffices to show that each Y_f is an A-peak set. Evidently the function $F = 1 - \varepsilon(f - f(y))^2$ peaks on Y_f if the positive number ε is small enough. [This result is also true, but harder to prove, for complex algebras. See the last theorem below.]

(iii) If Y is in $A(X)$ and K is a compact subset of X that misses Y, then there is an element a of A

such that $0 < a \leqslant 1$, $a(Y) = \{1\}$, and $a < 1/2$ on K. Since $X \setminus K$ is an open neighborhood of the compact set $Y = \cap_{f \in A} Y_f$, some finite number of the Y_f's have their intersection in $X \setminus K$ too. This intersection is an A-peak set by (i) above. Thus there is an $f \in A$ such that $Y \subset \{x \in X : f(x) = 1\} \subset X \setminus K$ and $|f(x)| < 1$ for $x \in X \setminus f^{-1}(\{1\})$. In particular, $\sup|f(K)| = \max|f(K)| < 1$ and so we can use $a = ((f + 1)/2)^m$ for some sufficiently large positive integer m.

LEMMA 1 [On Polynomial Separation—Robert Jewett (1963)]. *Let ε and δ with $0 < \varepsilon, \delta < 1/2$ be given. Then there exists a polynomial q with real coefficients such that*

$$\begin{aligned} &\text{(a)} \quad 0 \leqslant q(t) \leqslant 1 \quad \textit{for every} \quad t \in [0,1], \\ &\text{(b)} \quad 0 \leqslant q(t) < \delta \quad \textit{for every} \quad t \in [0,\varepsilon], \\ &\text{(c)} \quad 1 - \delta < q(t) \leqslant 1 \quad \textit{for every} \quad t \in [1-\varepsilon,1]. \end{aligned}$$

Proof. We'll find q of the form $q(t) = 1 - (1 - t^m)^n$. First choose a positive integer r so that $(3/4)^r < \delta$. Next choose a positive integer m so large that

$$1/(4(1-\varepsilon)^m) > 1 \tag{1}$$

and

$$((1-\varepsilon)/\varepsilon)^m(\delta/r) > 1 \tag{2}$$

(possible since $(1 - \varepsilon)/\varepsilon > 1$). Then $1/(1 - \varepsilon)^m - (3/4)/(1 - \varepsilon)^m > 1$, so an integer s can be found between these two:

$$(3/4)/(1-\varepsilon)^m \leqslant s < 1/(1-\varepsilon)^m < (\delta/r)(1/\varepsilon^m) \text{ (by (2))}. \tag{3}$$

Let $n = rs$ and note that by the second and third inequalities in (3)

$$n\varepsilon^m < \delta \tag{4}$$

and by the first and second inequalities in (3)

$$3/4 \leqslant s(1-\varepsilon)^m < 1. \tag{5}$$

Hence,

$$(1-\varepsilon^m)^n \geqslant 1 - n\varepsilon^m > 1 - \delta \text{ (by Bernoulli's inequality and (4))} \tag{6}$$

and

$$\begin{aligned} (1-(1-\varepsilon)^m)^n &= \left[(1-(1-\varepsilon)^m)^s\right]^r \leqslant \left[1 - s(1-\varepsilon)^m + (1/2)(s(1-\varepsilon)^m)^2\right]^r \\ &\overset{(5)}{<} \left[(1/4) + (1/2)(1)^2\right]^r = (3/4)^r < \delta. \end{aligned} \tag{7}$$

The inequality $(1 - x)^k \leqslant 1 - kx + \frac{1}{2}(kx)^2$, for $0 < x < 1$, used in the penultimate line is easily confirmed by induction on k.

Now set $p(t) = (1 - t^m)^n$ and note that $0 \leqslant t \leqslant \varepsilon$ implies that $p(t) > 1 - \delta$, and $1 - \varepsilon \leqslant t \leqslant 1$ implies that $p(t) \leqslant p(1 - \varepsilon) < \delta$. It remains to put $q(t) = 1 - p(t)$.

REMARKS. The polynomials q produced in the proof actually have integer coefficients and this is useful in some applications. See Ferguson [1980] for more on this theme. We use the lemma just to avoid appealing to the classical Weierstrass theorem, in order that the latter be a genuine corollary of the Principal Theorem.

LEMMA 2 [Special partition of unity]. *Let X be a compact Hausdorff space, B a unital subalgebra of $C_{\mathbb{R}}(X)$. To each Y in $B(X)$ let there be associated a compact subset D_Y of $X \setminus Y$. Then there exist elements $Y_1, \ldots, Y_n$ of $B(X)$ with the following property: given $\delta > 0$, nonnegative $b_1, \ldots, b_n$ can be found in B which satisfy $b_1 + \cdots + b_n = 1$, and $b_j < 2\delta$ on D_{Y_j} $(j = 1, \ldots, n)$.*

Proof. According to Remark 3 (iii) we can, for each Y in $B(X)$, find a function c_Y in B such that

$$0 < c_Y \leqslant 1, \qquad c_Y(Y) = \{1\}, \quad \text{and} \quad c_Y < 1/2 \text{ in } D_Y.$$

Then the number $u_Y = \sup\{c_Y(x) : x \in D_Y\}$ is less than $1/2$ and we can choose a number v_Y such that $0 < u_Y < 1 - u_Y < v_Y < 1$. The set Y lies in the open set $U_Y = \{x \in X : c_Y(x) > v_Y\}$ and so by compactness there exist $Y_1, \ldots, Y_n$ in $B(X)$ such that

$$X = U_{Y_1} \cup \cdots \cup U_{Y_n}.$$

Now let $\delta > 0$ be given. Cite Lemma 1 to get polynomials q_j with real coefficients such that

$$0 \leqslant q_j \leqslant 1 \text{ on } [0,1], \qquad 0 \leqslant q_j < \delta \text{ on } \left[0, u_{Y_j}\right], \quad \text{and} \quad 1 - \delta < q_j \leqslant 1 \text{ on } \left[v_{Y_j}, 1\right].$$

Define the functions g_j to be $q_j \circ c_{Y_j}$. These functions belong to B, since B is a real unital algebra. Define

$$b_1^* = g_1, b_2^* = (1 - g_1)g_2, \ldots, b_k^* = (1 - g_1)\cdots(1 - g_{k-1})g_k, \ldots,$$
$$b_n^* = (1 - g_1)\cdots(1 - g_{n-1})g_n.$$

These too are all functions in B. The equation

$$b_1^* + \cdots + b_k^* = 1 - (1 - g_1)\cdots(1 - g_k)$$

holds for $k = 1$ and if true for some $k < n$, then

$$\begin{aligned} b_1^* + \cdots + b_{k+1}^* &= 1 - (1 - g_1)\cdots(1 - g_k) + (1 - g_1)\cdots(1 - g_k)g_{k+1} \\ &= 1 - (1 - g_1)\cdots(1 - g_k)(1 - g_{k+1}). \end{aligned}$$

Hence by induction

$$b_1^* + \cdots + b_n^* = 1 - (1 - g_1)\cdots(1 - g_n).$$

From the properties of the q_j and the c_{Y_j} we clearly have $0 \leqslant g_j \leqslant 1$ for every j. Hence also $0 \leqslant 1 - g_j \leqslant 1$ and it follows that $0 \leqslant b_j^* \leqslant 1$. On D_{Y_j} we have by definition $c_{Y_j} \leqslant u_{Y_j}$, so $g_j = q_j \circ c_{Y_j} < \delta$ by the second property of q_j. Hence

$$0 \leqslant b_j^* \leqslant g_j < \delta \text{ on } D_{Y_j}.$$

Now for each $x \in X$ there is a $k(x)$ in $\{1, 2, \ldots, n\}$ such that $x \in U_{Y_{k(x)}}$. Then $1 \geqslant g_{k(x)}(x) \geqslant 1 - \delta$, since $c_{Y_{k(x)}}(x) > v_{Y_{k(x)}}$ and $q_{k(x)} > 1 - \delta$ on $[v_{k(x)}, 1]$. It follows that

$$\begin{aligned} 1 \geqslant (b_1^* + \cdots + b_n^*)(x) &= 1 - \left(1 - g_{k(x)}(x)\right) \prod_{j \neq k(x)} \left(1 - g_j(x)\right) \\ &\geqslant 1 - \delta \prod_{j \neq k(x)} \left(1 - g_j(x)\right) \geqslant 1 - \delta. \end{aligned}$$

Finally, set

$$b_1 = b_1^*, \ldots, b_{n-1} = b_{n-1}^*, b_n = 1 - b_1^* - \cdots - b_{n-1}^*.$$

Then $b_n \leqslant b_n^* + \delta < 2\delta$ on D_{Y_n} and $b_n \geqslant b_n^* \geqslant 0$ throughout X. Hence these functions have all the desired properties.

REMARK. Thanks to an observation of K. A. Ross, Lemma 2 here is a little stronger than the corresponding one used by Machado.

THEOREM. *Let E be a real or complex Banach space, B a unital subalgebra of $C_{\mathbf{R}}(X)$, G a vector subspace of $C(X, E)$ which is also a B-module. Then for each f in $C(X, E)$*

$$\inf\{\|f - g\|_E : g \in G\} = \sup_{Y \in B(X)} \inf\{\|f|Y - g|Y\|_E : g \in G\}.$$

Proof. One inequality is obvious: if we fix f and set

$$d(Y) = \inf\{\|f|Y - g|Y\|_E : g \in G\}, Y \in B(X),$$
$$d = \sup\{d(Y) : Y \in B(X)\},$$
$$d^* = \inf\{\|f - g\|_E : g \in G\},$$

then $d^* \geqslant d(Y)$ is obvious for each Y, so $d^* \geqslant d$ and what we must show is that $d^* \leqslant d$.

To this end let $\varepsilon > 0$ be given. For each Y choose $g_Y \in G$ such that

$$d(Y) + \varepsilon > \|(f - g_Y)|Y\|_E = \sup\{\|f(x) - g_Y(x)\|_E : x \in Y\}.$$

For each Y the set

$$D_Y = \{x \in X : \|f(x) - g_Y(x)\|_E \geqslant d(Y) + \varepsilon\}$$

is compact and disjoint from Y. Let $Y_1, \ldots, Y_n$ in $B(X)$ be as furnished by Lemma 2. Choose

$$K > \|f\|_E + \|g_{Y_1}\|_E + \cdots + \|g_{Y_n}\|_E$$

and

$$0 < \delta < \varepsilon/(2nK).$$

Consider the $b_1, \ldots, b_n$ in B provided by the lemma for this δ and set

$$g = b_1 g_{Y_1} + \cdots + b_n g_{Y_n} \in G$$

(since G is a module over B). We have

$$f - g = \sum_{j=1}^{n} b_j \cdot (f - g_{Y_j}),$$

since $b_1 + \cdots + b_n = 1$. Now consider a fixed $x \in X$. If $x \notin D_{Y_j}$, then the definition of the latter set shows that

$$\|b_j(x)(f(x) - g_{Y_j}(x))\|_E = \|f(x) - g_{Y_j}(x)\|_E b_j(x)$$
$$\leqslant (d(Y_j) + \varepsilon) b_j(x).$$

While if x is in D_{Y_j}, then

$$\|b_j(x)(f(x) - g_{Y_j}(x))\|_E = b_j(x)\|f(x) - g_{Y_j}(x)\|_E$$
$$\leqslant b_j(x) K \quad \text{(choice of } K\text{)}$$
$$\leqslant 2\delta K \quad \text{(properties of } b_j \text{ on } D_{Y_j}\text{)}.$$

Summing over j gives the estimates

$$\|f(x) - g(x)\|_E \leqslant \sum_{j=1}^{n} \|b_j(x)(f(x) - g_{Y_j}(x))\|_E$$
$$\leqslant \sum_{j=1}^{n} [(d(Y_j) + \varepsilon) b_j(x) + 2\delta K]$$
$$\leqslant \sum_{j=1}^{n} (d + \varepsilon) b_j(x) + 2n\delta K \leqslant d + \varepsilon + 2n\delta K$$
$$\text{(definitions of } b_j\text{'s and } d\text{)}$$
$$< d + 2\varepsilon \quad \text{(choice of } \delta\text{)}.$$

This holds for every $x \in X$, so $\|f - g\|_E < d + 2\varepsilon$, that is, $d^* \leqslant d + 2\varepsilon$. As $\varepsilon > 0$ is arbitrary, we

have $d^* \leqslant d$, as needed.

COROLLARY. *Under the hypotheses of the last theorem, the supremum there is a maximum, i.e., there exists a* $Y = Y_f \in B(X)$ *such that*

$$\inf\{\|f - g\|_E : g \in G\} = \inf\{\|f|Y - g|Y\|_E : g \in G\}.$$

Proof (suggested by S. Saeki). In the language of the last proof, there exists by definition of d for each $n \in \mathbb{N}$ a $Y_n \in B(X)$ such that $d(Y_n) > d - (1/n)$. Select $x_n \in Y_n$ and let x_0 be a cluster point of $\{x_n : n \in \mathbb{N}\}$; say $x_0 \in Y_0 \in B(X)$. We will show that $d(Y_0) = d$, which will complete the proof. To this end, consider any $d_0 > d(Y_0)$. Then by definition of $d(Y_0)$ there exists $g \in G$ such that

$$\|f|Y_0 - g|Y_0\|_E < d_0.$$

The set

$$U = \{x \in X : \|f(x) - g(x)\|_E < d_0\}$$

is open and contains Y_0. Since Y_0 is a maximal set of constancy for B-functions, for each $x \in X \setminus Y_0$ there exists $b_x \in B$ such that $b_x(x) \neq b_x(Y_0)$. Since B is a unital algebra, we can choose this function so that $b_x(x) = 1$ and $b_x(Y_0) = 0$. The open set $b_x^{-1}(]1/2, \infty[)$ is then a neighborhood of x, so by compactness

$$X \setminus U \subset \bigcup_{j=1}^{m} b_{y_j}^{-1}(]1/2, \infty[)$$

for finitely many $y_1, \ldots, y_m \in X \setminus U$. The function $b = b_{y_1}^2 + \cdots + b_{y_m}^2$ lies in B and satisfies $b(Y_0) = 0$, $b > \frac{1}{4}$ on $X \setminus U$. Since $b(x_0) = 0$ and x_0 is a cluster point of $\{x_n\}$, we have $b(x_n) < \frac{1}{4}$ for infinitely many n. For these n then $b(Y_n) = b(x_n) < \frac{1}{4}$, so Y_n cannot meet the set $X \setminus U$, in which $b > \frac{1}{4}$. That is, $Y_n \subset U$ for infinitely many n. For these n we then have

$$\|f(x) - g(x)\|_E < d_0$$

for every $x \in Y_n$, by definition of U. That is,

$$\|(f - g)|Y_n\|_E < d_0,$$

whence $d(Y_n) < d_0$. Since this holds for infinitely many n, it follows from the original choice of the Y_n that $d \leqslant d_0$. Since $d_0 > d(Y_0)$ is arbitrary, we infer that $d \leqslant d(Y_0)$. The reverse inequality being obvious from the definition of d, the proof is thus complete.

We're now ready to begin the proof of the Principal Theorem. The Axiom of Choice in the form of Zorn's Lemma will be used. The latter states that if " $\leqslant$ " partially orders a nonvoid set S and every linearly ordered subset of S (i.e., "chain") has an upper bound in S, then S possesses a maximal element. The proof below is from Machado (1977), but is reminiscent of the original argument of Bishop (1961).

Proof of Principal Theorem. Let $f \in C(X, E)$ be given. Set

$$d^* = \inf\{\|f - g\|_E : g \in G\}.$$

If $d^* = 0$, then $0 = d^* \geqslant \inf\{\|f|S - g|S\|_E : g \in G\}$ for every S in $A(X)$ shows that the conclusion is trivial in this case. So assume $d^* > 0$.

Let $\mathscr{D}$ be the set of all ordered pairs $(\mathscr{P}, S)$ such that

(i) $\mathscr{P}$ is a partition of X into disjoint, nonempty closed subsets,

(ii) $S \in \mathscr{P}$ and $d^* = \inf\{\|f|S - g|S\|_E : g \in G\}$.

The class $\mathscr{D}$ contains the pair $(\{X\}, X)$ and so $\mathscr{D} \neq \varnothing$. Partially order $\mathscr{D}$ by the decree

$$(\mathscr{P}, S) \leqslant (\mathscr{Q}, T) \text{ if and only if } \mathscr{Q} \text{ refines } \mathscr{P} \text{ and } T \subset S.$$

We want to show that any chain $\mathscr{C}$ in $\mathscr{D}$ has an upper bound in $\mathscr{D}$. We manufacture such an upper

bound as follows: For each $x \in X$ and $(\mathscr{P}, S) \in \mathscr{C}$ let $\mathscr{P}(x)$ be the unique set in $\mathscr{P}$ which contains x. Set

$$\mathscr{Q}(x) = \cap\{\mathscr{P}(x) : (\mathscr{P}, S) \in \mathscr{C}\} \supset \{x\}.$$

Evidently $\mathscr{Q} = \{\mathscr{Q}(x) : x \in X\}$ is a partition of X into nonempty, compact subsets. These subsets are disjoint:

$$\mathscr{Q}(x) \cap \mathscr{Q}(y) \neq \varnothing \Rightarrow \mathscr{P}(x) \cap \mathscr{P}(y) \neq \varnothing$$

for every $(\mathscr{P}, S)$ in $\mathscr{C} \Rightarrow \mathscr{P}(x) = \mathscr{P}(y)$ for every $(\mathscr{P}, S)$ in $\mathscr{C} \Rightarrow \mathscr{Q}(x) = \mathscr{Q}(y)$. Finally define

$$T = \cap\{S : (\mathscr{P}, S) \in \mathscr{C}\}.$$

We will show that $(\mathscr{Q}, T) \in \mathscr{D}$ and is an upper bound for the chain $\mathscr{C}$. To this end it suffices to show that T is nonempty and that

$$(*) \qquad d^* = \inf\{\|f|T - g|T\|_E : g \in G\}.$$

For each g in G and $(\mathscr{P}, S)$ in $\mathscr{C}$ the sets

$$K(g) = \{x \in T : \|f(x) - g(x)\|_E \geqslant d^*\},$$

$$K(g, \mathscr{P}, S) = \{x \in S : \|f(x) - g(x)\|_E \geqslant d^*\}$$

are compact and evidently

$$(**) \qquad K(g) = \cap\{K(g, \mathscr{P}, S) : (\mathscr{P}, S) \in \mathscr{C}\}.$$

Each $K(g, \mathscr{P}, S)$ is nonempty because $d^* = \inf\{\|f|S - h|S\|_E : h \in G\}$ and the set of these therefore enjoys the finite intersection property. Indeed if $(\mathscr{P}_1, S_1), \ldots, (\mathscr{P}_n, S_n) \in \mathscr{C}$, then, $\mathscr{C}$ being linearly ordered, we can suppose the notation chosen so that

$$(\mathscr{P}_1, S_1) \leqslant \cdots \leqslant (\mathscr{P}_n, S_n).$$

But then, from the definition of $\leqslant$, it is clear that $S_1 \cap \cdots \cap S_n = S_n$, so

$$K(g, \mathscr{P}_1, S_1) \cap \cdots \cap K(g, \mathscr{P}_n, S_n) = K(g, \mathscr{P}_n, S_n) \neq \varnothing.$$

We infer on compactness grounds and from (**) that $K(g) \neq \varnothing$. Since $K(g) \subset T$ for each g, we get

$$\begin{aligned} \inf\{\|f|T - g|T\|_E : g \in G\} &\geqslant \inf\{\|f|K(g) - g|K(g)\|_E : g \in G\} \\ &\geqslant d^*, \text{ by definition of } K(g). \end{aligned}$$

Of course this infimum is obviously not greater than d^*, by definition of d^*, so the desired equality (*) holds.

Zorn now provides a maximal element $(\mathscr{P}, S)$ in $\mathscr{D}$. We will prove that S is an A-antisymmetric set. Because of property (ii) of S, this will prove the theorem. To this end introduce $A_S = \{f \in A : f(S) \subset \mathbb{R}\}$ and $B = A_S|S$. The latter is a unital subalgebra of $C_{\mathbb{R}}(S)$. We now assume, contrary to what we want, that S is *not* an A-antisymmetric set. That means that the decomposition $B(S)$ is not trivial, i.e., consists of not just S alone. Now $G|S$ is a B-module and so the last theorem provides T in $B(S)$ such that

$$(***) \qquad d^* = \inf\{\|f|S - g|S\|_E : g \in G\} = \inf\{\|f|T - g|T\|_E : g \in G\}.$$

Of course $\varnothing \neq T \subsetneq S$. Consider the partition $\mathscr{Q}$ of X consisting of the elements of $\mathscr{P}$ different from S together with all the sets in $B(S)$. It is strictly finer than $\mathscr{P}$ and (***) shows that $(\mathscr{Q}, T) \in \mathscr{D}$. The maximality of $(\mathscr{P}, S)$ is contradicted, and therewith the A-antisymmetry of S is established.

As noted in the introduction, the Principal Theorem has many wonderful applications, some of which are treated in detail in Burckel (1972). A simple application which nevertheless secures a nontrivial result is the following one from Glicksberg (1962):

COROLLARY. *A closed ideal in a function algebra is conjugate-closed to the extent that the algebra itself is. In fact, if A is a subalgebra of $C_{\mathbb{C}}(X)$, I a uniformly closed ideal in A, and f and $\bar{f} \in A$, then the following are equivalent: $f \in I$, $|f|^2 \in I$, $\bar{f} \in I$.*

Proof. Without loss of generality we may suppose that $1 \in A$. Clearly it suffices to show that $|f|^2 \in I$ implies $f \in I$. To this end, consider $S \in A(X)$. The functions $f_1 = \operatorname{Re} f = (f + \bar{f})/2$ and $f_2 = \operatorname{Im} f = (f - \bar{f})/2i$ belong to A and are real-valued on S. Hence they are constant on S, so $f = f_1 + if_2$ is constant on S, say $f = c_S$. If $c_S = 0$, then $f|S = 0 \in I|S$. If $c_S \neq 0$, then on S, $f = (1/\bar{c}_S)f\bar{f} = (1/\bar{c}_S)|f|^2$, so $f|S \in I|S$. Thus $f|S \in I|S$ for every $S \in A(X)$. It follows from the Principal Theorem (with $E = \mathbb{C}$, $G = I$) that the distance from f to I in the uniform norm is 0. That is, $\|f - f_n\|_\infty \to 0$ for some $f_n \in I$. Since $f \in A$ and I is closed in A, we infer that $f \in I$.

Many applications of Bishop's Theorem, however, rely crucially on one other fact, which Bishop also noted, namely that each restriction algebra $A|Y$ is uniformly closed in $C_{\mathbb{C}}(Y)$. This is because for effective application the hypotheses on A should pass over to the restriction algebras, and a basic hypothesis when functional analysis techniques are involved is the completeness of A.

So perhaps it would be premature to close without indicating how this fact follows from elementary considerations.

THEOREM. *Let X be a compact Hausdorff space and A a uniformly closed unital subalgebra of $C_{\mathbb{C}}(X)$. Then each maximal A-antisymmetric set is the intersection of the A-peak sets which contain it.*

Proof. Consider $Y \in A(X)$. There are A-peak sets which contain Y, e.g., X. Let $\mathscr{P}$ denote the family of all of them, $Z = \cap\mathscr{P}$. We want to show that $Y = Z$; to this end we suppose that $Y \subsetneq Z$ and deduce a contradiction. If we look at the algebra $B = \{f|Z : f \in A \,\&\, f(Z) \subset \mathbb{R}\}$, then every function in B is constant on Y, so Y lies in some set $W \in B(Z)$. The maximality of Y shows that $W \neq Z$. According to Remark 3 (iii), there is then a function $h_0 \in B$ with $h_0(Z) \subset [0,1]$, $h_0(W) = \{1\}$ and $h_0(z_0) < \frac{1}{2}$ for some $z_0 \in Z \setminus W$. Let $f_0 \in A$ with $f_0|Z = h_0$ and for each positive integer n look at the set

$$V_n = \{x \in X : |f_0(x)| < 1 + 2^{-n}\}.$$

This is an open neighborhood of Z. Hence by the finite intersection property of $\mathscr{P}$ (Remark 3 (i)), V_n contains an element of $\mathscr{P}$. A sufficiently high power of a peaking function then provides us with an $f_n \in A$ such that

$$\|f_n\|_\infty = 1, \qquad f_n(Z) = \{1\} \,\&\, |f_n| < (2^n\|f_0\|_\infty)^{-1} \quad \text{on } X \setminus V_n. \tag{1}$$

Consider $f = f_0\sum_{n=1}^{\infty} 2^{-n}f_n$. Since A is a uniformly closed algebra, $f \in A$. Also $|f| \leqslant |f_0|$. Since $|f_0| \leqslant 1$ on $\cap_{n=1}^{\infty} V_n$ is clear from the definition of the V_n, we then see that

$$|f| \leqslant 1 \text{ on } \bigcap_{n=1}^{\infty} V_n. \tag{2}$$

Consider $x \in X \setminus \cap_{n=1}^{\infty} V_n$. Setting $V_0 = X$, there is then an $m \geqslant 1$ such that $x \in V_{m-1}$ while $x \notin V_n$ for $n \geqslant m$. Thus

$$|f_0(x)f_n(x)| \leqslant 1 + 2^{-m+1} \qquad \text{for } n = 1, 2, \ldots, m-1, \tag{3}$$

since $\|f_n\|_\infty = 1$ while (if $m - 1 \geqslant 1$) $|f_0| < 1 + 2^{-m+1}$ on V_{m-1}. Also

$$|f_0(x)f_n(x)| < 2^{-n} < 2^{-m+1} \qquad \text{for } n \geqslant m \tag{4}$$

by (1) and the fact that $x \notin V_n$. Combining (3) and (4) gives

$$\begin{aligned} |f(x)| &\leqslant (1 + 2^{-m+1})\sum_{n=1}^{m-1} 2^{-n} + 2^{-m+1}\sum_{n=m}^{\infty} 2^{-n} \\ &= (1 + 2^{-m+1})(1 - 2^{-m+1}) + 2^{-m+1}2^{-m+1} = 1, \end{aligned}$$

holding for each $x \in X \setminus \cap_{n=1}^{\infty} V_n$. Combining this inequality with (2) shows that $\|f\|_\infty \leqslant 1$. It follows that the function $\frac{1}{2}(f+1) \in A$ peaks on the set $P = f^{-1}(\{1\})$. Since $f_0(W) = h_0(W) = \{1\}$ and $f_n(Z) = \{1\}$ for all n, we have $f = 1$ on $Y \subset W \subset Z$, so $Y \subset P$. Consequently $P \in \mathscr{P}$ and so $Z = \cap\mathscr{P} \subset P$. But $z_0 \in Z$ and

$$|f(z_0)| \leqslant |f_0(z_0)| = |h_0(z_0)| < \tfrac{1}{2},$$

so $z_0 \notin f^{-1}(\{1\}) = P$. Contradiction.

COROLLARY. *Under the hypotheses of the Principal Theorem, if A and G are complete, then $G|Y$ is closed in $C(Y, E)$ for each maximal A-antisymmetric set Y.*

Proof. The usual maneuvers involving fast Cauchy subsequences show that it suffices to demonstrate that $g \in G|Y$ whenever $g = \Sigma_{n=1}^{\infty} g_n$ with $g_n \in G|Y$ and $\|g_n\|_E < 2^{-n}$. Let $f_n \in G$ satisfy $f_n|Y = g_n$. Then

$$V_n = \{x \in X : \|f_n(x)\|_E < 2^{-n}\}$$

is a neighborhood of Y and so by the preceding theorem and Remark 3 (i) this set contains an A-peak set over Y. A sufficiently high power of a peaking function provides us with a $\phi_n \in A$ such that

$$\|\phi_n\|_\infty = 1, \qquad \phi_n(Y) = \{1\} \;\&\; |\phi_n| < \left[2^n(\|f_n\|_E + 1)\right]^{-1} \quad \text{on } X \setminus V_n.$$

Then $\phi_n f_n|Y = g_n$ and

$$\|\phi_n(x) f_n(x)\|_E \leqslant \begin{cases} \|f_n(x)\|_E < 2^{-n} & \forall x \in V_n \\ \|f_n(x)\|_E \left[2^n(\|f_n\|_E + 1)\right]^{-1} < 2^{-n} & \forall x \in X \setminus V_n. \end{cases}$$

Therefore $\|\phi_n f_n\|_E < 2^{-n}$, so the series $\Sigma_{n=1}^{\infty} \phi_n f_n$ converges in the closed submodule G of $C(X, E)$ to a function F which clearly satisfies $F|Y = \Sigma g_n = g$.

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WHAT IS LISP?

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1. Introduction. LISP, according to some computer scientists, is the only language in which civilized people write computer programs. According to others, it is the embodiment of mathematical logic applied to computer programming. According to others, it is the language of choice for building highly complex systems. LISP stands for LISt Processing. Invented by John McCarthy in the early 1960's for use in symbol manipulations and artificial intelligence, LISP is at once among the oldest programming languages in common use today and among the newest. It occupies a special place at the junction between computer science and mathematics. In this article we discuss only portions of LISP. These portions have nice mathematical properties and are shared by most versions of the language. We shall refer to these portions as "idealized LISP."

2. Algorithms and Programs. The purpose of a computer is to execute our algorithms. One of the features which distinguish algorithms from related entities such as recipes is that an algorithm is "definite." Every step must be precisely and unambiguously defined. We must, in principle, be able to lay out each step in sufficient detail so that a student or clerk will be able to apply the algorithm without asking any questions. Similarly, when we wish to have one of these algorithms executed by a computer, we need to set forth every step in the algorithm precisely, in a language that the computer "understands." Such a precise expression of an algorithm is called a program. The language in which such a program might be written is called a programming language. Typical programming languages include FORTRAN, BASIC, and of course LISP.

If we are to have any confidence in our translation of an algorithm into a programming language, the meaning of each statement in the programming language must likewise be precise and unambiguous. In the next section, we shall see some of the difficulties that arise when we attempt to understand conventional programming languages precisely.

3. What's Wrong with Conventional Programming Languages? A program is a statement of an algorithm in a formal, precise language. When one runs a program on a machine, the machine executes the statements in the program, thus performing the algorithm. While many algorithms are familiar even in elementary mathematics, the student's first exposure to the idea of a formal language usually comes only when he takes a course in mathematical logic. The languages in which one writes computer programs, however, do not usually look like the language of predicate

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logic. A typical fragment of a computer program might look something like:

READ(X)
$Y = X$
$X = X + 1$
PRINT(X).

This fragment instructs the machine to "read" a value (perhaps by waiting for a human to type it in) and to set the variable X equal to that value. The second line instructs the machine to set the value of the variable Y equal to the value of X. The third line instructs the machine to take the value of X, add 1 to it, and make that the new value of X. The last line tells the machine to print the value of X. Thus, if one typed in "3", this program would cause the machine to type back "4".

The problem with this program, from a mathematical point of view, is that it has five occurrences of the variable X, which mean several different things. The third and fifth occurrences of X seem to refer to the value 4. The fourth occurrence refers to the value 3. It is not clear what the first occurrence refers to, if anything. Because a symbol like X may have different meanings at different places in a program, one may not even perform as elementary an operation as substituting equals for equals in a program. For instance, one may not change the last line in the fragment to PRINT(Y) (despite the presence of $Y = X$ on line 2), since the value of Y which will be printed is still 3, not 4. The symbols X and Y, which we have called "variables," simply do not behave the way variables do in mathematics.

It is possible to construct mathematical models for the behavior of such programs. A better tack, however, is to use a radically different programming language, such as LISP, in which a variable behaves as a mathematical variable should. Programming languages in which variables behave mathematically are called "functional languages". APL, for example, is a functional language organized around the concept of an array. LISP combines the elegance of a functional language with the extremely flexible data structure of lists. We shall explore each of these concepts in turn.

4. Organization of a LISP program. Before discussing the details of LISP, we need to sketch the environment in which LISP operates. Typically, the programmer sits at a terminal with a typewriter keyboard and a TV screen or typewriter-like printer. He types on the keyboard, and the characters typed in, along the LISP's responses, are displayed on the screen or printed on the printer. We say "LISP prints such-and-such" just as we said "the machine prints such-and-such" before: LISP can be thought of as a machine itself, not just as a language. A LISP can be built either directly out of electronic components or by writing a program in a conventional programming language which simulates a LISP; the details of how this is done are of little concern mathematically.

In LISP, the basic units from which programs are built are functions, in the familiar mathematical sense. The things one types when sitting at the terminal are divided into function definitions and function evaluations. When LISP is ready to listen to the programmer's typing, it prints an asterisk. Thus a fragment of a conversation with LISP might look as follows:

*define *double*[x] = $x + x$

double defined

**double*[3]

6

*define *abs*[z] = if $z < 0$ then $-z$ else z

abs defined

```
*abs[-2]
2
*double[abs[double[-2]]]
8.
```

The lines following the asterisks were typed by the programmer; the others were printed by LISP. In the first line, the programmer has defined a function called "*double*". LISP responds by printing the message "*double* defined". The programmer then tests the function by evaluating it with argument 3. LISP responds with 6, as expected. The programmer then defines another function of one argument, called "*abs*," and tests it on the argument -2; LISP responds with 2, as desired. Last, the programmer does a more complicated evaluation: *double* of -2 is -4; since -4 is less than 0, *abs* of -4 is 4; *double* of 4 is 8, which is printed. Note the use of a conditional expression in the definition of *abs*; this is an important feature of LISP programming. Again, since we are illustrating idealized LISP, the particular syntax we have used here does not correspond to any real LISP; indeed, it is quite different from the syntax used in most LISPs, but the concepts are the same.

A second important feature of LISP programming is the ubiquity of recursive definitions. For example:

$$\text{*define } fib[n] = \text{if } n < 3 \text{ then } 1 \text{ else } fib[n-1] + fib[n-2].$$

This is a perfectly well-defined recurrence relation: $fib[1] = 1$, $fib[2] = 1$, $fib[3] = 2$, $fib[4] = 3$, and so on. Applying *fib* to the natural numbers in sequence yields the familiar Fibonacci sequence which starts out with two 1's and in which each subsequent element is the sum of the two preceding elements: 1, 1, 2, 3, 5, 8, 13, 21,.... While such a definition is foreign to most conventional programming languages, and probably looks like magic to the programmer used to FORTRAN or BASIC, all but the simplest programs in LISP are written as such recurrences. It is of no immediate concern how this magic is accomplished; it suffices that it works. Functions of several arguments may also be defined. For example, one version of a famous, very rapidly growing function discovered by Ackermann may be written as:

$$\begin{aligned}&\text{*define } a[m, n] = \text{if } m = 0, \text{ then } n + 1\\ &\text{* else if } n = 0, \text{ then } a[m-1, 1]\\ &\text{* else } a[m-1, a[m, n-1]].\end{aligned}$$

It is easy to see that any computable function may be written in this way. LISP is not, of course, the only programming language that allows recursive definitions; PASCAL and C, for example, would allow similar definitions.

5. Proving Recursive Programs Correct. We may take advantage of the fact that LISP programs behave like mathematical definitions to prove properties of LISP programs. Take for example the following definition:

$$\text{*define } f[x, y] = \text{if } y = 0 \text{ then } 0 \text{ else } x + [x, y-1].$$

We can prove by induction that $f[x, y] = xy$ for any nonnegative integer y. Let $P(k)$ be the predicate "for any x, $f[x, k] = xk$". P is true at 0, since $f[x, 0] = 0 = x0$. For the induction step, assume that P is true at k. To check the truth of P at $k + 1$, we calculate:

$$\begin{aligned}f[x, k+1] &= x + f[x, k] && \text{(def'n of } f\text{)}\\ &= x + xk && \text{(induction hypothesis)}\\ &= x(k+1) && \text{(algebra)}.\end{aligned}$$

We may prove other properties of LISP functions by similar inductions. This example

illustrates the close connection between LISP and mathematical induction: recursive programs provide a concrete way to experiment with induction, and induction provides a way of reasoning about recursive programs. This connection has proved fruitful in the classroom.

6. Lists. Though LISP stands for LISt Processing, we have used only numerical examples so far. In this section we will define lists and the basic operations on them.

A list is a sequence of items, where each item may be a basic object called an atom or another list. We represent lists as strings of symbols as follows. Let us first define an atom as a numeral (that is, a textual representation of an integer) or the special symbol TRUE or the special FALSE. Then a list is a string of the form

$$(a_1 \quad a_2 \quad \ldots \quad a_n)$$

for some $n > 0$, where each a_i is an atom or a list. More precisely, the set of lists is the smallest set of strings satisfying the preceding condition.

Thus the following strings are all lists:

()	a list of no elements
(3)	a list of one element, the atom 3
(37 15 −4)	a list of three elements, all atoms
(3 () TRUE)	a list of three elements, one of which is the list ()
(1 2 3 6 5 (−37) (2 7 TRUE 8 FALSE))	a list of seven elements
(1 () 2 (()) 3 ((())))	a list of six elements, whose even-numbered elements are all lists of one element each.

The three basic functions on lists are head, tail, and cons. The head of a list $(a_1 \ldots a_n)$ is its first element: a_1. The tail of a list $(a_1 \ldots a_n)$ is the list consisting of all the elements but the first: $(a_2 \ldots a_n)$. The head or tail of the null list () is undefined. Thus:

**head*[(4 5 6 7)]

4

**tail*[(4 5 6 7)]

(5 6 7).

The function *cons*(from "construct") takes two arguments. The first is either an atom or a list, and the second is a list. The list *cons*$[a, x]$ is the list obtained by inserting the value a just before the first element of x. Thus *cons*$[a, x]$ always has exactly one more element than x does. Our imagined dialogue at the terminal might continue:

**cons*[3, (4 5)]

(3 4 5)

**cons*[(3), (4 5)]

((3) (4 5))

**cons*[3, ()]

(3)

**cons*[(TRUE FALSE), ()]

((TRUE FALSE)).

The basic properties of *head*, *tail*, and *cons* may be summarized in the following proposition:

PROPOSITION. *If x is an atom or list, and y is a list, then:*

if x is a nonempty set then $cons(head(x), tail(x)) = x$.

$head(cons(x, y)) = x$.

$tail(cons(x, y)) = y$.

$cons(x, y) \neq ()$.

We can use the operations head, tail, and cons to write some useful LISP functions:

```
*define upto[n] = if n = 0 then ( ) else cons[n, upto[n − 1]]

upto

*upto[0]

( )

*upto[3]

(3 2 1)

*define append[x, y] = if x = ( ) then y
*    else cons[head[x], append[tail[x], y]]

append defined

*append[( ), (1 2 3)]

(1 2 3)

   *append[(3), (4 5)]

   (3 4 5)

   *append[(3 2 1), (TRUE FALSE)]

   (3 2 1 TRUE FALSE)

   *cons[(3 2 1), (TRUE FALSE)]

   ((3 2 1) TRUE FALSE).
```

The function *append* concatenates two lists: if the first list is empty, then the answer is just the second. If the first list is nonempty, then the answer is a list whose first element is the same as the first element of the first list and whose remainder may be found by concatenating the tail of the first list onto the second list. If x denotes a list with j elements, and y denotes a list with k elements, then $append[x, y]$ has $j + k$ elements, while $cons[x, y]$ has $k + 1$ elements. Thus $append[(3\ 2\ 1), (TRUE\ FALSE)] = (3\ 2\ 1\ TRUE\ FALSE)$, while $cons[(3\ 2\ 1), (TRUE\ FALSE)] = ((3\ 2\ 1)\ TRUE\ FALSE)$.

We may prove properties of functions on lists by induction, just as we did for integers. Induction on lists may take several forms. The simplest is:

Tail-induction: *Let P be a predicate on lists such that*

(1) (*Base Step*) $P[()]$ *is true*

(2) (*Induction Step*) *for any list x and any list or atom a, if $P[x]$ is true, then $P[cons[a, x]]$ is true.*

Then P is true on every list.

This is called tail-induction because in the induction step we show P at $cons[a, x]$ using the induction hypothesis on its tail x. It is equivalent to induction on the length of a list. Other induction principles may be stated which are not so obviously equivalent to a numerical induction. Let us use tail-induction to show that append is associative. We write out the proof not because of any cleverness, but because it illustrates the usefulness of writing computer programs as functions.

PROPOSITION. *For any lists* x, y, *and* z,

$$append[append[x, y], z] = append[x, append[y, z]].$$

Proof. By tail-induction on x.

Base Step: When $x = ()$, we have

$$\begin{aligned} &append[append[(\,), y], z] \\ &= append[y, z] \quad (since\ append[(\,), y] = y) \\ &= append[(\,), append[y, z]]. \end{aligned}$$

Induction Step. Assume $append[append[x, y], z] = append[x, append[y, z]]$. We need to show

$$append[append[cons[a, x], y], z] = append[cons[a, x], append[y, z]].$$

We calculate:

$$\begin{aligned} &append[append[cons[a, x], y], z] \\ &= append[cons[a, append[x, y]], z] \\ &= cons[head[cons[a, append[x, y]]], append[tail[cons[a, append[x, y]]], z]] \\ &\quad (since\ cons[a, append[x, y]] \neq (\,)) \\ &= cons[a, append[append[x, y], z]] \\ &= cons[a, append[x, append[y, z]]] \quad (by\ induction\ hypothesis) \\ &= append[cons[a, x], append[y, z]]. \end{aligned}$$

■

7. Handling Complex Data. In this section we shall see how these techniques may be extended to handle lists involving not only numbers but also symbolic data. To do this, we introduce a new kind of atom, the symbol. A symbol, in idealized LISP, is a sequence of upper-case alphabetic characters. As before, we put enough spaces around symbols so we can differentiate between adjacent ones. Thus, the following strings are lists

(FOO BAR)	a list of two atoms, the symbols FOO and BAR
(FOOBAR)	a list of one atom, the symbol FOOBAR
(1 1 (JOHN))	symbols may appear anywhere an atom can
(11 (JOHN))	This list's head is eleven

We can now build quite complicated lists to represent complex data. For example, in a genetic counseling program a typical list might be something like:

```
((NAME (JOHN DOE))
 (BLOODTYPE (A POS))
 (FATHER
   ((NAME (JACK DOE))
     (SUFFEREDFROM MYESTHENIAGRAVIS
       (ONSET (3 58))
       (DIED (7 62))))
 (MOTHER
   ((NAME (JANE DOE))
   (EYECOLOR BLUE)
   (BLOODTYPE OF (O NEG))))))).
```

By using symbols to label particular pieces of data, we make it possible to have different pieces of data for different people or to have the data appear in different orders. This flexibility is extremely useful.

Another use of symbols is in LISP's internal representation of functions. The append function, for example, might be represented as the list

```
(LAMBDA (X Y) (COND
    ((NULL X) Y)
    (T (CONS (HD X) (APPEND (TL X) Y))))).
```

Here the symbol LAMBDA indicates that this list represents a function and COND indicates a conditional expression. A programmer who is familiar with the internal representation of functions as lists can write programs to build and execute other programs quite easily. While it is possible to write programs in FORTRAN or BASIC which manipulate other programs, it is generally difficult to do so. In LISP it is easy, and is a very useful tool.

Once functions can be represented as lists, we may use them as data even if we do not know the details of the representation. Consider the following:

```
*define maphead[f, x] = if x = ( ) then ( )
*    else cons[f[head[x]], maphead[f, tail[x]]].
```

The function maphead applies the function f to each element of the list x, and returns the list of results. Thus:

```
maphead[double, (4 0 2)] = (8 0 4)
maphead[upto, (4 0 2)] = ((4 3 2 1) ( ) (2 1)).
```

Another useful function is *apphead*, which is like *maphead* except that it uses *append* instead of *cons*:

```
*define apphead[f, x] = if x = ( ) then ( )
*    else append[f[head[x]], apphead[f, tail[x]]].
```

The effect is to have one fewer level of parentheses in the answer (much as one might take the union of a family of sets). Thus *apphead*[*upto*, (4 0 2)] = (4 3 2 1 2 1). Note that since *append* [(), y] = y, the () resulting from evaluation of *upto*[0] made no contribution to the answer. Functions like maphead or apphead, which apply a function to the elements of a list, are called "mapping functions."

It turns out to be convenient to be able to use maphead or apphead without giving a name to the function to be applied to the list elements. LISP borrows from the lambda calculus a technique for describing "anonymous functions." The expression

```
lambda[x].cons[x, (ANDA)]
```

describes a function of one argument. This function, when applied to the list (FOO BAR), for example, yields ((FOO BAR) ANDA). Thus

```
*maphead[lambda[x].cons[x, (ANDA)], (1 2 3))
((1 ANDA) (2 ANDA) (3 ANDA)).
```

8. Two Short Examples. As an example of the use of lambda expressions with mapping functions, let us write a function which we will call *insertall*. This will be a function of two arguments, a and x. It is supposed to return the list of all lists obtained by inserting the element a into all positions in list x. Thus, If x has k elements, there are $k + 1$ positions at which a may be inserted: after each of the k elements, and before the first element. So *insertall*[a, x] should yield the list of these $k + 1$ lists:

```
insertall[4, (1 2 3)] = ((4 1 2 3)(1 4 2 3)(1 2 4 3)(1 2 3 4))
```

We do not care, however, about the order in which these $k + 1$ lists occur in the answer, so long as they are all there exactly once.

We analyze this task as follows: If x is (), there is exactly one place to insert a; this may be done using the expression *cons*[a, x]. Since we are to find the list of all such insertions, our answer should be the list whose only element is *cons*[a, x]. Hence the answer should be *cons*[*cons*[a, x], ()].

If x is not (), then a may be inserted at the front of x (again using *cons*[a, x]), or it may be inserted somewhere later in x. So we can write, as a first try:

```
define insertall[a, x] = if x = ( ) then cons[cons[a, x], ( )]
  else cons[cons[a, x],
    ???].
```

We now make the key observation: all the other possible insertions consist of an insertion of a into the tail of x, with *head*[x] grafted back on. So *insertall*[a, *tail*[x]] is almost the right thing to put in place of the ??? in our trial program. What we need to do is to graft *head*[x] back on to each element of *insertall*[a, *tail*[x]]. For this the mapping function *maphead* does admirably:

```
*define insertall[a, x] = if x = ( ) then cons[cons[a, x], ( )]
*    else cons[cons[a, x],
*        maphead[lambda[z].cons[head[x], z],
*            insertall[a, tail[x]] ]].
```

This completes the program for *insertall*.

As a second example, we can use *insertall* to help us write a permutation generator. We are to write a function *perms* which, given a positive integer n, returns the list of all permutations of n. We shall represent a permutation as a list, e.g., (4 2 3 1) will represent the permutation $\pi(1) = 4$, $\pi(2) = 2$, $\pi(3) = 3$, $\pi(4) = 1$.

We would like to write *perms* [n] as a recursive function, i.e., in terms of *perms*[$n - 1$]. If we had a permutation of $n - 1$ (represented as a list of integers from 1 to $n - 1$), how could we get a permutation of n? One obvious way is to insert the integer n somewhere in the list. So, if z is a permutation of $n - 1$, then *install*[n, z] gives the list of all permutations of n which are built from z in this way. Furthermore, every permutation of n is uniquely determined by the way it can be built from some permutation of $n - 1$. So our strategy is to use some mapping function with *insertall* on *perms*[$n - 1$].

In this case, the relevant mapping function is *apphead*, since *apphead* is analogous to union. Putting this all together, we get:

```
*define perms [n] = if n = 1 then ((1))
*    else apphead[lambda[z].insertall[n, z],
*            perms[n - 1]].
```

This is, of course, not the only way to generate permutations. Many other algorithms have been used; but this is probably the shortest and most obviously correct: our one-paragraph analysis above constitutes a proof of the correctness of this algorithm.

9. What's Wrong with LISP? One maxim in computer science is "One man's bug is another man's feature." Put more stolidly, both bugs and features represent discrepancies between what a program or algorithm was intended to do and what it does in practice. To call such a discrepancy a bug or a feature is to make a value judgement as to whether the actual behavior or the original intended behavior is preferable. In this section we will consider two common bugs-or-features in LISP.

The actual syntax of most LISPs is much closer to the list-structure representation of the *append* function in Section 7 than it is to the transcripts we have used. Many people find this syntax annoying. It has even been alleged that LISP really stands for "Lots of Irritating Single

Parentheses." It was McCarthy's original intention that the list-structure syntax was to be merely a stopgap until a more conventional syntax was devised. In the meantime, however, programmers discovered that the list-structure syntax was, in practice, quite easy to use. It is easy both for writing and reading, since it is easy to express which parts of an expression depend on which other parts. This property, of course, is shared with conventional functional notation in mathematics. Furthermore, because the same format is used for both data and program, one can easily write programs which create other LISP programs. This is quite a bit more difficult in languages with more conventional syntax.

Another common perception is that LISP is inefficient for some applications, such as numerical calculations. This depends on the implementation. A good LISP compiler can generate code for which is just as good, or better, than most FORTRAN or PL/I compilers. Indeed, in the last few years several firms have developed machines in which all of the operating system code, except for a very small kernel, is written in LISP. The real problem with many numerical applications is that they require a lot of memory as well as a lot of time. When memory is the scarce resource, then it may be better to use another language which enables the programmer to use more compact data structures than lists. Even here, however, many modern LISPs include arrays and other more conventional data structures as well as lists.

10. Conclusions. We have presented a short overview of the programming language LISP. LISP differs from other programming languages in that it is built around the mathematical concept of a function, and uses the data structure of lists. We have sketched techniques for proving properties of functions written in LISP, and suggested how lists allow extremely complex data objects to be manipulated.

11. For Further Reading.

Section 1. A discussion of the early history of LISP may be found in [**McCarthy 81**]. A good, very elementary textbook on LISP is [**Friedman 74**]. Another fairly elementary book, with good discussions of applications in artificial intelligence, is [**Winston & Horn 81**]. Some more advanced books, with rather different orientations, are [**Abelson & Sussman**] (LISP in the general context of computer science), [**Allen 78**] (implementation of LISP), and [**McCarthy & Talcott**] (proofs of correctness). LISP is available for most large computers and many microcomputers. More information about LISP, oriented toward the personal computer, can be found in [**Byte 79**].

Section 3. Technically, the problem with conventional programming languages is that they are "referentially opaque" [**Quine 60**]. Mathematical models of such programming languages may be constructed using modal logic [**Hoare 69, Pratt 76, Ashcroft & Wadge 76**] or lattice theory [**Gordon 79, Scott 77, Stoy 77**].

Section 4. For Ackermann's function, see [**Ackermann 28, Rice 65**]. For the notion of a computable function, see [**Minsky 67, Brainerd & Landweber 74**].

There are many interesting mathematical questions concerning the correspondence between LISP programs, Church's lambda-calculus, and recurrence relations when the recurrence relation fails to have a unique solution. In general, one gets a partial function which is the smallest (as a set of ordered pairs) among all solutions to the recurrence relation [**Manna & Vuillemin 72**]. This is the so-called "least fixed point." Different implementation techniques may, however, give different solutions [**deBakker 76, Friedman & Wise 76**]. The study of partial orders of these functions generalizes to lattice-theoretical models of programming languages [**Manna 74, Stoy 77, Scott 77**].

Section 6. For more on formal languages and their role in the definition of programming languages, see [**Barrett & Couch 79**]. For more on various induction principles, see [**McCarthy 63, McCarthy & Talcott, Morris & Wegbreit 77, Burstall 68**]. The logic of lists and similar structures have been studied [**Boyer & Moore 79, Oppen 80**].

Section 7. The usefulness of LISP for program-manipulation programs is discussed in [**Sandewall 78**].

Section 8. For other algorithms for generating permutations, see [**Sedgewick 77**]. The use of different representations of permutations is discussed in [**McCarthy & Talcott**].

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N IN4.*

COLORINGS OF CIRCLES

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In this paper we will consider some unsolved problems concerning colorings of circles in the plane which are related to the famous Four Color Theorem (4CT). We will present some results on these problems and mention some conjectures. The problems we present involve only basic graph theory and geometry and can be understood easily but they are also challenging enough to be of interest to mathematicians.

1. Graph Theory. A **graph** consists of a finite set of **vertices** and a set of **edges** satisfying the following two conditions: Each edge joins two different vertices, and no two different edges join the same pair of vertices. The easiest way to describe a graph is to draw a picture. For instance, the 4 vertices and 6 edges of a tetrahedron form a graph and are pictured in Fig. 1. In three-dimensional Euclidean space each graph can be represented so that each vertex is a point and the edges are simple curves which do not intersect.

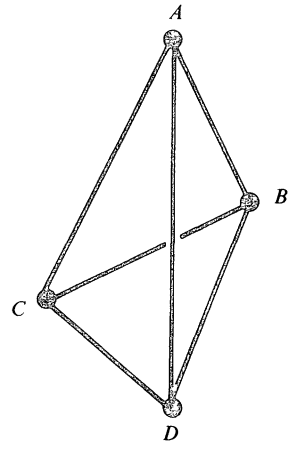


FIG. 1

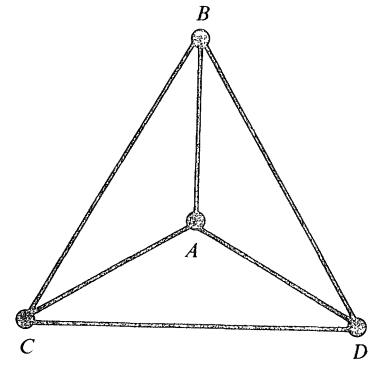


FIG. 2

The important feature of a graph is its combinatorial structure. If, in a graph G, the two vertices A, B are joined by the edge c, then we say that the two vertices A and B are **adjacent** and that A and B are **incident** to c. The **valence** of a vertex A is the number of edges incident to A. A graph is called **complete** if every pair of vertices is adjacent. The graph in Fig. 1 is a complete graph on 4 vertices. The symbol K_n will be used to represent a complete graph on n vertices.

A graph which can be drawn in the plane so that the edges have no intersections except at the vertices is called a **planar** graph. The complete graph on 4 vertices is a planar graph (see Fig. 2) but it is well known that the complete graph on 5 vertices cannot be drawn in the plane. Another example of a planar graph is pictured in Fig. 4 and another nonplanar graph is shown in Fig. 9.

2. **The Four Color Theorem.** A graph is called **n-colorable** if each vertex can be assigned one of n colors in such a way that no two vertices of the same color are adjacent. We usually represent different colors by numbers. The graph in Fig. 9 is colored by 5 colors and thus 5-colorable. The reader is invited to show that this graph is not colorable by less than 5 colors. The **chromatic number** $\chi(G)$, of a graph G, is the minimum number k such that G is k-colorable. Thus $\chi(G) = k$ if G is k-colorable but not $(k-1)$-colorable. So the graph in Fig. 4 has chromatic number 4 and the graph in Fig. 9 has chromatic number 5. The graph K_n is an example of a graph with chromatic number n.

One of the most famous mathematical theorems of all time, the Four Color Theorem, states that every planar graph is 4-colorable. It is easy to see that some planar graphs cannot be colored with less than 4 colors. The graph K_4 is such a graph, as is the graph in Fig. 4. But for over 100 years mathematicians couldn't prove that there were no graphs that required more than 4 colors. Finally, in 1976, Wolfgang Haken and Kenneth Appel [1] of the University of Illinois solved this famous problem. Their proof, however, made unprecedented use of computers. Although the theorem is now known to be true, its proof can only be verified by many hours of computer time. This seemingly simple problem turned out to be very hard indeed. In the remainder of this paper we will introduce some coloring problems that are related to the 4CT.

3. **Constellations and Circles.** Consider two different circles in the plane. They will meet in either 0, 1 or 2 points. If one circle meets the interior of another, then we say that the two circles overlap. If two circles meet in exactly one point Q, we say that the two circles are **adjacent** at Q. A **constellation** will be a finite set of circles in the plane so that no three circles are adjacent at the same point. However note that it is still possible for three circles to be mutually adjacent at three different points. See Fig. 3.

With every constellation one can associate a graph. For every circle in the constellation there will be one vertex in the graph. Two vertices in the graph will be adjacent, if the corresponding

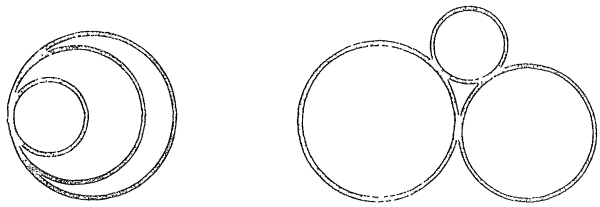


FIG. 3

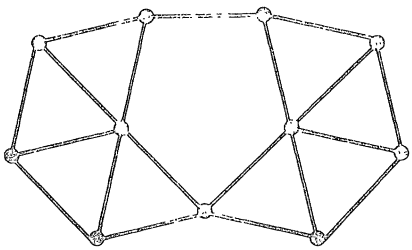


FIG. 4

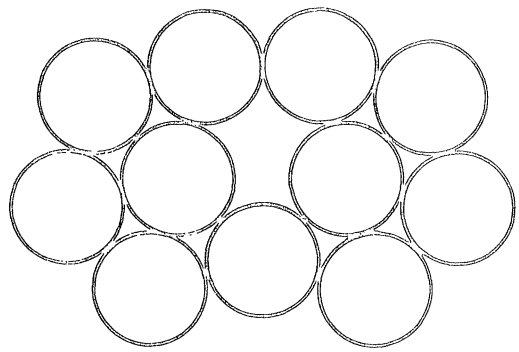


FIG. 5

circles are adjacent. For example, a constellation is pictured in Fig. 5 and the corresponding graph is pictured in Fig. 4.

4. **The Penny Problem.** There are several different problems in which constellations can arise. The easiest of them is the so called "penny problem." Suppose we arrange a finite number of pennies in the plane so that no two pennies overlap, such as the arrangement in Fig. 5. The penny problem then asks what is the minimum number of colors one would need to color any arrangement of pennies so that no two adjacent (touching) pennies would be the same color. An arrangement of pennies gives one a constellation of a special kind. First of all every circle has the same radius and secondly no two circles overlap. For example, the constellation in Fig. 5 could be obtained from an arrangement of 11 pennies. The graph of such a constellation can be constructed in a particularly nice way. One puts a vertex at the center of each penny, and if two pennies are adjacent, one draws an edge between the two centers. In this way one gets a planar graph (see Fig. 4) which is the graph of the constellation. Obviously the minimum number of colors required to color the graph of the constellation coincides with the minimum number of colors needed to color the pennies and the 4CT tells us that any planar graph can be colored with 4 colors. This allows one to say that any arrangement of pennies can be colored with 4 colors. This means that the answer to the penny problem is 4 colors, since some arrangements require 4 colors. We leave it to the reader to show that the arrangement in Fig. 5 cannot be colored with less than 4 colors. But the penny problem can be proved easily without relying on the very difficult proof of the Four Color Theorem. The proof is as follows.

Suppose it is not true that every constellation of pennies can be colored with 4 colors. Then there are constellations which can only be colored by 5 or more colors. We choose such a constellation S with the minimum number of pennies. In that constellation S, look at two pennies A, B whose geometric distance is as large as possible. The penny A cannot have more than 3 pennies adjacent to it. For if it has 4 or more pennies adjacent to it, as in Fig. 6, then no matter how they are arranged one of the pennies C or D would be further from penny B than penny A is, and that's impossible. Therefore penny A has three neighbors E, F, G (or possibly less than three).

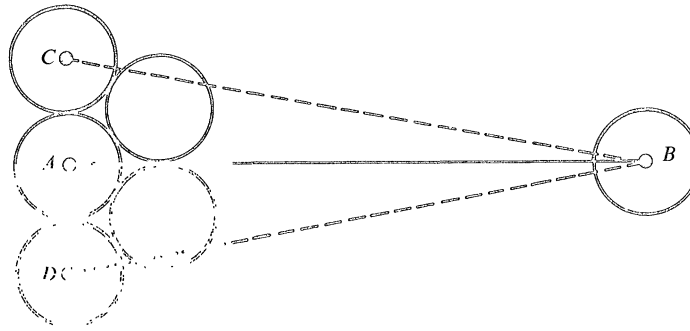


FIG. 6

If penny A is removed, one gets a constellation with one less penny and so by assumption it can be colored with 4 colors. In this coloration the pennies E, F, G get at most three of the colors, say 1, 2, 3. Put back penny A and give it color 4. This is a coloration of the original constellation S with only 4 colors. So there don't exist constellations which are not 4-colorable. Therefore the answer to the penny problem is 4 colors.

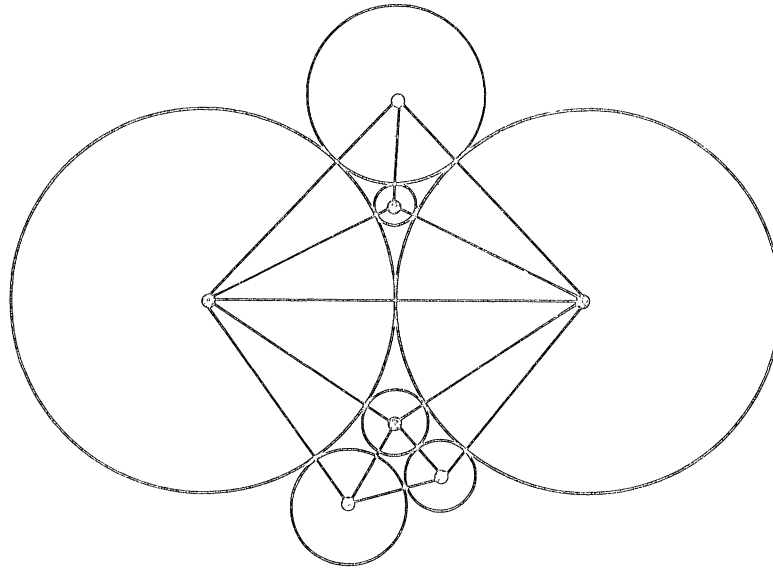


FIG. 7

5. The Coin Problem. Now instead of arranging just pennies in the plane, we arrange many coins which may have different sizes. The coin problem asks what is the minimum number of colors which allows one to color any arrangement of coins so that adjacent coins always receive different colors. The constellations that we now consider may have circles of different radii but the circles will still not overlap. As before, the graph of such a constellation can be constructed by putting a vertex at the center of each circle and, when two circles are adjacent, drawing a straight line between their respective centers. In this way one obtains a planar graph with straight line edges which is the graph of the constellation. See Fig. 7 for an example of a constellation and its graph. Again one can rely on the Four Color Theorem to show that the answer to the coin problem is four. This time there is no easier proof that we know of. In fact an interesting question arises. It is known by a theorem of Wagner [5] that every planar graph can be drawn in the plane so that each edge is a straight line. Now the question arises as to whether the coin problem is

equivalent to the four color problem. We think the answer is that they are equivalent, but we haven't been able to prove it. In other words we think that the following conjecture is true.

CONJECTURE. *Every planar graph can be obtained as the graph of a constellation without overlapping circles.*

If this conjecture is true, then the coin problem and the four-color problem are in fact equivalent. This conjecture would be a stronger statement about planar graphs than Wagner's theorem.

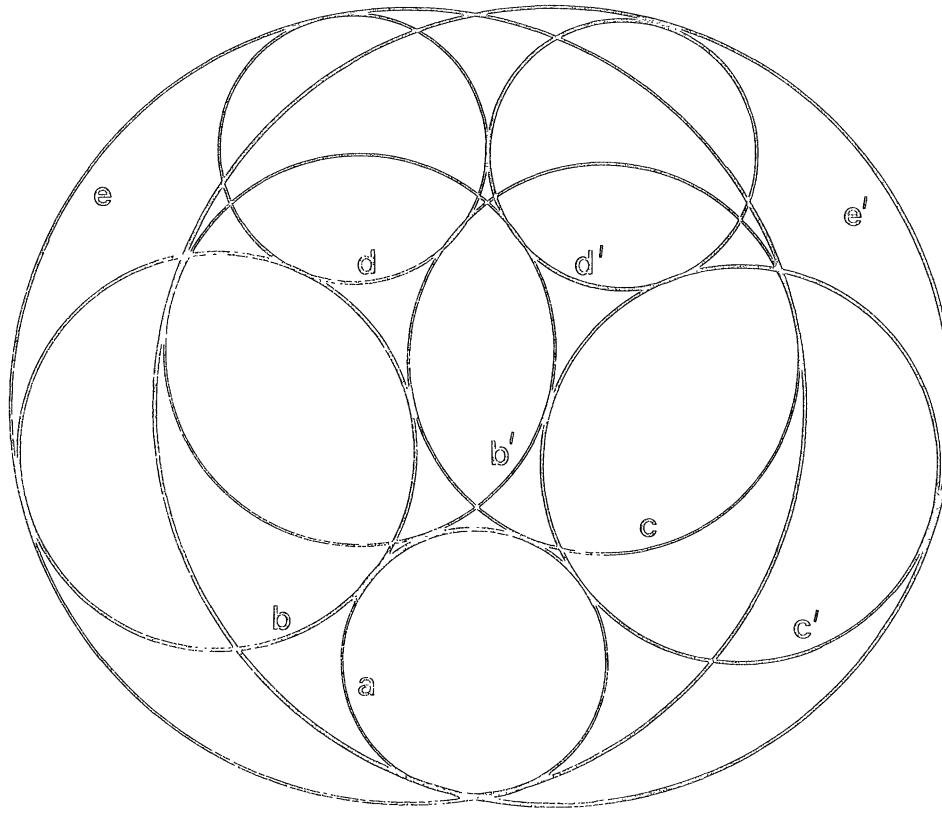


FIG. 8

6. The Circle Problem. Now we consider constellations where the circles can have different radii and they are also allowed to overlap. The circle problem asks what is the minimum number, k, of colors that one needs to color any constellation so that any two adjacent circles have different colors. This problem was first mentioned by Ringel [4] in 1959. It is obvious that 5 circles cannot be mutually adjacent. So K_5 is not the graph of a constellation just as K_5 was not a planar graph. This might lead one to suspect that the answer to the circle problem is also four. However Fig. 8 shows a constellation of 9 circles which cannot be colored with 4 colors. A coloration of the graph of the constellation with 5 colors is pictured in Fig. 9. There is also a constellation with only 8 circles requiring 5 colors which is illustrated in Fig. 10. The graph of the constellation in Fig. 10 can be obtained from Fig. 9 by identifying the top two vertices. The constellation in Fig. 8 is obtained by rotating the arrangement of 5 circles a, b, c, d, e around the center of circle a until the two circles d and d' touch in only one point. By modifying the sizes of circles a, b, c, d, one can get the circles b' and c to coincide and this gives us the constellation in Fig. 10 with 8 circles. One can show that every constellation with 7 circles or less can be colored with 4 colors, but we omit the proof. So we know that k is at least 5. Remarkably enough, we don't yet know whether k is finite.

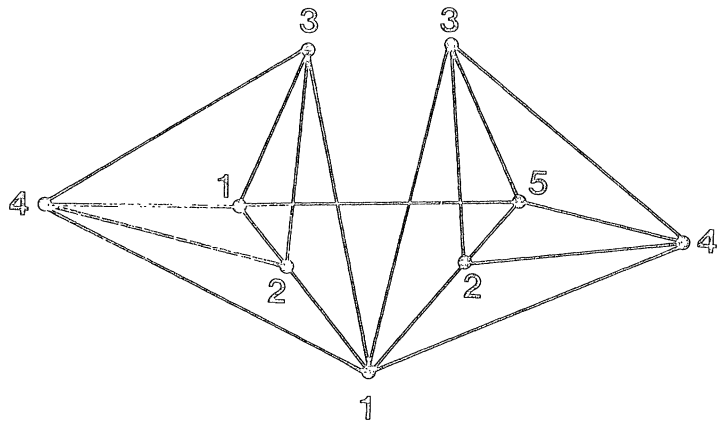


FIG. 9

The usual way to attack a coloring problem is to find a bound for the minimum valence in the set of graphs to be colored. For example in considering the penny problem we showed that some penny always had three neighbors or less. But this method fails here because of the following theorem.

THEOREM. *For any integer n, there exists a constellation where each circle is adjacent to at least n circles.*

This theorem can be proved by the following construction which is shown in Fig. 11. Starting with any constellation S with minimum valence t, we construct a new constellation S' with minimum valence $t + 1$. Choose a point P exterior to every circle in the constellation S. Rotate the constellation S around P t times, so we then have $t + 1$ copies of S. Then for every circle C in the constellation S, draw a new circle centered at P which is adjacent to every image of C. Fig. 11 shows the situation when $t = 3$. This new constellation S' has minimum valence $t + 1$. Actually one must be careful in choosing P and rotating S so that no multiple adjacencies occur.

7. The Overlapping Penny Problem. If we consider constellations where every circle has the

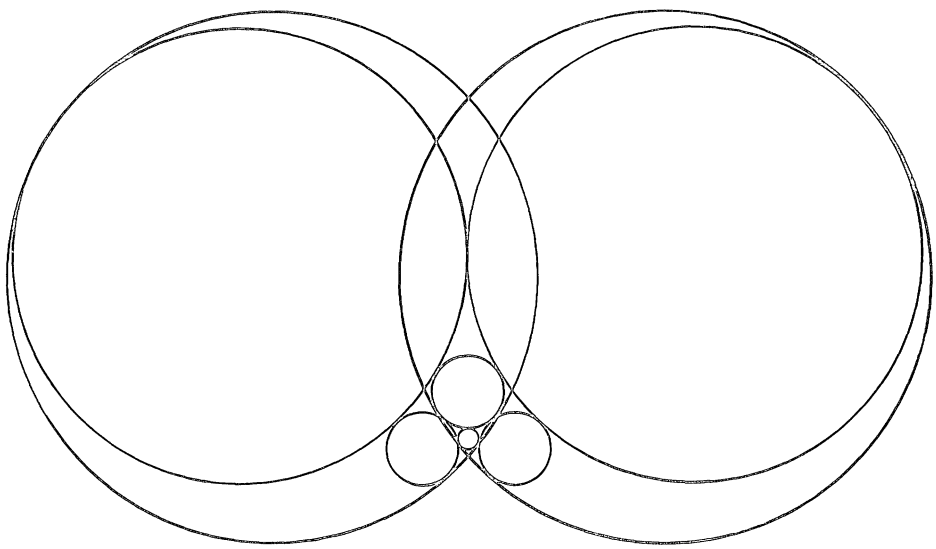


FIG. 10

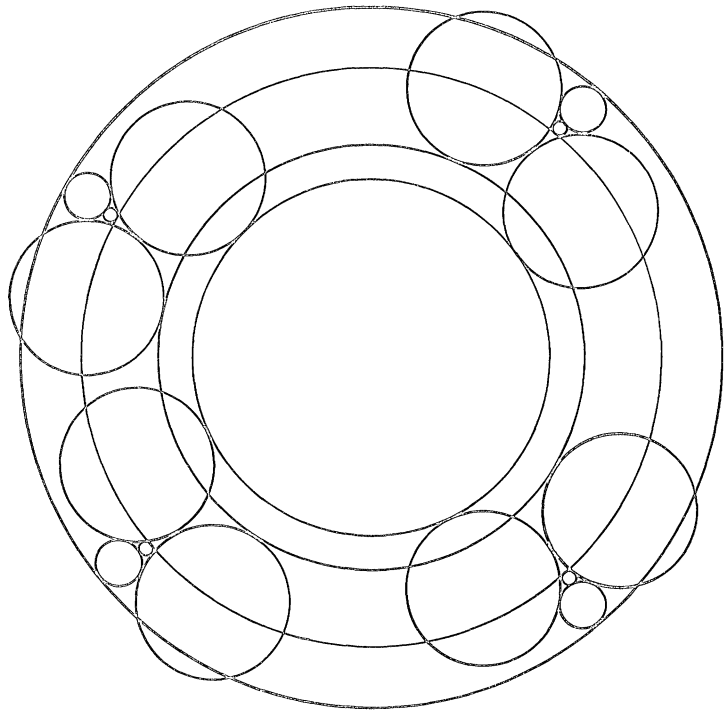


FIG. 11

same radius but overlapping is still allowed, we would have the overlapping penny problem. Even this special case is very far from being solved. The minimum number of colors required will be at least 4. Again we are able to construct constellations where the minimum valence is arbitrarily high, but in this case Hadwiger [3] has shown that the minimum number of colors required will not exceed seven.

The overlapping penny problem can be reformulated in another way using graphs rather than circles. Suppose one chooses a finite number of points in the plane and joins by an edge any two points which are distance one apart. Then the minimum number of colors required to color each of these graphs will be the same as the number of colors required in the overlapping penny problem.

This problem is thus related to the idea of the dimension of a graph, defined in [2]. The **dimension** of a graph G is the minimum number n such that G can be embedded into Euclidean n-space E_n with every edge having length 1, so that the vertices of G are mapped onto distinct points of E_n, but there is no restriction on the crossing of the edges. The chromatic number $\chi(E_n)$ is the least integer k such that for any graph G embedded in E_n as above, the vertices of G can be colored using k colors so that no two adjacent vertices have the same color. Thus in this terminology the overlapping penny problem is equivalent to finding $\chi(E_2)$. It is an unpublished result of Klee that for all n, $\chi(E_n)$ is finite, and one can easily show that $\chi(E_n) \geqslant n + 2$.

Call a graph **critical** of dimension n, if G has dimension n and for any proper subgraph H, the dimension of H is less than n. Another interesting unsolved problem is to characterize the critical n-dimensional graphs for $n \geqslant 3$.

8. **Spherical Coloring Problems.** All of the preceding problems concerning colorings of circles can be restated for dimension 3 and colorings of spheres. If one looks at arrangements of nonoverlapping spheres, each having the same radius, that would lead to the **ping pong ball problem**. Here we can show that every arrangement can be colored with 9 colors and some

arrangements require 5 colors. So the minimum number of colors required for the ping pong ball problem is 5, 6, 7, 8 or 9.

If we allow the spheres to have different radii, but still require that they don't overlap, then that would lead to the **ball problem**. If we allow the spheres to overlap, then we would have the **sphere problem**. If we allow overlapping spheres but require the radii to be the same, we have the **overlapping ping pong ball problem**. Each of these three problems is still unsolved as far as we know.

In general one can consider the corresponding problems for $n - 1$ dimensional spheres in E_n.

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NOTES

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Material for this department should be sent to Professor J. Arthur Seebach, Jr., Department of Mathematics, St. Olaf College, Northfield, MN 55057.

GEOMETRIC SERIES IN INCOMPLETE NORMED ALGEBRAS*

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A. MARQUINA
Facultad de Matematicas, University of Valencia, Dr. Moliner s/n, Burjasot, Valencia, Spain

By a normed algebra we shall mean a linear associative algebra that, as a vector space, is a real or complex normed space with norm satisfying the multiplicative inequality $\|fg\| \leq \|f\| \cdot \|g\|$ for all f and g and with unit element, 1, $\|1\| = 1$.

The main invertibility criterion in a Banach algebra A involves "geometric series" and it says the following:

If f is an element in a Banach algebra A with $\|f - 1\| < 1$, then the inverse of f exists, and this inverse can be represented as the geometric series

$$f^{-1} = \sum_{n=0}^{\infty} (1 - f)^n$$

(*where* $(1 - f)^0 = 1$ *as usual*).

The purpose of this note is to present simple examples of incomplete normed algebras satisfying the invertibility criterion. That means, surprisingly, that a large class of absolutely

*See Advanced Problem 6423 [1983, 289].

CENTER SECTION
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T:	Textbook	13-18:	Grade Level
S:	Supplementary Reading	1-4 :	Time in Semesters
P:	Professional Reading	** :	Special Emphasis
L:	Undergraduate Library	?? :	Questionable

General. VIC 20 User Guide. John Heilborn, Ran Talbott. Osborne/McGraw-Hill, 1983, xi + 388 pp, $14.95 (P). [ISBN: 0-931988-86-1] Elementary operating instructions, BASIC language reference, graphics, sound, peripherals (particularly disk drive). Alternate and user-defined character sets discussed, including multicolor graphics characters. Pinout, memory map appendices are largely unsupported by text; no instructive reference to assembly language. Emphasis on features desirable for programming games. RB

General, S, L.** Mathematical Snapshots, Third American Edition, Revised and Enlarged. H. Steinhaus. Oxford U Pr, 1983, 311 pp, $7.95 (P). [ISBN: 0-19-503267-5] Originally published in 1938, this classic uses visual reasoning to "stretch the imagination without unduly straining the mind:" shortest paths, geometric algebra, Platonic solids, tessellations, etc. This is a reprinting of the 1969 Revised Third Edition (TR, May 1969). LAS

General, S(16-18), L. Bypasses: A Simple Approach to Complexity. Z.A. Melzak. Wiley, 1983, xvi + 245 pp, $34.95. [ISBN: 0-471-86854-X] A survey of examples in mathematics (usually graduate-level), science, engineering, literature, theology, social science, and everyday life of the conjugacy strategy (called a "bypass") of transforming, manipulating, then inverting. Melzak stresses that bypasses reduce complexity, hence are useful for pedagogy, for research, and for invention. LAS

General, T*(13: 1, 2). Mathematics for Business and Social Sciences, An Applied Approach, Third Edition. Abe Mizrahi, Michael Sullivan. Wiley, 1983, xv + 864 pp, $27.95. [ISBN: 0-471-05396-1] Improvements include reorganization of key chapters through additions, deletions and shifting of topics. Some new exercises and examples. Questions from recent CPA, CMA and Actuarial examinations have been appended. Numerous articles for relevant applications have been referenced. (First Edition, TR, December 1976; Second Edition, TR, August-September 1979.) JK

Precalculus, T?(13: 1), S. Intermediate Algebra. Linda and Jimmie Gilbert. Prentice-Hall, 1983, xiv + 498 pp, $21.95. [ISBN: 0-13-469536-4] Exercises and material arranged in order of increasing difficulty. Over 4000 exercises (!) provide plenty of drill. The exposition is primarily by example--theory seems only sketched in brief passages. RB

Precalculus, T(13: 1, 2). College Algebra and Trigonometry. Chris Vancil. Macmillan Pub, 1983, xiv + 557 pp, $24.95. [ISBN: 0-02-422400-6] Standard precalculus text. Trigonometry introduced via triangles. Exercises integrated into expository material. MW

Precalculus, T(13). College Algebra and Trigonometry. Charles C. Carico. Wiley, 1983, ix + 500 pp, $23.95. [ISBN: 0-471-07700-3] Standard precalculus topics, including exponential, logarithmic,and trigonometric functions, plus systems of equations and inequalities, combinations and the binomial theorem, and analytic geometry. Readable, many routine problems. RM

Education, S(15-18). Topics for Mathematics Clubs. Ed: LeRoy C. Dalton, Henry D. Snyder. NCTM, 1983, vi + 106 pp, $4.75 (P). [ISBN: 0-87353-208-2] Revision of 1973 edition (TR, February 1974) with two new chapters on computers and an updated bibliography. Ideas on Fibonacci numbers, Pascal's triangle, groups, projective and non-Euclidean geometry, transfinite numbers, topology, and number theory. MW

Education, S(15-18). How to Choose and Create Good Problems for Primary Children, Second Edition. Doyal Nelson, Joan Worth. NCTM, 1983, 40 pp, $3 (P). [ISBN: 0-87353-205-8] Material from the 1975 NCTM Yearbook, Mathematics Learning in Early Childhood. Seven characteristics of good problems including mathematical significance, variety of physical embodiments, possibility of solutions on several levels and opportunity for child involvement. Examples of good problems and ideas for classroom organization. MW

History, P, L^. Automation of Reasoning. Ed: Jörg Siekmann, Graham Wrightson. Springer-Verlag, 1983. Volume 1: Classical Papers on Computational Logic 1957-1966, xii + 525 pp, $35 [ISBN: 0-387-12043-2]; Volume 2: Classical Papers on Computational Logic 1967-1970, xii + 637 pp, $39. [ISBN: 0-387-12044-0] Reprints of papers on the mechanization of reasoning by computers that have had a decisive influence on the field (29 in Volume 1 from 1957-1966, and 33 in Volume 2 from 1967-1970). Introduced by three special historical essays that provide general context. Each volume concludes with a comprehensive bibliography of related works. LAS

History, S(15-17), L^. Classics of Mathematics. Ed: Ronald Calinger. Moore Pub, 1982, xxv + 742 pp, $18 (P). [ISBN: 0-935610-13-8] A comprehensive yet concise source book containing 133 brief extracts from mathematical source documents ranging from ancient Greece to the early 20th century (Lebesgue, Gödel, Birkhoff), each translated into English and introduced by sectional commentaries. Selections are designed to illustrate for intelligent non-specialists major developments of mathematics; numerous explanatory notes provide valuable historical interpretations. LAS

Foundations, T(16-17), S, P. The Logic of Decision, Second Edition. Richard C. Jeffrey. U of Chicago Pr, 1983, xiv + 231 pp, $20. [ISBN: 0-226-39581-2] Uses elementary logical and mathematical means to study such philosophical issues as subjective probability and subjective desirability which form the core of Bayesian decision theory. Most of the book is devoted to a new theory of preferences between propositions. The mathematics involves only elementary principles of probability. LCL

Number Theory, P. Computational Methods in Number Theory, Part I. Ed: H.W. Lenstra, Jr., R. Tijdeman. Math. Centre Tracts, No. 154. Math Centrum, 1982, v + 198 pp, Dfl. 26,40 (P). [ISBN: 90-6196-248-X] A collection of nine papers dealing with primality testing, factorization of integers and polynomials, perfect numbers, and public-key cryptography. SG

Number Theory, S^(15-16), L^. Higher Mathematics from an Elementary Point of View. H. Rademacher. Ed: D. Goldfeld. Birkhauser Boston, 1983, 138 pp, $29 [ISBN: 3-7643-3064-3] A lightly edited exposition in 11 chapters of a 1947 series of lectures, ranging from prime numbers through functions, decimals, and irrational numbers to modular groups, inversions and mechanical linkages to perform inversions. Each lecture is a concise, insightful gem, applying sophisticated reasoning to elementary concepts. LAS

Linear Algebra, S(14). Fundamentals of Linear Algebra and Analytic Geometry. Ya. S. Bugrov, S.M. Nikolsky. Transl: Leonid Levant. MIR, 1982, 189 pp, $3.45 (P). A first course in a Russian mathematics curriculum for engineering students, to be followed by calculus, etc. Theories of determinants, matrices and the vector spaces R^n (general vector spaces excluded) intertwined with analytic geometry: topic arrangement cohesive yet unusual in America. High quality translation; familiar terminology. No exercises. RB

Linear Algebra, T(14-15). Matrix Methods for Engineers and Scientists. Stephen Barnett. McGraw-Hill, 1979, xi + 185 pp, $11.95 (P). [ISBN: 0-07-084084-9] A how-to text that gets a little further than most first course texts (Sylvester's law of inertia, numerical methods, matrix functions) by passing over proofs and rushing over without comment such questions as using vectors i, j, k as entries in a determinant. AWR

Linear Algebra, T(13-14: 1), S, L. Linear Algebra Through Geometry. Thomas Banchoff, John Wermer. Undergrad Texts in Math. Springer-Verlag, 1983, x + 257 pp, $24. [ISBN: 0-387-90787-4] The authors take seriously the basic tenet of good teaching: "Proceed from the concrete to the abstract." Important ideas of linear algebra (e.g., dot product, linear transformation, eigenvalues, quadratic forms) are introduced in the familiar setting of the geometry of the plane. The very same outline is used to present the linear algebra of three dimensional space, and then again with the geometry of four dimensions. The concluding material abstracts the ideas to n-dimensional vector algebra. A clear alternative to the present market in linear algebra texts. LCL

Algebra, P. Universal Algebra and Applications. Ed: Tadeusz Traczyk. Banach Center Pub., V. 9. PWN, 1982, 454 pp. [ISBN: 83-01-02145-4] Papers presented at the Stefan Banach International Mathematical Center during the special semester named in the title. Applications mostly mean logic, set theory, and recursive functions. JAS

Finite Mathematics, T(13: 1, 2). Applied Mathematics for Business, Economics, and the Social Sciences. Ann J. Hughes. Richard D Irwin, 1983, xx + 753 pp, $28.95. [ISBN: 0-256-02829-X] Topics in finite mathematics (sets, counting, probability, matrices, linear programming including the simplex algorithm), calculus (including exponential, logarithmic, multivariable functions), mathematics of finance. Emphasis on intuition and model building with applications to business. LCL

Calculus, S(13), L. Calculus: Basic Concepts for High Schools. L.V. Tarasov. Transl: V. Kisin, A. Zilberman. MIR Pub, 1982, 183 pp, $3.50 (P). A brief but rigorous traditional approach to calculus (through limits of sequences) couched in the form of 14 dialogues between an enthusiastic "author" and a skeptical, questioning, but highly intelligent "reader." Two-thirds is devoted to preparation, only one-third to derivatives, integrals and differential equations (LC circuits). LAS

Calculus, T(13: 2), S, L. Applied Calculus. Laurence D. Hoffmann. McGraw-Hill, 1983, xi + 658 pp, $25.95. [ISBN: 0-07-029319-8] Includes most topics of standard calculus course; order of coverage is nonstandard--e.g., trigonometric functions covered in last chapter. Mathematical development is intuitive, motivated by examples--e.g., formal treatment of limits and continuity in appendix.

Applied topics: numerical methods (with error estimation), total differential, Lagrange multipliers. Excellent word problems throughout stress modelling; not many graphical problems or much appeal to graphical intuition. PZ

Complex Analysis, T(16-17: 1, 2), S, P. Holomorphic Functions of One Variable. Serge Colombo. Gordon & Breach, 1983, ix + 274 pp, $39.50. [ISBN: 0-677-05950-7] Directed at physicists, this text contains at least the basics of a first-year graduate course in "pure" complex analysis--the applied character is mainly evident in the final chapter on special functions and in the general emphasis on calculation. Develops much of the necessary analysis background. Relatively few exercises; frequent typos. PZ

Differential Equations, P. Functional Integration and Semiclassical Expansions. F. Langouche, D. Roekaerts, E. Tirapegui. Math. & Its Applic., V. 10. D. Reidel Pub, 1982, xii + 313 pp, $39.50. [ISBN: 90-277-1472-X] Fundamental solutions to certain partial differential equations (e.g., Schrodinger and Fokker-Planck type) can be represented as path integrals in phase space. Functional integrals, the objects of study, are defined as limits of discretized approximations to path integrals. An important property is multiplicity: since these approximations can be chosen in different ways, formally different functional integral representations of solutions can arise. PZ

Differential Equations, P. Partial Differential Equations: Time-Periodic Solutions. Otto Vejvoda. Martinus Nijhoff Pub, 1982, xiii + 358 pp, $69.50. [ISBN: 90-247-2772-3] A comprehensive consideration of the relevant methods and theory of time-periodic solutions to non-linear partial differential equations. Large bibliography, most entries of which are described in the text. LCL

Differential Equations, T(16-18), S, P, L. Geometrical Methods in the Theory of Ordinary Differential Equations. V.I. Arnold. Transl: Joseph Szücs, Mark Levi. Grund. der math. Wissenschaften, B. 250. Springer-Verlag, 1983, xi + 334 pp, $36. [ISBN: 0-387-90681-9] A qualitative survey of examples and methods in structural stability, perturbation theory, bifurcation theory with applications to turbulence and other nonlinear phenomena. Exposition is informal, concrete, and graphic, yet rather choppy and disconnected: "As much as possible we avoid references from one chapter to another, even from one paragraph to another." LAS

Differential Equations, P. Nonlinear Phenomena in Mathematical Sciences. Ed: V. Lakshmikantham. Academic Pr, 1982, xxvi + 1035 pp, $94.50. [ISBN: 0-12-434170-5] Over 100 invited papers and research reports from a June 1980 international conference held at the University of Texas at Arlington. Papers emphasize nonlinear functional analysis, nonlinear wave theory, and applications to medical and life sciences. LAS

Numerical Analysis, P. Difference Methods and Their Extrapolations. G.I. Marchuk, V.V. Shaidurov. Applic. of Math., No. 19. Springer-Verlag, 1983, x + 334 pp, $54. [ISBN: 0-387-90794-7] This monograph discusses the use of Richardson extrapolation to improve the accuracy of solutions of ordinary and partial differential equations found by standard methods. Covers both the theoretical and the practical aspects. AO

Numerical Analysis, T(15-16: 1). Introduction to Numerical Computation in Pascal. P.M. Dew, K.R. James. Springer-Verlag, 1983, xi + 291 pp, $16 (P). [ISBN: 0-387-91216-9] Covers nonlinear equations in one variable, systems of linear equations, and numerical integration. An unusual aspect of this text is its emphasis on the development of good-quality mathematical software. AO

Numerical Analysis, T(15-16: 1). An Introduction to Numerical Methods with Pascal. L.V. Atkinson, P.J. Harley. Addison-Wesley, 1983, xii + 300 pp (P) [illegible] [ISBN: [illegible]-0788-7] An elementary introduction to numerical analysis. Topics covered include the solution of nonlinear and linear algebraic equations, eigenvalue and eigenvector calculations, discrete function approximation, numerical differentiation and integration, and the solution of ordinary differential equations. AO

Functional Analysis, P. Lecture Notes in Mathematics-975: Radical Banach Algebras and Automatic Continuity. Ed: J.M. Bachar, et al. Springer-Verlag, 1983, viii + 470 pp, $23.00 (P). [ISBN: 0-387-11985-X] Proceedings of a conference held at California State University, Long Beach, July 17-31, 1981. JAS

Analysis, P. Lecture Notes in Mathematics-983: Nonstandard Analysis-Recent Developments. Ed: A.E. Hurd. Springer-Verlag, 1983, 213 pp, $12 (P). [ISBN: 0-387-12279-6] Nine papers illustrating current research in applications of nonstandard analysis in such diverse areas as statistical mechanics, partial differential equations, Banach space theory, and probability theory. AO

Analysis, L. Handbook of Applied Mathematics: Selected Results and Methods, Second Edition. Ed: Carl E. Pearson. Van Nostrand Reinhold, 1983, xiii + 1307 pp, $67.50. [ISBN: 0-442-23866-5] A collection of twenty-one articles surveying areas of mathematics most useful in applied work, including differential and integral equations, linear algebra, approximation theory, and optimization theory. Emphasis is on techniques. AO

Algebraic Geometry. Lecture Notes in Mathematics-980: Fonctions thêta et théoreme du cube. Lawrence Breen. Springer-Verlag, 1983, xiii + 112 pp, $8.50 (P). [ISBN: 0-387-12002-5]

Differential Geometry, P. Differential Geometry. Ed: Gy. Soós, J. Szenthe. North-Holland, 1982, 829 pp, $136.25. [ISBN: 0-444-86197-1] Proceedings of a colloquium organized by the János Bolyai Mathematical Society and held in Budapest September 3-7, 1979. JAS

Geometry, T*(10: 11). Geometry, A High School Course. Serge Lang, Gene Murrow. Springer-Verlag, 1983, xxiii + 470 pp, $24 (P). [ISBN: 0-387-90727-0] A preliminary edition which varies the presentation according to the topic; e.g., the presentation of triangles is similar to Euclid's but coordinate and algebra are used elsewhere. Covers the rigid motions and includes discussion of changes [illegible] volume under dilation, vectors and dot product. JNC

Geometry, T(18), P. The Geometry of Discrete Groups. Alan F. Beardon. Grad. Texts in Math., No. 91. Springer-Verlag, 1983, xii + 337 pp, $39. [ISBN: 0-387-90788-2] An introduction to the geometry of the action of discrete groups of Möbius transformations (includes a comprehensive chapter on analytical hyperbolic geometry). As much as possible, the explanations are geometric rather than purely algebraic. LCL

Algebraic Topology, T(17-18: 1, 2), S, P. Vector Bundles, Volume 1: Foundations and Stiefel-Whitney Classes. Howard Osborn. Pure & Appl. Math., V. 101-1. Academic Pr, 1982, xii + 371 pp, $49.50. [ISBN: 0-12-529301-1] Part of a comprehensive 3-volume introduction to vector bundles, characteristic classes, K-theory. Focus on Z/2 classes of vector bundles, with applications to non-oriented manifolds. Intended for self-study or the classroom after a term in singular homology and cohomology. Over 200 remarks survey the literature in detail; lengthy, cross-referenced bibliography; 42 exercises. RB

Topology, T(16: 1), S, L. Introduction to Topology. Theodore W. Gamelin, Robert Everist Greene. Saunders Coll Pub, 1983, xii + 196 pp, $28.95. [ISBN: 0-03-062476-2] Metric spaces, general topology and homotopy theory, focusing on concrete applications, shunning algebraic formalism. Proofs of implicit function, ham sandwich, Jordan curve theorem (via covering spaces), noncontractibility of S^n (via Stokes theorem); numerous exercises. Concise, motivational introduction to the topics, emphasizing results of interest to analysts. RB

Optimization, T(16-17), L. Variational Calculus with Elementary Convexity. John L. Troutman, W. Hrusa. Undergrad. Texts in Math. Springer-Verlag, 1983, xiv + 364 pp, $29.80. [ISBN: 0-387-90771-8] Motivation is rooted in the historic problems of the calculus of variations. The goal is to use convexity and an emphasis on global sufficiency so as to make solutions accessible through elementary methods. To this reviewer, the admirable goal may be hindered somewhat by what seems excessive notational demands, but the idea is intriguing. Book deserves to be looked at. AWR

Optimization, P. Lecture Notes in Mathematics-979: Mathematical Theories of Optimization. Ed: J.P. Cecconi, T. Zolezzi. Springer-Verlag, 1983, 268 pp, $14.50 (P). [ISBN: 0-387-11999-X] Fourteen contributed papers at a 1981 conference at the Instituto per la Mathematica Applicata del C.N.R., Genova. Subject content ranges from optimal control to calculus of variations, from operations research to non-smooth optimization. AWR

Optimization, P. Lecture Notes in Economics and Mathematical Systems-206: Redundancy in Mathematical Programming, A State-of-the-Art Survey. Mark H. Karwan, et al. Springer-Verlag, 1983, vii + 286 pp, $19 (P). [ISBN: 0-387-11552-8] A collection of articles reviewing the major approaches to identifying and removing redundancy in mathematical programming problems. Results of extensive tests comparing the methods are presented. AO

Optimization, T(16-17: 1), P, L. Numerical Methods for Unconstrained Optimization and Nonlinear Equations. J.E. Dennis, Jr., Robert B. Schnabel. Prentice-Hall, 1983, xiii + 378 pp, $28.95. [ISBN: 0-13-627216-9] A survey of methods for small- to medium-size problems. Methods not based on Newton's method (e.g., conjugate direction and Brown-Bent methods) are not covered. An extensive appendix contains pseudocode for a library of programs based on the algorithms in the book. AO

Optimization, S(13-14), L. Optima for Animals. R. McNeill Alexander. Edward Arnold, 1982, vii + 112 pp, $13.95 (P). [ISBN: 0-7131-2843-7] A concise catalogue of simple models for optimization of animal form (bones, eggshells), motion (gaits, burst swimming), behavior (feeding, territories), and reproduction (breeding, sex ratios), concluding with a brief synopsis of mathematical optimization methods. LAS

Optimization, P. Mathematical Programming: The State of the Art. Ed: A. Bachem, M. Grötschel, B. Korte. Springer-Verlag, 1983, viii + 655 pp, $55. [ISBN: 0-387-12082-3] Includes twenty-one comprehensive survey papers presented at the Eleventh International Symposium on Mathematical Programming (Bonn, West Germany, 1982). AO

Probability, P. Lecture Notes in Mathematics-976: Ecole d'Eté de Probabilités de Saint-Flour XI-1981. X. Fernique, et al. Springer-Verlag, 1983, xi + 465 pp, $23.50 (P). [ISBN: 0-387-11987-6]

Probability, P. Lecture Notes in Mathematics-982: Stability Problems for Stochastic Models. Ed: V.V. Kalashnikov, V.M. Zolotarev. Springer-Verlag, 1983, xvii + 295 pp, $17 (P). [ISBN: 0-387-12278-8] Proceedings of the Sixth International Seminar held in Moscow, USSR, April 1982. LAS

Probability, S(17-18), P. Lectures on Topics in Stochastic Differential Equations. Daniel W. Stroock. Springer-Verlag, 1982, ii + 91 pp, $6.90 (P). [ISBN: 0-387-11549-8] Lectures on solutions to stochastic differential equations as a function of their starting point, and on the trajectories of these solutions. AO

Statistics, S(18), P. Lecture Notes in Statistics-18: Conjugate Duality and the Exponential Fourier Spectrum. Wray Britton. Springer-Verlag, 1983, 226 pp, $16.80 (P). [ISBN: 0-387-90826-9] Results

of a numerical experiment: a sufficiently smooth spectral density was reconstructed by computer from estimated trigonometric moments, using the minimum negentropy criterion. The problem was linked to general theory in convex analysis via conjugate duality. Includes 163 pages of tables, bibliography and printouts. RB

Statistics, T(13: 1). Elementary Statistical Concepts, Second Edition. Ronald E. Walpole. Macmillan Pub, 1983, xiv + 442 pp, $22.95 (P) [ISBN: 0-02-424020-6]; Solutions Manual, iii + 97 pp, (P). [ISBN: 0-02-424030-3] Written for senior high school students and college freshmen of limited mathematical ability. Several additions make this edition more flexible: inclusion of nonparametric statistics, additional material in most of the chapters, additional real-life examples and exercises. Exercises follow sections (rather than chapters as in the First Edition, TR, May 1976). A Student Study Guide is available for this edition. LCL

Statistics, P. Probability and Statistical Inference. Ed: Wilfried Grossmann, Georg Ch. Pflug, Wolfgang Wertz. D Reidel Pub, 1982, viii + 389 pp, $49.50. [ISBN: 90-277-1427-4] Proceedings of 2nd Pannonian Symposium at Bad Tatzmannsdorf, Austria, June 1981. LAS

Statistics, T(14-17: 1, 2). Statistics for Research. Shirley Dowdy, Stanley Wearden. Wiley, 1983, xiv + 537 pp, $34.95. [ISBN: 0-471-08602-9] Presupposes no statistics. Supplies probabilistic and mathematical ideas and techniques as needed. The usual topics plus Poisson distributions, analysis of covariance, and multiple regression. Tries for intuitive understanding without becoming involved in mathematical theory. FLW

Statistics, T(13: 1, 2). Statistics and Probability in Modern Life, Third Edition. Joseph Newmark. Saunders College Pub, 1983, xi + 630 pp. [ISBN: 0-03-058407-8] Revision of the author's 1977 Second Edition (TR, March 1978). Relatively more space is devoted to probability topics than in most elementary texts. RSK

Computer Literacy, S(13). The Compleat Computer, Second Edition. Dennie L. and Cynthia L. Van Tassel. SRA, 1983, 264 pp, $11.95 (P). [ISBN: 0-574-21415-1] An anthology of over 100 excerpts from both fiction and expository writing intended to reflect society's view of computers. Includes excerpts from many well-known computer pioneers as well as from contemporary commentators on the computer age. (First Edition, TR, June-July 1976.) LAS

Computer Literacy, S(9-13). Encyclopedia of Computer Terms. Douglas Downing. Barron's Educ Ser, 1983, v + 148 pp, $6.95 (P). [ISBN: 0-8120-2519-9] An elementary and somewhat dated dictionary of common computer jargon, including most BASIC commands. Useful only for complete novices. LAS

Computer Programming, T(13: 1). BASIC by Design: Structured Computer Programming in BASIC. Andrew Kitchen. Prentice-Hall, 1983, xvii + 489 pp, $18.95 (P). [ISBN: 0-13-060269-8] Easy-to-read introduction to BASIC emphasizing design and documentation techniques. Uses block charting with repeat loops and conditional blocks. Very complete, from uses of the delete key through functions, character strings and sequential files using BASIC-PLUS. Other systems described in the appendix. MW

Computer Programming, S*. Machine Code and Better BASIC. Ian Stewart, Robin Jones. Birkhauser Boston, 1983, x + 189 pp, $11.95 (P). [ISBN: 3-7643-3115-1] A sequel to the authors' elementary Timex/Sinclair 1000 Programs, Games and Graphics (TR, March 1983), this spritely yet literate book uses Basic as a launching pad for exploring data structures (arrays, stacks, linked lists, trees), structured programming (via three extended programs), and Z80 machine code. An excellent transition from beginning to advanced programming. LAS

Computer Programming, S(13). A Guide to Programming in Level II BASIC. Bruce Presley. Van Nostrand Reinhold, 1982, 253 pp, $12.95 (P). [ISBN: 0-442-25892-5] Introductory text for Level II BASIC for the TRS-80 computer. Standard treatment of topics. Suitable for self study. RM

Computer Programming, T(13). A Structured Approach to FORTRAN. J. Winston Crawley, Charles E. Miller. Reston Pub, 1983, xvi + 750 pp. [ISBN: 0-8359-7092-2] This book contains an introduction to the FORTRAN language using the dialect called FORTRAN-77. It uses a spiral approach in which each topic is covered many times, each time at a greater level of complexity and in more depth. It has extensive examples, lots of pedagogic aids and lots of exercises at the end of each chapter. It must also be, at 750 pages, the longest introductory programming text ever written! MS

Computer Programming, S(13). A Guide to Programming: IBM Personal Computer. Bruce Presley. Van Nostrand Reinhold, 1982, 288 pp, $16.95 (P). [ISBN: 0-442-26015-6] An elementary introduction to programming in IBM Basic written specifically for users of the IBM Personal Computer. AO

Computer Programming, T(12-14), S. C Programming Guide. Jack Purdum. Que Corp, 1983, xvi + 250 pp, (P). [ISBN: 0-88022-022-8] A carefully written, gentle introduction to C using examples from Basic as comparisons in the opening chapters. Designed for small C compilers on microcomputers, the first half is restricted to programs that can be run on any system. Advanced features (floating point data types, disk file operations, structures, and unions) are covered in the second half. An appendix lists several commercial C compilers. LAS

Software Systems, S(15-16), P. Lecture Notes in Computer Science-150: Enduser Systems and Their Human Factors. Ed: A. Blaser, M. Zoeppritz. Springer-Verlag, 1983, 138 pp, $8.50 (P). [ISBN: 0-387-12273-7] Proceedings of the scientific symposium conducted on the occasion of the 15th anniversary of the Science Center Heidelberg of IBM Germany, March 18, 1983. General psychological

discussions not requiring much if any mathematics, mathematical maturity, or computer science. JAS

Computer Science, P. Lecture Notes in Computer Science-151: CONLAN Report. R. Piloty, et al. Springer-Verlag, 1983, xi + 174 pp, $10 (P). [ISBN: 0-387-12275-3] CONLAN (CONsensus LANguage) is intended to be a formally defined consensus or common base hardware description language for all levels and aspects of computer hardware description. This report from the CONLAN Working Group is a self-contained introduction to CONLAN as it now exists, and is aimed at developers of software for computer-aided design of digital systems. RM

Computer Science, P. An Algebraic Analysis of Storage Fragmentation. Terry Betteridge. Computer Science: Systems Prog., No. 15. UMI Research Pr, 1982, xx + 212 pp, $44.95. [ISBN: 0-8357-1364-4] A theoretical analysis of some allocation schemes. Presents exact results obtained using Markov chain techniques. A revised version of the author's Ph.D. thesis. AO

Computer Science, P. Search Mechanisms for Large Files. Marie-Anne Kamal Neimat. Computer Science: Distr. Database Syst., No. 11. UMI Research Pr, 1981, xi + 114 pp, $34.95. [ISBN: 0-8537-1231-1] A study of data structures and search methods for files that are sorted and sequentially stored in secondary storage. A revised version of the author's Ph.D. thesis. AO

Computer Science, P. Secrecy, Authentication, and Public Key Systems. Ralph C. Merkle. Computer Science: Systems Prog., No. 18. UMI Research Pr, 1982, 104 pp, $34.95. [ISBN: 0-8357-1384-9] Describes results of research in computational cryptography, specifically in the areas of public key distribution and digital signatures. A revised version of the author's Ph.D. thesis. AO

Computer Science, T(16-17: 1), P. Concepts for Distributed Systems Design. Gregor von Bochmann. Springer-Verlag, 1983, xi + 259 pp, $19. [ISBN: 0-387-12049-1] Definitions, examples and problems of parallel and distributed computer systems. Layered protocol architectures, compatibility issues, message transport, and data transmission networks. Formal specification methods. RWN

Computer Science, P. Lecture Notes in Computer Science-152: Specification and Design of Software Systems. Ed: E. Knuth, E.J. Neuhold. Springer-Verlag, 1983, 152 pp, $10 (P). [ISBN: 0-387-12284-2] Contains papers presented at the Conference on Operating Systems held in Visegrad, Hungary, January 23-27, 1982. RJA

Computer Science, S, P, L. Studies in Computer Science. Ed: Seymour V. Pollack. Studies in Math., V. 22. MAA, 1982, xvii + 388 pp, $29. [ISBN: 0-88385-124-5] A short course in computer science: eight topical essays on programming language, formal languages, analysis of programs, computational complexity, artificial intelligence, numerical analysis, simulation, and data analysis, introduced by an historical essay on the development of computer science. Some essays give evidence of being dated (e.g., discussion of ACM's Curriculum '68, but no reference to Curriculum '78). LAS

Computer Science, P. Mathematical Experiments on the Computer. Ulf Grenander. Pure & Appl. Math. Academic Pr, 1982, xviii + 525 pp, $39.50. [ISBN: 0-12-301750-5] From the preface: "What we have in mind is using the computer as the mathematician's laboratory, in which he can perform experiments in order to augment his intuitive understanding of a problem, or to search for conjectures, or to produce counterexamples, or to suggest a strategy for proving a conjecture." Includes case studies, an introduction to APL, and a library of subroutines. AO

Computer Science, S(15-17), P. Introduction to Local Area Networks. Digital Equipment Corp, 1982, vi + 159 pp, (P). A brief introduction to local area networks (LANS). Covers the basic terms and concepts. Places LANs in the context of distributed processing and networking in general. AO

Systems Theory, P. Homotopy Methods and Global Convergence. Ed: B. Curtis Eaves, et al. NATO Conf. Ser., V. 13. Plenum Pr, 1983, viii + 318 pp, $45. [ISBN: 0-306-41127-X] Papers from a June 1981 NATO Advanced Research Institute on the application of homotopy techniques to analyzing nonlinear systems, especially to fixed point calculations used in systems science. LAS

Applications, P. Transactions of the Twenty-Eighth Conference of Army Mathematicians. US Army (P.O. Box 12211, Research Triangle Park, NC 27709), 1982, xv + 533 pp, (P). Papers on diffusion transport and mixing from a June 1982 conference in Bethesda, Maryland. LAS

Applications (Artificial Intelligence), S, P, L. Chess Skill in Man and Machine, Second Edition. Ed: Peter W. Frey. Texts & Mono. in Comp. Sci. Springer-Verlag, 1983, xiv + 329 pp, $28. [ISBN: 0-387-90790-4] The current crop of chess playing machines are performing at very high levels--far better than that envisioned ten years ago when the material for the first volume was collected. This updated edition documents these developments. It also includes a summary of recent games, an expanded bibliography, and two additional chapters on the relative merits of search-based and knowledge-based programs. (First Edition, TR, June-July 1977.) LCL

Applications (Cryptography), P. Lecture Notes in Computer Science-149: Cryptography. Ed: Thomas Beth. Springer-Verlag, 1983, viii + 402 pp, $18.50 (P). [ISBN: 0-387-11993-0] Papers from an April 1982 international conference on cryptography near Erlangen, Germany. An expository introduction leads to nine groups of related papers, ranging from classical (WW II) schemes to the new public-key methods. LAS

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-211: The Stability of a Macroeconomic System with Quantity Constraints. Paul van den Heuvel. Springer-Verlag, 1983,

vii + 169 pp, $13 (P). [ISBN: 0-387-11992-2] The Barro, Grossman, Malinvaud model of economics considers the interactions between the consumption sector and the production sector. Prices and stocks are rigidly fixed in the short term, but give rise to adjustments in the long term. These adjustments can be described by a system of differential equations. The purpose here is to study the stability properties of equilibria of this system. LCL

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-209: Essays and Surveys on Multiple Criteria Decision Making. Ed: Pierre Hansen. Springer-Verlag, 1983, vii + 441 pp, $27.50 (P). [ISBN: 0-387-11991-4] Proceedings of the Fifth International Conference on Multiple Criteria Decision Making at Mons, Belgium, August 9-13, 1982. JAS

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-210: Technology, Organization and Economic Structure. Ed: Ryuzo Sato, Martin J. Beckmann. Springer-Verlag, 1983, viii + 195 pp, $16 (P). [ISBN: 0-387-11998-1] Essays in honor of Isamu Yamada, ranging from cotangent bundles and variational principles to informed critiques of Reaganomics. LAS

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-212: Invariance Principles and the Structure of Technology. Ryuzo Sato, Takayuki Nôno. Springer-Verlag, 1983, 94 pp, $10.50 (P). [ISBN: 0-387-12008-4] Applications of Lie groups to estimating parameters in economic models of technological change. Special focus is on distinguishing between increased productivity due to increasing returns to scale vs. true increases due to technological advances. LAS

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-213: Aspiration Levels in Bargaining and Economic Decision Making. Ed: Reinhard Tietz. Springer-Verlag, 1983, viii + 406 pp, $27.50 (P). [ISBN: 0-387-12277-X] Proceedings of the third conference on experimental economics, Winzenhohl, Germany, September 1982. Focus is on the strategies and results of human negotiations, rather than on idealized rational optimization theory. LAS

Applications (Engineering), P. Progress in Boundary Element Methods, V. 2. Ed: C.A. Brebbia. Springer-Verlag, 1983, 217 pp, $44. [ISBN: 0-387-91217-7] 8 papers on applications of boundary element methods to such problems as fracture mechanics, linear isotropic elasticity, viscoplasticity and creep. RWN

Applications (Engineering), P. Structure of Complex Turbulent Shear Flow. Ed: R. Dumas, L. Fulachier. Springer-Verlag, 1983, xix + 444 pp, $39. [ISBN:0-387-12156-0] Proceedings of a September 1982 symposium in Marseilles, France. Main conclusion: further progress depends on high accuracy experiments in simple situations. LAS

Applications (Engineering), T(16: 1). Finite Element Primer. Bruce Irons, Nigel Shrive. Halsted Pr, 1983, 157 pp, $39.95. [ISBN: 0-470-27414-7] An elementary introduction to finite element calculations. The physical basis of the technique is emphasized and the reader's attention is called to what can go wrong. AO

Applications (Fluid Mechanics), T(16-18), S, L. Inviscid Fluid Flows. Hilary Ockendon, Alan B. Tayler. Appl. Math. Sci., V. 43. Springer-Verlag, 1983, viii + 146 pp, $16 (P). [ISBN: 0-387-90824-2] A modeling course in fluid mechanics, including such examples as the mathematical description of tidal waves and sonic booms. It assumes standard elementary material on inviscid incompressible hydrodynamics and an introduction to partial differential equations and wave motion. LCL

Applications (General), T(14-16), S*, L.** Discrete and System Models. Ed: William F. Lucas, Fred S. Roberts, Robert M. Thrall. Modules in Appl. Math., V. 3. Springer-Verlag, 1983, xx + 353 pp, $28. [ISBN: 0-387-90724-6] This third volume presents 14 models or systems of models illustrating a wide variety of discrete techniques. Activities for the reader vary from routine exercises to significant projects. The first chapter "Foresight-Insight-Hindsight" presents 31 "jokes" which illustrate the art of model formulation. These attention-getting morsels may be useful for instructors in a wide variety of courses. JAS

Applications (Life Sciences), T(15-17: 1), S, L.** Life Science Models. Ed: Helen Marcus-Roberts, Maynard Thompson. Modules in Appl. Math., V. 4. Springer-Verlag, 1983, xx + 366 pp, $28. [ISBN: 0-387-90739-4] Concluding volume in the series of CUPM modules in applied mathematics: four models of the time evolution of population, three on epidemiology, three on ecology, et al. Problems, references, notes for instructors help to make this an appealing volume for a modelling seminar. LAS

Applications (Management), T?, S(13-14). Executive Planning with Basic. X.T. Bui. Sybex, 1982, xiii + 196 pp, $13.95 (P). [ISBN: 0-89588-083-0] A spoonful of sugar makes the medicine go down. Some rather significant mathematical ideas, e.g., linear programming and multiple linear regression, are simply but understandingly presented in an environment where a microcomputer BASIC program can be a powerful tool. Its value as a text is reduced by having no exercises and only superficial "theory." JAS

Applications (Medicine), P*. Clinical Trials: Issues and Approaches. Ed: Stanley H. Shapiro, Thomas A. Louis. Statistics, V. 46. Dekker, 1983, x + 209 pp, $34.50. [ISBN: 0-8247-1741-4] Collection of eight articles dealing with the fundamental design, conduct (including ethical aspects), analysis, and reporting issues involved in the use of randomized clinical trials. RSK

Applications (Modelling), P. Air Pollution Modeling and Its Application II. Ed: C. de Wispelaere. NATO Challenges of Modern Soc., V. 3. Plenum Pr, 1983, xi + 873 pp, $95. [ISBN: 0-306-41115-6] Proceedings of the 12th International Technical Meeting on Air Pollution Modeling held during August 1981 at SRI International in Menlo Park. 51 papers, with a special emphasis on physical and chemical reactions in plumes. LAS

Applications (Pattern Recognition), S(16-18), P. Pattern Models. Narendra Ahuja, Bruce J. Schachter. Wiley, 1983, xiv + 309 pp, $39.95. [ISBN: 0-471-86194-4] The study of planar patterns (e.g., tessellations, mosaics, covering models, wave models, image models) which are useful in analyzing and synthesizing spatial data. Attention is given to the analysis of the stochastic properties of geometrical patterns (e.g., what are the expected sizes of regions, etc.) as well as to the development of efficient algorithms for handling spatial data (e.g., displaying spatial patterns in perspective). LCL

Applications (Pattern Recognition), P. Image Sequence Processing and Dynamic Scene Analysis. Ed: T.S. Huang. Computer and System Sci., No. 2. Springer-Verlag, 1983, ix + 749 pp, $59.20. [ISBN: 0-387-11997-3] The proceedings of a 1982 NATO Advanced Study Institute held in Braulage/Harz, West Germany. Includes thirty-eight papers on motion estimation, pattern recognition and artificial intelligence techniques in dynamic scene analysis, and applications. AO

Applications (Physics), P. Théorie de l'Élasticité: Manuel de Résolution des Problèmes. V. Rékatch. MIR, 1980, 264 pp. A comprehensive monograph on mathematical elasticity theory. General theory and results are presented, but most of the book is devoted to problems of classical and practical interest. Each chapter ends with exercises. PZ

Applications (Physics), P. Statistical Theory and Random Matrices. Moshe Carmeli. Pure & Appl. Math., V. 74. Dekker, 1983, ix + 203 pp, $35. [ISBN: 0-8247-1779-1] Summarizes the fundamentals of a statistical theory of energy levels, which can be applied to highly excited states of a complex physical system, relating the physical aspects of the theory to the mathematical theory of random matrices. RSK

Applications (Physics), T(16-18: 1, 2), P. The Elements of Mechanics. Giovanni Gallavotti. Texts and Mono. in Physics. Springer-Verlag, 1983, xiv + 575 pp, $48. [ISBN: 0-387-11753-9] A relatively self-contained exposition of mathematical mechanics. The text begins with simple problems in the qualitative theory of ordinary differential equations and proceeds through the more modern theory of stability. AO

Applications (Physics), P*. Exactly Solved Models in Statistical Mechanics. Rodney J. Baxter. Academic Pr, 1982, xii + 486 pp, $81. [ISBN: 0-12-083180-5] A survey of the two-dimensional lattice models in statistical mechanics that have been solved exactly. AO

Applications (Social Science), T(17: 1), P. Multidimensional Scaling. Mark L. Davison. Wiley, 1983, xiv + 242 pp, $25.95. [ISBN: 0-471-86417-X] In the Wiley Series in Probability and Mathematical Statistics. Designed "to show what MDS is, to explain how to use the major techniques, and to prepare researchers to read more technical literature." Uses a narrow definition to avoid overlapping with cluster analysis and factor analysis techniques. Good set of references. RSK

Applications (Social Science), S(14-16), P, L.** Approval Voting. Steven J. Brams, Peter C. Fishburn. Birkhauser Boston, 1983, xix + 198 pp, $14.95 (P); $24.95. [ISBN: 3-7643-3124-0; 3-7643-3108-9] Just in time for the 1984 political season: a definitive apologia for approval voting, a system for multicandidate elections in which voters vote for as many candidates as they approve of. Brams and Fishburn argue the merits of approval voting, first in popular nontechnical terms using many examples from U.S. congressional and primary elections, then in more rigorous language of combinatorial and probabilistic proof. LAS

Applications (Social Science), T(15-17), S, P, L. Mathematical Models in the Social and Behavioral Sciences. Anatol Rapoport. Wiley, 1983, xii + 507 pp, $49.95. [ISBN: 0-471-86449-8] A demonstration of the "integrative function of the mathematical approach to social science," with the hope of "restructuring habits of thinking about social phenomena." Organized around classes of mathematical tools: differential equations, stochastic models, structural models (social) choice theory, game theory, and "problems of quantification." LAS

Reviewers

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Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

Missouri Section

The annual spring meeting of the Missouri Section was held on April 22-23, 1983 at Missouri Western State College, St. Joseph, Missouri. There were 125 registrants.

Invited Lectures:

"Some Bridges To and From Mathematics," by Alfred B. Willcox, Executive Director of M.A.A.
"Something Old, Something New, Something Borrowed, and Something Blue," by Paul Humke, St. Olaf College, Northfield, Minnesota.
"MATHFILE: Introduction and Demonstration," by Gabrielle Carr, Librarian, Missouri Western State College.

Short Presentations:

"Simple Links in Locally Compact Connected Hausdorff Spaces are Nondegenerate," by David John, Missouri Western State College.
"Two Problems in Recreational Mathematics," by Curtis Cooper, Central Missouri State University.
"An Inverse--Limit Proof of Keller's Theorem," by Mark Michael, Southeast Missouri State University.
"The Risk Generalization Model and Series--Summability Approach to the Pascal Triangle Diagonal Matrix," by Bev Harris, Southwest Baptist University.
"Niven Numbers: Past, Present and Future," by Robert E. Kennedy, Central Missouri State University.
"Introducing Random Samples in Elementary Statistics Using the PDP-11 and Poise," by Ben Budde, Westminster College.
"Generalized Differences," by Russ Euler, Northwest Missouri State University.
"The Importance of Three Years of High School Mathematics: A Perspective from Students," by Charles Mitchell, Northwest Missouri State University.
"A Summer Math Institute for High School Students," by Larry Campbell, The School of the Ozarks.
"Plotting Along Using Graphics for Beginning Programming," by Janet Fite, teacher of the gifted at Bode and Spring Garden Schools, St. Joseph.
"The Art and Science of Computerized Combat Simulations," by Kent Pickett, Missouri Western State College.

Workshops:

Planetarium Demonstration, by Chris Godfrey, Missouri Western State College.
"Computer Applications in Mathematics Education," by Bill Huston (Moderator), Missouri Western State College.
Novice's Microcomputer Workshop, by George Bishop and Ken Johnson (Instructors), Missouri Western State College.
Microcomputer Exhibit, by Missouri Western State College Mathematical Sciences Society. Bruce Kelley, Faculty Advisor.

Louisiana-Mississippi Section

The Louisiana-Mississippi Section held its annual meeting at Delta State University at Cleveland, Mississippi on February 18-19, 1983. There were 100 faculty registrants. The meeting was the 60th joint meeting with the Louisiana-Mississippi Branch of the NCTM.

Invited Lectures:

"Pressures for Curriculum Modification," by R.D. Anderson, Past President, M.A.A.
"Geometrical Proofs of Some Theorems About Differentiable Functions," by James R. Dorrah, Louisiana State University.

Student Papers:

"Force and Velocity in the English Frog Sartorius Muscle," by Bob Philpot, Belhaven College.
"Rose's Rose," by John A. Bailey, Millsaps College. (Winner of $50 cash prize.)
"Trapezoidal Properties," by Scott Bowie, Millsaps College.
"Sequential Verbalization for the Inverse of a One-to-One Function," by Laurel C. Eskridge, Millsaps College.
"Fibonacci Numbers and Patterns of Independent Sets in Cycles and Paths," by Mary Harrington, University of Mississippi. (Winner of $50 cash prize.)

Panel Discussions:

"Entrance Requirements for Public Universities," by A.J. Hulin, University of New Orleans, and John L. Tilley, Mississippi State Univerity.
"Mathematics Placement of Freshmen Students," by Nancy Dunn (Moderator), University of New Orleans.
"Junior College Transfer Students--Follow-Up," by Ken Johnston (Moderator), Hinds Junior College.

Contributed Papers:

"The Space of Entire Functions Revisited," by L. Swetharanyam, McNeese State University.
* "A Generalization of Euler's ϕ-Function," by S. Leigh, University of Southwestern Louisiana.
* "Fixed Point Theorems for Certain Classes of Multi-Valued Mappings," by H. Kaneko, Mississippi State University.
"Fibonacci Numbers and Graphs," by W. Staton, University of Mississippi.
* "A Concrete Experience Approach to Concept Formation in Developmental Arithmetic," by R. Yellott, McNeese State University.
"Microcomputers in the Junior High School Classroom," by R. Grantham, University of Mississippi.
"Microcomputer Aids in Teaching Trigonometry," by D. Cook, University of Mississippi.
* "Z-Closed Spaces," by T. Thompson, University of Southwestern Louisiana.
* "Semi-T_0-Identification Spaces and S-Essentially T_0 Spaces," by C. Dorsett, Louisiana Tech University.
* "Orderly Neighborhoods in a Disorderly Row," by M. Maxfield and J. Maxfield, Louisiana Tech University.
"A Discrimination Procedure for Bivariate Probability Models," by B. Asrabadi, Nicholls State University.

Oklahoma-Arkansas Section

This meeting was held at the University of Oklahoma on March 18-19, 1983. Approximately 100 were in attendance.

Invited Lectures:

"Some Mathematical Surprises," by Victor Klee, University of Washington.
"Report on the National Science Board Commission on Precollege Education and Mathematics, Science and Technology," by Katherine Layton, Beverly Hills, California.

Contributed Papers:

"Continued Square Roots," by Bianca M. Hearn, Hendrix College.
"p-Adic Numbers," by Daryl Ezzo, Oral Roberts University.
"A Child's Garden of Quaternions," by Bobby Winters, East Central University.
"Elementary Transfer Orbits," by Tim Koster, Oklahoma State University.
"Log-Convexity and the Gamma Function," by Karen Anderson, Hendrix College.
"Functional Equations Arising from Notational Ambiguities in Calculus," by Karen Shirley, Hendrix College.
"Two Ancient Greek Construction Problems in Euclidean and Hyperbolic Geometry," by Jack M. Rau, Oklahoma State University.
"The Diffusion Index as an Indicator of Stock Market Trends," by Dennis Bertholf, Harry Comeskey, and Wayne B. Powell, Oklahoma State University.
"Robust Autoregressive Time Series Parameter Estimation," by John Fingerlin, G.O. Scan-AMCO.
"Digital Image Processing," by Frank Chimenti, Southwestern State University.
"Some Closed Graph Theorems," by Ray Hamlett, East Central University.
"Problem Solving," by Doug Foster, Oral Roberts University.
A Mathematical System of Single Fingerprint Classification," by William R. Orton, University of Arkansas at Fayetteville.
"An Applied Linear Algebra Course," by Tom Cairns and Bill Coberly, University of Tulsa.
"Functions of Mathematics Augmented with a Micro," by Kelvin Casebeer, Southwestern State University.
"Survey of the Mathematics Shortage in Oklahoma," by Harold Huneke, University of Oklahoma.
"Problem Solving for Gifted Junior High Students," by Paul Duvall, Oklahoma State University.
"Apportionment: A Model with Means," by Donald L. Pattern, University of Oklahoma.
"Joint Graphs and the Traversability," by Mahesh M. Hiremath, University of Arkansas at Fayetteville.
"Supercyclic Hilbert Space Operators," by Bill Stockwell, Central State University.
"Constructing Amicable Pairs from Known Amicable Pairs: An Alternative Approach," by Dale Woods and Joe D. Flowers, Central State University.
"Maximum Likelihood Estimation of Linear Sufficient Statistics," by Bill Coberly, University of Tulsa.

A one-hour course on Hofstadter's Godel, Escher, Bach by Dale Alspach, Michael Folk, and Joel Haack, Oklahoma State University.

Panel Presentation:

"Precollege Mathematics Education in the United States," by Katherine Layton (Presider), Beverly Hills High School, California; R.D. Anderson, Louisiana State University; Dale Franks, Hope, Arkansas; Dave McCurdy, US Congressman, 4th District Representative, Oklahoma; and Charles Watson, Department of Education, Arkansas.

Intermountain Section

The spring meeting of the Intermountain Section met in conjunction with the meeting of the American Mathematical Society at the University of Utah, Salt Lake City on April 29-30, 1983.

Invited Addresses:

"Some Recent Results in Infinite-Dimensional Topology," by Richard D. Anderson, Louisiana State University and MAA President.
"School Mathematics in the Calculator-Computer Age," by Richard D. Anderson, Louisiana State University and MAA President.
* "Approximating Regular n-gons by Folding Paper: An Interplay of Analysis, Geometry, and Number Theory," by Jean J. Pedersen, University of Santa Clara.

Short Presentations:

* "The Power Series Solution of Power Series Autonomous Ordinary Differential Equations," by R. Bryce Parry, Utah State University.
* "Reflections in the Plane," by Steve H. Heath, Southern Utah State College.
* "How to Bias a Random Selection," by Jay Huber, Ricks College.
* "A Relationship Between the Modified Euler Method and e," by Thomas P. Dence, California State University, Los Angeles.
"Mathematics Honor Club: Striving for Excellence," by Mila Baldazo, Northwest Intermediate School, Salt Lake City.
"Using Computers in University-Level Mathematics," by Donald R. Snow, Brigham Young University.

Panel Discussions:

"Industrial Applications and Opportunities," by Ronn Carpenter (Moderator), Thiokol Corporation; Klancy deNevers, Christensen, Inc.; Tom Jensen, Evans and Sutherland; David Pilcher, Hercules, Inc.; Robert Shoop, Sperry.
"Trends in State Administrations Affecting Mathematics Education," by Donald Clark (Moderator), State of Utah Mathematics Specialist.

Student Session:

"Determination of the Point of Intersection of Two Lines From Four Given Points Using Only Compass Construction," by John Bossard, University of Utah.
"The Volterra Mapping Technique," by Ed Phillips, Idaho State University.
"Finding Polynomial Roots in One Variable Using the Continuation Method," by Greg Kellar, Brigham Young University.
"Differential Equations of an Electron in the Presence of a Magnetic Monopole," by Leon Arriola, Idaho State University.
"Group Representation Theory and Quantum Mechanics," by Thomas Saxton, University of Utah.
"Introduction to the Electronic Neuron," by Dean Mumme, Idaho State University.
"Solution of Monthly Problem E2946," by Jeff Loveland, Sky View High School, Logan.
"A Rearrangement of the Alternating Harmonic Series," by Fon Brown, Utah State University.

Workshop:

"Computer-Aided Geometric Design," following a special session on the same topic organized as part of the AMS program by Peter Alfeld and Robert Barnhill, University of Utah.

New Jersey Section

The New Jersey Section met jointly with the Mathematics Association of Two Year Colleges on April 23, 1983 at the Glassboro State College, Glassboro, New Jersey. There were approximately 50 people in attendance.

Invited Addresses:

"Franklin's Magic Squares," by Edward Moore, Rancocas Valley High School.
"A Mathematical Modeling Approach to the Evaluation of Biomechanical Quality in the Human Foot," by Phil Demp, Glassboro State College.

Panel Discussions:

"Pre-College Preparation in Mathematics," by Ashby Foote, Rutgers; Gabriella Wepner, Ramapo College; Roman Voronka, New Jersey Institute of Technology; Mike Schero, Bloomfield College; Bohdan Lukaschewsky, Union County College; Robert Shanks, President of Association of Mathematics Teachers of New Jersey; Frederic Kreisler, Department of Higher Education; Mark Chamberlain, Glassboro State College.
A general discussion of how being in mathematics affects social interaction both in and out of the mathematical team, by Cathy Folio and Pat Kenachaft, Association for Women in Mathematics.

Student Paper:

"Counter Examples in Finite Groups," by J. Schiffman, CUNY.

Indiana Section

The fall meeting of the Indiana Section was held on October 15, 1983 at the campus of Indiana Univ.-Purdue Univ. at Fort Wayne, Indiana. The meeting was attended by approximately 40 people.

Invited Address:

"Competing Risks in Survival Analysis," by Melvin L. Moeschberger, Ohio State University.

Contributed Papers:

"Groups and Graphs," by Marc Lipman, Indiana University-Purdue University at Fort Wayne.
"Mathematics in Literature," by David Neuhouser, Taylor University.
"An Overview of Decomposing 3-Manifolds," by Roger B. Nelson, Ball State University.

Panel Discussion:

"Mathematics Education and Other Things in China," by Herb Bailey, Rose-Hulman Institute of Technology; Emma Garnett, Ball State University; Don Clark, Purdue University, Hammond Campus.

Pacific Northwest Section

The annual spring meeting of the Pacific Northwest Section was held at the University of Idaho, Moscow, on June 16-18, 1983. Approximately 135 persons attended.

Invited Lectures:

"Mathematics at Work in Society (MAWIS)," by Gail Adele Williams, University of Idaho.
"Modeling Measle Epidemics," by John Reay, Western Washington University.
"Mathematicians Are People, Too," by Donald J. Albers, Menlo College.
"The Early History of Computers," by Peter Hilton, SUNY at Binghamton.
"Making Math Video Tapes," by John Loughlin, Lane Community College.
"Geometry of Moorish Ornamentation," by Branko Grunbaum, University of Washington.
"Discovering Fibonacci Identities," by Calvin T. Long, Washington State University.

Short Presentations:

"Why are Matrices So Useful?" by Chang-Li Yiu, Pacific Lutheran University.
"Basing Courses on UMAP Modules," by Larry Anderson, Whitman College.
"A Mathematical Problem from Model Airplanes," by Larry Bobisud, University of Idaho, Moscow.
"Age Structure and the Paradox of Enrichment," by David Wollkind, Washington State University.
"Combinatorial Topology and Commutative Algebra," by Steve Johnson, Seattle Pacific University.
"Computing in an Elementary Number Theory Class," by Rod Hansen, Whitworth College.
"An Inner Product for Sequences," by Herb Holden, Gonzaga University.
"A General Notion of Independence of Sequences of Integers," by John Burke, Gonzaga University.
"Inclusion Considerations in Summability Methods," by Stanley Luke, Seattle Pacific University.
"A Report on an Elementary Differential Equations Microcomputer Laboratory," by Tyre Newton, Washington State University, and Leonard G. Henscheid, Washington State University.
"Some Imperfect Maintenance Models," by Bob Fontenot, Whitman College.

Panel Discussion:

"Honors Program in Mathematics," by George McCrae, University of Montana; William Mech, Boise State University; Donald Tucker, University of Utah.

Short Course:

"Computer Graphics in Mathematics Instruction," by David Moursund, University of Oregon.

arrangements require 5 colors. So the minimum number of colors required for the ping pong ball problem is 5, 6, 7, 8 or 9.

If we allow the spheres to have different radii, but still require that they don't overlap, then that would lead to the **ball problem**. If we allow the spheres to overlap, then we would have the **sphere problem**. If we allow overlapping spheres but require the radii to be the same, we have the **overlapping ping pong ball problem**. Each of these three problems is still unsolved as far as we know.

In general one can consider the corresponding problems for $n - 1$ dimensional spheres in E_n.

References

1. Kenneth I. Appel & Wolfgang Haken, Every planar map is four colorable, Part I. Discharging, Kenneth I. Appel, Wolfgang Haken & J. Koch, Part II. Reducibility, Illinois J. Math., 21 (1977) 429–567; MR 58#27598ab.

2. Paul Erdős, Frank Harary & William T. Tutte, On the dimension of a graph, Mathematika, 12 (1965) 118–122; MR 32#5537.

3. H. Hadwiger, Ungeloste Probleme No. 40, Elemente der Math., 16 (1961) 103–104.

4. G. Ringel, Färbungsprobleme auf Flächen und Graphen, Mathematische Monographien, 2. VEB Deutscher Verlag der Wissenschaften, Berlin, 1959; MR 22#235; esp. p. 26.

5. K. Wagner, Bemerkungen zum Vierfarbenproblem, J.-ber. Deutsch. Math.-Vereinig. 46 (1936) 26–32; Zbl. 14,181b.

NOTES

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GEOMETRIC SERIES IN INCOMPLETE NORMED ALGEBRAS*

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By a normed algebra we shall mean a linear associative algebra that, as a vector space, is a real or complex normed space with norm satisfying the multiplicative inequality $\|fg\| \leq \|f\| \cdot \|g\|$ for all f and g and with unit element, 1, $\|1\| = 1$.

The main invertibility criterion in a Banach algebra A involves "geometric series" and it says the following:

If f is an element in a Banach algebra A with $\|f - 1\| < 1$, then the inverse of f exists, and this inverse can be represented as the geometric series

$$f^{-1} = \sum_{n=0}^{\infty} (1 - f)^n$$

(*where* $(1 - f)^0 = 1$ *as usual*).

The purpose of this note is to present simple examples of incomplete normed algebras satisfying the invertibility criterion. That means, surprisingly, that a large class of absolutely

*See Advanced Problem 6423 [1983, 289].

convergent series, the geometric series, are also convergent in such incomplete normed algebras (recall that a normed space is a Banach space if and only if every absolutely convergent series is convergent). The existence of such algebras does not appear to be well known in the literature on normed algebras (see [1], [2] and [3]). Our purpose is justified since the invertibility criterion need not be true in an incomplete normed algebra, as the following example shows:

EXAMPLE. Let $\mathbb{T}$ be the unit circle. Let $\mathcal{P}$ be the complex normed algebra consisting of all polynomials (with complex coefficients)

$$\sum_{n=0}^{N} \alpha_n z^n$$

on $\mathbb{T}$, provided with the pointwise operations and the supremum norm, $\| \|_\infty$. We define $p(z) = 1 - (z/2)$. It is immediate that there is no polynomial $q \in \mathcal{P}$ such that $p(z)q(z) = 1$ for all z in $\mathbb{T}$. On the other hand,

$$\|1 - p(z)\|_\infty = \tfrac{1}{2} < 1.$$

Thus, $\mathcal{P}$ does not satisfy the invertibility criterion. The reason is that the geometric series

$$\sum_{n=0}^{\infty} (z/2)^n$$

defines a function that does not belong to $\mathcal{P}$.

Let A be a normed algebra. We shall denote by $G(A)$ the group of invertible elements. If f is an element in A and if δ is a fixed real number with $0 < \delta \leqslant 1$, then the series

$$\sum_{n=0}^{\infty} f^n$$

is called a *geometric δ-series* if $\|f\| < \delta$. A geometric δ-series is absolutely convergent, i.e.,

$$\sum_{n=0}^{\infty} \|f^n\| < \infty$$

since $\|f^n\| \leqslant \|f\|^n$ for all n.

PROPOSITION. *The following conditions are equivalent*:

(1) *$G(A)$ is open in A.*

(2) *There is a real number δ with $0 < \delta \leqslant 1$ such that if f is in A and $\|f - 1\| < \delta$, then f is invertible.*

(3) *There is a real number δ with $0 < \delta \leqslant 1$ such that every geometric δ-series converges in A.*

Proof. It is obvious that (1) implies (2). Further (2) implies (1), by the same proof as in [3], p. 5 for instance, by replacing 1 by δ. (2) implies (3), for let us choose the same δ as in (2). Let f be such that $\|f\| < \delta$. Let us define

$$s_N = \sum_{n=0}^{N} f^n, \quad N \text{ nonnegative integer.}$$

By virtue of (2) we have that $1 - f$ is invertible. Let $g = (1 - f)^{-1}$. Since $g(1 - f) = 1$ we have

$$\begin{aligned} \|s_N - g\| &= \|g(1 - f)s_N - g\| \\ &= \|g[(1 - f)s_N - 1]\| \\ &\leqslant \|g\| \cdot \|s_N - s_N f - 1\| \\ &= \|g\| \cdot \|f^{N+1}\| \leqslant \|g\| \cdot \|f\|^{N+1}. \end{aligned}$$

Thus, if N tends to infinity, $\|s_N - g\|$ tends to zero. Finally, by the usual proof (3) implies (2).

COROLLARY. *Every geometric* 1*-series converges in* A *if and only if every* f *with* $\|f - 1\| < 1$ *has an inverse.*

The proof follows from that of the Proposition.

REMARK. The reader may check easily that $G(\mathcal{P})$ is not open for the normed algebra $\mathcal{P}$ of the Example.

We now present examples of incomplete normed algebras satisfying the condition that every geometric 1-series is convergent.

Let K be an infinite Hausdorff compact space. We recall that K is said to be zero-dimensional if the topology of K has a base of open-closed subsets of K. Let us suppose that K is *zero-dimensional.* Let $C(K)$ be the Banach algebra of all real or complex continuous functions defined on K. A function in $C(K)$ is called *simple* if the range of f is finite. Let $C_s(K)$ be the uniformly dense subalgebra of all simple functions of $C(K)$. We now have the following result:

THEOREM. $C_s(K)$ *is an incomplete normed algebra such that every geometric* 1*-series is convergent. In particular,* $G(C_s(K))$ *is open.*

Proof. We shall show that if f is in $C_s(K)$ with $\|f - 1\|_\infty < 1$, then f is invertible, and then invoke the Corollary. There is a finite partition of K, consisting of open-closed sets, A_j, $j = 1,\ldots,n$, i.e., pairwise disjoint finite open cover of K, such that

$$f = \sum_{j=1}^{n} \alpha_j \chi_{A_j}$$

where $\alpha_j, j = 1,\ldots,n$ are scalars and where χ_{A_j} is the characteristic function of A_j. Since

$$\|f - 1\|_\infty = \max_{1 \leqslant j \leqslant n} |\alpha_j - 1| < 1$$

for every j we have that $\alpha_j \neq 0$. Thus, we obtain

$$f^{-1} = \sum_{j=1}^{n} (1/\alpha_j)\chi_{A_j},$$

and, therefore, f^{-1} is a simple function.

From the Theorem we obtain the following consequence:

COROLLARY. *The normed algebra* m_0 *of all scalar sequences with finite range, provided with the supremum norm, satisfies the invertibility criterion.*

Proof. Apply the Theorem to the Stone-Čech compactification of the positive integers with the discrete topology. Alternatively, note that the proof of the theorem applies directly to the present case.

References

1. S. K. Berberian, Introduction to Functional Analysis and Operator Theory, Springer, Berlin, Heidelberg, New York, 1974.

2. A. Brown and C. Pearcy, Introduction to Operator Theory I (Elements of Functional Analysis), Springer, Berlin, Heidelberg, New York, 1977.

3. E. L. Stout, The Theory of Uniform Algebras, Bogden and Quigley, New York, 1971.

ANSWER TO PHOTO ON PAGE 18

Reading from left to right: Walter Feit, John Thompson, Daniel Gorenstein. Photo by Joseph Gallian.

A SIMPLE PROOF OF THE FUNDAMENTAL THEOREM ON FINITE ABELIAN GROUPS

C. R. MACCLUER
Department of Mathematics, Michigan State University, East Lansing, MI 48824

I have long been unhappy with the standard proofs for the Fundamental Theorem on Finite Abelian Groups. Either the proofs are detailed calculations or involve much deeper notions such as modules over principal ideal domains. The proof I present here (uniqueness portion omitted) is more in character with the elementary nature of the result. We recall that a group is called indecomposable if it cannot be written as the direct sum of two proper subgroups.

THEOREM. *Cyclic groups of prime power order are indecomposable. All other finite abelian groups can be decomposed nontrivially.*

Proof. Let us proceed immediately to the only difficult case, a group G of prime power exponent p^e. Let H be a maximal cyclic subgroup, necessarily of order p^e. Let us now prove that H is a direct summand of G by induction on the order of G.

Choose a subgroup L with

$$L \cap H = 0.$$

Assume for the moment that $L \neq 0$. Then by induction, the faithful copy of H in G/L is a direct summand, and hence, pulling back, a direct summand of G.

And now for the detail: We must choose L so that $L \cap H = 0$ and $L \neq 0$. For L, use the cyclic subgroup generated by any element of order p not in H. There must be such elements. For if not, the kernel of the map

$$x \mapsto px$$

is exactly the subgroup of H of order p and so successive applications of this map yield the series

$$G \supset pG \supset p^2G \supset \cdots \supset p^{e-1}G \supset 0$$

whose steps are each of index p forcing $G = H$.

By applying this method to a finite dimensional vector space, thought of as a module over the ring of polynomials in an operator, and using induction rather on the dimension of the vector space, an immediate and elegant proof of the existence of the rational canonical form results, free of the Invariant Factor Theorem. The method does not seem to have much scope in more general settings.

MISCELLANEA

118.

By and by comes Mr. Cooper... of whom I entend to learn Mathematiques; and so begin with him today.... After an hour's being with him at Arithmetique, my first attempt being to learn the Multiplicacion table, then we parted till tomorrow.

—Samuel Pepys, Diary entry for 4 July 1662.

Remark: Pepys was at that time something like a modern Secretary of the Navy.

Students should be asked to check that indeed $f(f(x)) = x$. In this case the concept of the principal square root will be reinforced, i.e., that $\sqrt{t^2} = -t$ if $t \leqslant 0$.

We conclude with still another approach for finding functions of the form $f(x) = (ax + b)/(cx + d)$ which satisfy $f(f(x)) = x$ for all x for which $f(f(x))$ is meaningful, with $c \neq 0$ and $ad - bc \neq 0$. As has been mentioned, many students were able to guess the solution $f(x) = 1/x$. A reasonable generalization is the function $y = k/x$. This function is of course symmetrical with respect to the line $y = x$. Since the graphs of $y = (ax + b)/(cx + d)$ (if $ad - bc \neq 0$) and $y = k/x$ are hyperbolas, it is reasonable to try to transform $y = k/x$ in such a way that the curve in its new position will remain a hyperbola symmetrical with respect to $y = x$. Since point (a, a) is on $y = x$, the curve $y - a = k/(x - a)$ seems to be still symmetrical with respect to $y = x$ (this can be easily verified). Hence, we obtain the function $y = k/(x - a) + a$ or $y = (k - a^2 + ax)/(x - a)$. Since $k - a^2$ is an arbitrary constant, let $k - a^2 = b$. Thus $y = (ax + b)/(x - a)$. It is instructive to ask students to show that this agrees with the previously obtained solution $f(x) = (ax + b)/(cx - a)$ for $c \neq 0$, because if $c \neq 0$, we may write

$$f(x) = \frac{\frac{a}{c}x + \frac{b}{c}}{x - \frac{a}{c}}$$

which has the form $(Ax + B)/(x - A)$.

MISCELLANEA

WHICH PRESIDENT OF THE U.S. WROTE THIS?

119.

"Dear Sir

I have to acknolege the receipt of your favor of May 1st in which you mention that you have finished the 6 first books of Euclid, plane trigonometry, surveying & algebra and ask whether I think a further pursuit of that branch of science would be useful to you. There are some propositions in the latter books of Euclid, & some of Archimedes, which are useful, & I have no doubt you have been made acquainted with them. Trigonometry, so far as this, is most valuable to every man. There is scarcely a day in which he will not resort to it for some of the purposes of common life; the science of calculation also is indispensible as far as the extraction of the square & cube roots, algebra as far as the quadratic equation & the use of logarithms is often of value in the ordinary cases: but all beyond these is but a luxury; a delicious luxury indeed; but not to be involved in by one who is to have a profession to follow for his subsistence."

Later he wrote as follows.

"Having to conduct my grandson through his course of mathematics, I have resumed that study with great avidity. It was ever my favorite one. We have no theories there, no uncertainties remain on the mind; all is demonstration and satisfaction. I have forgotten much, and recover it with more difficulty than when in the vigor of my mind I originally acquired it."

The punctuation and spelling are reproduced exactly. See p. 72 for the answer.

My career has taken still another turn away from mathematics; I am now mainly interested in economics, especially monetary theory and libertarianism. Indeed, I increasingly become convinced that many of the problems with economics today result from the overuse of mathematics and the underuse of thought by economists."

The Experimental Musician. "I am probably not at all typical of the class in which I find myself [five years after a second winning of the Olympiad, a graduate student]. In fact, my progress through college has been quite uneventful, except for occasional Putnam citations and the like. By contrast, many of my Olympiad colleagues have published papers while I find myself scanty in scholarly attributes. I am a B student in a B-centered school, and still deficient in basic areas such as differential equations, complex analysis and functional theory.

I have long given up the idea of being a leading mathematician. The talent might still be there, but the motivation is lacking—there is no real appeal for me in a life spent bandying symbols. Most of us from the Olympiad have had incredible training opportunities and are well prepared to go out into the world and do something. There are real questions out there. Can a dozen nations develop nuclear weaponry without destroying each other? Do pesticides actually do more good than harm? Is welfare aiding the nations's poor or destroying them? The prevailing opinion is that these are not math problems, hence beyond the scope of the practicing mathematician. I disagree. Certainly we can contribute something, given a desire to do so, and any such progress is worth fifty theorems in algebraic topology."

The Computer Consultant. "I often [two years past the Olympiad, a junior] find that people assume that I will automatically get a degree in math or some closely allied field like physics and become a professional mathematician, just because I was in the Olympiad program. Personally, I don't like math that much; I detest physics; and I think that engineering is boring. I decided to be a math major partly because it was the expected thing to do; now I think that I may have made a mistake. The Olympiad program gave me an early shove in the direction of mathematics, and this may have been a good thing. On the other hand, most Olympiad winners I know seem quite content to be math or physics majors; so perhaps I am an exceptional case. I don't mean to say that the Olympiad program was bad. On the contrary, I thought it was great. The problem is just that I let it force me into becoming a math major when I should have resisted the pressure."

I am confident that all three young men will end up making contributions, in some way, to mathematics, but that is to be seen.

Nura D. Turner
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ANSWER TO QUESTION ON PAGE 56

The letters partly quoted on p. 56 are copied from *Thomas Jefferson and mathematics*, by David Eugene Smith, Scripta Mathematica, 1 (1932); the first is on p. 88, the second on p. 9.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS),
H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk () indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by May 31, 1984. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3031. *Proposed by Gengzhe Chang, University of Science and Technology of China, Hefei, Anhui, China.*

Suppose function f together with its derivatives of order up to $n + 1$ is defined and continuous in a neighborhood of $x = 0$ and satisfies $f(0) = 0$. Prove that

$$\lim_{x \to 0} \frac{d^k}{dx^k}\left[\frac{f(x)}{x}\right] = \frac{1}{k+1} f^{(k+1)}(0)$$

for $k = 0, 1, 2, \ldots, n$.

E 3032. *Proposed by J. O. Shallit, University of California, Berkeley.*

Let d, r be integers with $d \geqslant 2, r \geqslant 1$. Let p be a polynomial with real coefficients, and $\deg(p) < r$. Show how to partition the set

$$S = \{p(0), p(1), p(2), \ldots, p(d^r - 1)\}$$

into d disjoint subsets whose union is S, such that the sum of each subset is the same.

E 3033. *Proposed by M. McAsey, Bradley University and L. A. Rubel, University of Illinois.*

Let f be a twice differentiable function on $(-\infty, \infty)$ with f, f', f'' increasing. Fix numbers a and b with $-\infty < a \leqslant b < \infty$. For each $x > 0$, define $\xi = \xi(x)$ so that

$$\frac{f(b+x) - f(a-x)}{b - a + 2x} = f'(\xi)$$

by the Mean Value Theorem. Prove that $\xi(x)$ is an increasing function of x. Can the hypothesis f'' increasing be replaced with f'' positive?

E 3034. *Proposed by David Cox, Battelle Memorial Institute.*

Let $0 < x_0 < 1$ and $x_{n+1} = x_n - x_n^2$, for $n = 0, 1, 2, \ldots$. Prove that

$$\lim_{n \to \infty} \frac{n(1 - nx_n)}{\log n} = 1.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

The "Intersection Character" of Subfamilies of 2^U, U Finite

E 2887 [1981, 349]. *Proposed by Arnold Adelberg, Grinnell College.*

Let $\mathcal{C}$ be a collection of subsets of the finite nonempty set U. (i) Characterize $\mathcal{C}$ if each proper subset of U meets an even number of sets in $\mathcal{C}$. (ii) Characterize $\mathcal{C}$ if each proper subset of U meets an odd number of sets in $\mathcal{C}$. (See E 2792 [1979, 702].)

Solution by Douglas E. Cameron, The University of Akron. We assume that the empty set $\varnothing$ is improper and A meets $C \in \mathcal{C}$ if $A \cap C \neq \varnothing$.

For convenience we shall use INCH $\mathcal{C}$ to denote the intersection character of $\mathcal{C}$; that is, for $\mathcal{C}$ in (i) INCH $\mathcal{C}$ means "even," for $\mathcal{C}$ in (ii) INCH $\mathcal{C}$ means "odd." We prove that if INCH $\mathcal{C}$ is even, then $\mathcal{C} = P(U) - \{\varnothing\}$ whereas if INCH $\mathcal{C}$ is odd, then $\mathcal{C} = P(U) - \{U, \varnothing\}$ or $\mathcal{C} = \{U\}$.

LEMMA 1. *If INCH $\mathcal{C}$ is even, then $|\mathcal{C}|$ is odd if INCH $\mathcal{C}$ is odd, $|\mathcal{C}|$ is even unless $\mathcal{C} = \{U\}$.*

Proof. It is obvious that for $\mathcal{C} = \{U\}$, INCH $\mathcal{C}$ is odd. Let $m(\mathcal{C}) = \min\{|\mathcal{C}| : C \in \mathcal{C}\}$ and $C_1 \in \mathcal{C}$ such that $|C_1| = m(\mathcal{C}) \neq |U|$. Then $U - C_1 \neq \varnothing$ intersects all $C \in \mathcal{C}$ but C_1, i.e., $|\mathcal{C}| - 1$ members of $\mathcal{C}$.

LEMMA 2. *If $\mathcal{C} \neq \{U\}$, then $C \supseteq P(U) - \{U, \varnothing\}$.*

Proof. If $\{x\} \notin \mathcal{C}$ for $x \in U$, then $U - \{x\}$ meets all members of $\mathcal{C}$ and thus INCH $\mathcal{C}$ = evenness of $\mathcal{C}$ which contradicts Lemma 1. Thus $\{x\} \in \mathcal{C}$ for all $x \in U$.

Assume that for $B \subseteq U$ such that $|B| < k < |U|$, $B \in \mathcal{C}$ and $A \subseteq U$, $|A| = k$, $A \notin \mathcal{C}$. Then $U - A$ meets all but $2^{|A|} - 2 = |P(A) - \{A, \varnothing\}|$ members of $\mathcal{C}$ which contradicts INCH $\mathcal{C}$. Thus $A \in \mathcal{C}$.

If INCH $\mathcal{C}$ is even, then $\varnothing \notin \mathcal{C}$ (since otherwise $U - \{x\}$ meets $|\mathcal{C}| - 2$ members of $\mathcal{C}$ which contradicts Lemma 1). Thus $\mathcal{C} = P(U) - \{\varnothing\}$.

If INCH $\mathcal{C}$ is odd, then by Lemma 2, $\mathcal{C} = P(U)$ or $\mathcal{C} = P(U) - \{U, \varnothing\}$ or $\mathcal{C} = \{U\}$. But if $\mathcal{C} = P(U)$, $\{x\}$ meets $2^{|U|-1}$ members of $\mathcal{C}$. Thus $\mathcal{C} = P(U) - \{U, \varnothing\}$ or $\mathcal{C} = \{U\}$.

Also solved by K. L. Bernstein, B. Cheng & D. T. Hung (students), V. Hernandez (Spain), and the proposer.

Inductively Defined Sequences

E 2927* [1982, 130]. *Proposed by Clark Kimberling, University of Evansville.*

Define sequences $\{a_n\}, \{b_n\}, \{c_n\}$ inductively as follows: $a_1 = 1$, $b_1 = 2$, $c_1 = 4$, and take

a_n = least positive integer not among $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$

b_n = least positive integer not among $a_1, \ldots, a_{n-1}, a_n, b_1, \ldots, b_{n-1}, c_1, \ldots, c_{n-1}$

$c_n = 2b_n + n - a_n$.

Prove or disprove that $0 < n(1 + \sqrt{3}) - b_n < 2$ for all n.

Solution by William J. Gilbert, University of Waterloo, Canada. We will prove that $\alpha < n(1 + \sqrt{3}) - b_n < \beta$ for all n, where $\alpha = (9 - 5\sqrt{3})/3 > 0.113$ and $\beta = (12 - 4\sqrt{3})/3 < 1.691$.

First, prove by induction that $1 \le b_n - a_n \le 2$, $1 \le a_{n+1} - b_n \le 2$ and $2 \le c_{n+1} - c_n \le 6$ for all n, using the equation

$$(*) \qquad c_{n+1} - c_n = 2(b_{n+1} - a_{n+1}) + (a_{n+1} - b_n) - (b_n - a_n) + 1.$$

Secondly, let $\gamma_n = n(1 + \sqrt{3}) - b_n$. We will prove by induction that $\alpha < \gamma_n < \beta$. It is true that $\alpha < \gamma_1 < \beta$. Suppose $\alpha < \gamma_n < \beta$ for all $n < m$. By the previous result, b_m, a_m, b_{m-1}, a_{m-1}, b_{m-2}, a_{m-2} cannot be six consecutive integers. Let c_n be the first integer missing in this decreasing sequence. Then $n < m$ and either (a) $c_n = a_{m-j} - 1$ for $j = 0$ or 1 or (b) $c_n = b_{m-j} - 1$ for $j = 0$, 1 or 2. Hence $a_1, \ldots, a_m, b_1, \ldots, b_m, c_1, \ldots, c_n$ are the first b_m integers and either (a) $2m + n = b_m = c_n + 2 + 2j$ or (b) $2m + n = b_m = c_n + 1 + 2j$. Now $c_n = b_n + n + (b_n - a_n)$ so, in all cases, $2m + n = b_m = b_n + n + k$ where $2 \le k \le 7$. If $j = 2$, then $c_{n+1} - c_n = 6$ and from (*) it follows that $b_n - a_n = 1$. Hence k cannot be 7. Therefore $2m = b_n + k$ for $2 \le k \le 6$.

Now $\gamma_m = (1 + \sqrt{3})m - b_m = (1 + \sqrt{3})(b_n + k)/2 - (b_n + n + k) = (k - \gamma_n)(\sqrt{3} - 1)/2$. Using the induction hypothesis, if $2 \le k \le 6$, then $\gamma_m > (2 - \beta)(\sqrt{3} - 1)/2 = \alpha$ and, if $2 \le k \le 4$, then $\gamma_m < (4 - \alpha)(\sqrt{3} - 1)/2 < \beta$. If $k = 6$, it follows from (*) that $b_{n+1} - a_{n+1} = a_{n+1} - b_n = 2$ and so $b_{n+1} - b_n = 4$. Now $\gamma_{n+1} - \gamma_n = 1 + \sqrt{3} - (b_{n+1} - b_n) = \sqrt{3} - 3$. However $n + 1 < m$ so, using the induction hypothesis, $\gamma_n = \gamma_{n+1} + 3 - \sqrt{3} > \alpha + 3 - \sqrt{3}$. Therefore, if $k = 6$, $\gamma_m < (6 - \alpha - 3 + \sqrt{3})(\sqrt{3} - 1)/2 = \beta$. Similarly, if $k = 5$, it follows from (*) that $b_{n+1} - b_n = 3$ or 4. Hence $\gamma_{n+1} - \gamma_n \le \sqrt{3} - 2$ and $\gamma_n > \alpha + 2 - \sqrt{3}$. Therefore $\gamma_m < (5 - \alpha - 2 + \sqrt{3}) \cdot (\sqrt{3} - 1)/2 = \beta$ which completes the induction.

Edmund Butler, Tim Keller and Paul Pudaite also established the stronger inequalities on $x_n = n(1 + \sqrt{3}) - b_n$ given in the above solution.

It is claimed that numerical evidence suggests these bounds are the best possible. For example, at $n = 17{,}135$, $x_n = 1.690588$ while $x_n = 0.113253$ at $n = 23{,}407$.

Also solved by D. M. Bloom, R. Breusch, E. Butler, D. Finkel, Y. Hong, T. Keller, L. E. Mattics, and P. Pudaite.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by May 31, 1984. The solver's full post-office address should be on each sheet.

6448. *Proposed by Henry E. Fettis, Mountain View, CA.*

Show that

$$2\sqrt{3}\, Cl_2\left(\frac{\pi}{3}\right) = \psi'\left(\frac{1}{3}\right) - \frac{2\pi^2}{3}$$

where $Cl_2(\theta) = -\int_0^\theta \ln(2 \sin \frac{1}{2}t)\, dt$ is Clausen's integral and $\psi(z) = d/dz \ln \Gamma(z)$.

6449. *Proposed by Kent D. Boklan (student), Massachusetts Institute of Technology.*

Determine the smallest closed subinterval $[a, b]$ of $[0, 1]$ such that if p is a polynomial of degree at most 6 that increases on $[a, b]$, then necessarily $p(1) \geqslant p(0)$.

6450. *Proposed by J. O. Shallit, University of California, Berkeley.*

(a) Show that

$$\sum_{k=1}^{\infty} \frac{k - n[k/n]}{k(k+1)} = \log n$$

where n is a positive integer.

(b) Let $S(n, k)$ denote the sum of the digits of k when expressed in base n. Evaluate

$$\sum_{k=1}^{\infty} \frac{S(n, k)}{k(k+1)}.$$

SOLUTIONS OF ADVANCED PROBLEMS

Best Rank-k Approximation for a Matrix

6125 [1976, 818; 1980, 495; 1982, 503]. *Proposed by Simeon Reich, Tel Aviv University, Tel Aviv, Israel.*

It was pointed out by several solvers and the proposer that the original formulation of the problem was incorrect. The problem should be stated as follows (correction by the proposer):

For a given $n \times n$ complex normal matrix A of rank r, and an integer k, $1 \leqslant k \leqslant r$, let $M(A, k)$ denote the set of all normal $n \times n$ matrices of rank k that commute with A. A best normal rank-k approximation of A is a matrix $A_{(k)}$ in $M(A, k)$ satisfying

$$\|A - A_{(k)}\| = \inf\{\|A - X\| : X \in M(A, k)\} \quad \text{where } \|A\| = (\operatorname{trace} A^*A)^{1/2}$$

Show that $A^j_{(k)}$ is a best normal rank-k approximation of A^j for all $j \geqslant 1$.

Solution by the proposer. Since $A_{(k)}$ belongs to $M(A, k)$, $A^j_{(k)}$ is in $M(A^j, k)$. Now suppose there is a matrix B in $M(A^j, k)$ such that $\|A^j - B\| < \|A^j - A^j_{(k)}\|$. There are unitary matrices U and V such that $U^*AU = D$, $U^*A_{(k)}U = F$, $V^*A^jV = D^j$, and $V^*BV = E$, where D, E and F are diagonal. Since $\|\cdot\|$ is unitarily invariant, it follows that $\|D^j - E\| < \|D^j - F^j\|$. But this is impossible because F^j is certainly a best diagonal rank-k approximation to D^j.

The Derivatives of x^x

6392 [1982, 429]. *Proposed by C. Ward Henson, Bruce Reznick and Lee A. Rubel, University of Illinois.*

Is there some $x_0 > 0$ such that if $f(x) = x^x$, then $f^{(n)}(x) \geqslant 0$ for all $x \geqslant x_0$ and all $n = 0, 1, 2, \ldots$?

Solution by Paul R. Chernoff, University of California, Berkeley, California. No such x_0 exists because f is not an entire function. Suppose on the contrary that there is such an x_0. Then for $h > 0$ and any positive n, Taylor's theorem gives

$$f(x_0 + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} h^k + \frac{f^{(n+1)}(\xi)}{(n+1)!} h^{n+1}$$

where $x_0 < \xi < x_0 + h$. Since all the terms on the right side are nonnegative, it follows that all the partial sums of the Taylor series for $f(x_0 + h)$ are bounded by $f(x_0 + h)$. Hence the series converges. Since $h > 0$ is arbitrary, the Taylor series has an infinite radius of convergence. Then $f(x) = x^x$ is entire, a contradiction.

Also solved by Ulrich Abel, Miroslav D. Asic (Yugoslavia), Chico Problem Group, Chr. A. Meyer (Switzerland), C. C. Rousseau, Bertram Walsh, and the proposers.

The published solution uses only the elementary real variable form of Taylor's theorem. Bertram Walsh and the proposers obtain the stronger result that there is no $x_0 > 0$ for which $f^{(n)}(x_0) \geqslant 0$ for all $n \geqslant 0$, by using a deeper theorem of Pringsheim (see §7.2 of E. C. Titchmarsh, *The Theory of Functions*, second edition, Oxford University Press, 1939) which states that a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ with real nonnegative coefficients and positive radius of convergence r has a singular point at $x = x_0 + r$.

On Harmonic Functions

6393 [1982, 502]. *Proposed by G. A. Edgar, Ohio State University.*

I. Let $D = \{z \in \mathbb{C}: |z| < 1\}$ and $\overline{D} = \{z \in \mathbb{C}: |z| \leqslant 1\}$. Prove or disprove: Let $f_1: \overline{D} \to \mathbb{R}$ be continuous on $\overline{D}$ and superharmonic on D; let $f_2: \overline{D} \to \mathbb{R}$ be continuous on $\overline{D}$ and subharmonic on D; and suppose that $f_1(z) \leqslant f_2(z)$ for all $z \in D$. Then there exists a function $h: \overline{D} \to \mathbb{R}$, continuous on $\overline{D}$ and harmonic on D, with $f_1(z) \leqslant h(z) \leqslant f_2(z)$ for all $z \in D$.

II. Prove or disprove: Let μ be a positive measure of total mass 1 defined on the Borel sets of $\mathbb{C}$, with $\mu(\{z: |z| > 1\}) = 0$. Suppose μ satisfies $\int h(z)\, d\mu(z) = h(0)$ for all harmonic functions h: $\mathbb{C} \to \mathbb{R}$. Then μ also satisfies $\int s(z)\, d\mu(z) \geqslant s(0)$ for all subharmonic functions s: $\mathbb{C} \to \mathbb{R}$.

Solution by I. Netuka and J. Veselý, Charles University, Prague, Czechoslovakia. Both assertions are false. Recall that for every positive harmonic function f on D and z, $v \in \mathbb{C}$, $|z| \leqslant 1/2$, $|v| \leqslant 1/2$, the Harnack inequality (see L. Helms, *Introduction to Potential Theory*, Wiley-Interscience, New York 1969; Theorem 2.14) implies that

$$(*) \qquad f(z)/f(v) \leqslant 12.$$

Let G be the Green function for D with pole at the point $1/2$. Let $a = \sup\{G(z): z \in D, \operatorname{Re} z \leqslant 0\}$; and define $f_1(z) = 0$, $f_2(z) = 15a$ for $z \in \overline{D} \setminus D$, and $f_1(z) = \min(G(z), 14a)$, $f_2(z) = \max(15a - G(-z), a)$ for $z \in D$. Now it is easily seen that both f_1, f_2 enjoy all the properties stipulated in I. Any harmonic function h on D for which $f_1 \leqslant h \leqslant f_2$ has to be positive on D and such that $h(1/2)/h(-1/2) = (14a)/a = 14$. This contradiction with $(*)$ shows that assertion I is false.

To disprove II denote by H the set of functions on $\overline{D}$ which have harmonic extensions on $\mathbb{C}$. For any $g \in C(\overline{D})$, the space of continuous functions on $\overline{D}$, put

$$g^*(z) = \inf\{h(z): h \in H, h \geqslant g\}, z \in \overline{D}$$

(cf. H. Bauer, *Approximation and abstract boundaries*, this MONTHLY 85 (1978), 632–647). It is easily seen that

$$(f + g)^* \leqslant f^* + g^*, (\alpha f)^* = \alpha f^*$$

whenever $f, g \in C(\overline{D})$ and $\alpha \geqslant 0$, and hence $p(f) = f^*(0)$ is a sublinear functional on $C(\overline{D})$. For $z \in \overline{D}$ define

$$f(z) = \min(-\log((2/3)|z - 1/2|), 24 \log 3).$$

Then $f \in C(\overline{D})$ and $f > 0$ on D. It follows by $(*)$ that

$$f^*(0) \geqslant (24/12)\log 3 > \log 3 = f(0).$$

On the linear subspace $\{\alpha f: \alpha \in \mathbb{R}\}$ of $C(\overline{D})$, the mapping $\mu_0: \alpha f \to \alpha \log 9$ defines a linear functional majorized by p; indeed, for $\alpha \geqslant 0$,

$$\mu_0(\alpha f) = \alpha \log 9 \leqslant \alpha f^*(0) = (\alpha f)^*(0) = p(\alpha f)$$

while for $\alpha < 0$,

$$\mu_0(\alpha f) = \alpha \log 9 < 0 = (\alpha f)^*(0) = p(\alpha f).$$

By the Hahn-Banach theorem, there is a linear functional μ on $C(\overline{D})$ such that $\mu(f) = \mu_0(f)$ and $\mu(g) \leqslant p(g)$ whenever $g \in C(\overline{D})$. In particular, $g \in C(\overline{D})$ and $g \leqslant 0$ implies that

$$\mu(g) \leqslant p(g) = g^*(0) \leqslant 0.$$

Thus μ can be regarded as a positive measure on $\mathbb{C}$ with support in $\overline{D}$. Of course, $h^* = h$ for every $h \in H$. Consequently,

$$\mu(h) \leqslant p(h) = h^*(0) = h(0).$$

Since H is a vector space, we have $\mu(h) = h(0)$ whenever $h \in H$. Now define

$$s(z) = \max(\log((2/3)|z - 1/2|), -24\log 3), z \in \mathbb{C}.$$

Then s is continuous and subharmonic on $\mathbb{C}$ and $s = -f$ on $\overline{D}$. It follows that

$$\mu(s) = -\mu(f) = -\mu_0(f) = -\log 9 < -\log 3 = -f(0) = s(0).$$

We see that $\int s\, d\mu < s(0)$ and so assertion II is false.

REMARK. A modification of arguments from J. Bliedtner and W. Hansen, *Simplicial cones in potential theory II*, Inventiones Math., 46 (1978) 255–275, (cf. Proposition 4.5) leads to the same conclusion about assertion II.

Part I was also solved by the proposer.

REVIEWS

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A History of Greek Mathematics (2 vols.). By Sir Thomas Heath. Dover Publications, New York, 1981. Vol. 1 (From Thales to Euclid), xv + 446 pp., $8.50 (paperback); vol. 2 (From Aristarchus to Diophantus), xi + 586 pp., $8.50 (paperback).

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A FOUNTAINHEAD OF MODERN MATHEMATICS

It has been remarked, more than once, that modern mathematics is Greek mathematics. By this, probably a consciously hyperbolized statement, it is meant that much of modern mathematics finds its origin in the mathematics of the ancient Greeks.

The thesis is easily defended, and it would make an edifying and interesting study to supply the details. One might approach the study by considering chronologically the great mathematicians of the Greek period, starting with Thales of about 600 B.C. and running through Pappus of about A.D. 300, examining those roots of modern mathematics found in the works of these scholars. Or, perhaps better, one might develop the study by considering Greek roots of various important areas and concepts of mathematics, such as axiomatics, number theory, the integral calculus, projective geometry, transformation theory and the Erlanger Programm, modern geometries, metric space, the evolution of symbolic algebra, the structure of the real number system, plane and spherical trigonometry, the calculus of variations, logic, etc. The mountain of collected evidence, in either approach, would clearly establish ancient Greek mathematics as the fountainhead of a very large part of modern mathematics.

Let us briefly illustrate the second approach with a couple of examples, one broad and one narrow. Certainly one of the broadest creations in Greek mathematics was the invention of the axiomatic method. According to tradition, the first significant conscious injection of deduction into mathematics was made by Thales of Miletus. Thales is said to have replaced the essentially laboratory procedures employed in the geometry of pre-Hellenic times by short chains of deductive reasoning stemming from deeper basic assumptions. This idea was extended by Pythagoras and members of his school, by attaching in tandem numbers of such short chains, obtaining in each case a connected collection of geometrical results deduced sequentially from a

common initial set of fundamental assumptions. Then someone, probably a member of the Pythagorean school, entertained the grand idea of developing all of geometry, link by link, in a single long chain emanating from a given small initial platform of reasonably acceptable assumptions. After a number of partially successful attempts, Euclid essentially achieved the desired goal in his famous *Elements*. But one of Euclid's initial assumptions (his parallel postulate) evoked objections in that it did not seem reasonably acceptable. Accordingly, almost immediately, efforts were made to replace the objectionable postulate by a more acceptable one, or to eliminate it altogether from the initial platform of assumptions by deriving it as a consequence of the other assumptions. These efforts, after noteworthy investigations by Khayyam, Saccheri, Lambert, Legendre, and others, led, in the early half of the nineteenth century, to the independent discovery of a non-Euclidean geometry by Gauss, Lobachevsky, and János Bolyai. Then came the second non-Euclidean geometry of Riemann. Geometry was liberated from its original Euclidean mold, and the path was cleared for the creation of many new geometries. Similar investigations were applied to algebra, giving birth, also in the first half of the nineteenth century, to the first nontraditional algebra, thus opening the floodgates of modern abstract algebra. Further properties of a postulate set, besides that of independence, were studied, and metamathematics came into being. In particular, under the leadership of Hilbert, the original material axiomatics of the early Greeks was sharpened into the formal axiomatics of modern times. The axiomatic method has now penetrated all parts of modern mathematics. Much more can be said. In summary, it would be difficult to overstate the consequences to modern mathematics of the axiomatic method created by the early Greeks.

As a much narrower example illustrating the thesis that modern mathematics is largely Greek mathematics, consider the Pythagorean concepts of perfect, deficient, and abundant numbers. These concepts arose among the Pythagoreans in conjunction with the brotherhood's basic philosophy that the whole numbers control the universe. Subsequent Greek mathematicians seized upon and developed these Pythagorean concepts; Euclid, for example, gave a remarkable formula that yields even perfect numbers. But, from our present point of view, we note how mathematicians of the modern era also have seized upon the same concepts. Euler, in the eighteenth century, showed that every even perfect number must be of Euclid's form. Only twelve perfect numbers were known prior to 1952. In 1952 and following years, with the aid of high-speed electronic computers, fifteen more perfect numbers have been found, most of these being numbers of incredibly large size. The concept of perfect numbers has been generalized to that of *k-tuply perfect numbers*. In 1944, the concept of *superabundant numbers* was created. Other numbers related to perfect, deficient, and abundant numbers that have been added in recent times are *practical numbers*, *quasiperfect numbers*, *semiperfect numbers*, and *weird numbers*. The creation and study of these extended concepts illustrate how ancient Greek number work has inspired many related modern investigations. The existence or nonexistence of odd perfect numbers is at present a teasing and challenging open problem; it is known that there is no such number containing less than 100 digits. Other ancient number concepts, such as those of amicable numbers, polygonal numbers, and prime numbers, have similarly stimulated much modern research. One could write a sizeable book devoted only to modern researchers connected with prime numbers.

Since so much of modern mathematics has its roots in ancient Greek mathematics, considerable interest has been generated for the study of the history of the older body of material. For this purpose, the English-speaking population is particularly fortunate in having available the extraordinary treatise, *A History of Greek Mathematics*, by Sir Thomas Heath. Here, in a monumental work of over 1000 pages, one finds one of the most scholarly, most complete, and most charmingly written treatments of the subject, a treatment certain to kindle a deep appreciation of that early period of mathematical development and a genuine admiration of those who played leading roles in it. As the eminent Harvard geometer, Julian Lowell Coolidge, was wont to say, "There were giants in the land then."

Sir Thomas has written a great deal on the Greek mathematical genius. In addition to the

above two-volume work, there is his shorter one-volume *A Manual of Greek Mathematics*, and a number of separate volumes devoted specially to Archimedes and his work, to Euclid and his *Elements* (3 volumes), to Aristarchus, to Apollonius, and to Diophantus. The works of Sir Thomas Heath constitute a veritable treasurehouse of material for the interested reader.

We certainly owe deep thanks to Dover Publications for rescuing and preserving for us, in excellent and easily available form, Heath's *A History of Greek Mathematics*, and a number of Heath's other works.

A Course in Universal Algebra. By Stanley Burris and H. P. Sankappanavar. Springer-Verlag, New York, 1981. xvi + 276 pp.

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Ideals play the same basic role in ring theory that normal subgroups play in group theory; they determine the congruence relations that define quotients or, equivalently, they are the kernels of homomorphisms. This analogy and others like it (think of the isomorphism theorems for example) between various branches of algebra provide the raw material for universal algebra, the goal of which is to study, at the appropriate level of generality, those concepts and methods that recur throughout algebra. The appropriate level of generality for most purposes is given by the following definitions. A *type* of algebra is a set of operation symbols, each of which is assigned a rank indicating the number of arguments to which the operation is to apply (e.g., for group theory, the type consists of $\times, e, ^{-1}$ of ranks 2, 0, 1 respectively). An *algebra* of such a type is a nonempty set together with operations, one for each operation symbol, and having the specified number of arguments. (A zero-place operation is a distinguished element.) One reason for introducing types, rather than just letting an algebra be a set with some operations, is that, to define homomorphisms, one must know which operation on the domain is supposed to match which operation on the codomain. For two algebras of the same type, the intended matching is indicated by the use of the same operation symbols. A second use for the operation symbols is in writing identities, i.e., formal equations, like $x^{-1} \times x = e$, between expressions (called terms) built up from operation symbols and variables; one can compare different algebras of the same type by considering the identities they satisfy.

In this setting, it is easy to define such concepts as homomorphism, isomorphism, subalgebra, and congruence relation and to prove the basic theorems about these concepts, for example, the isomorphism theorems. The possibility of describing a congruence relation by specifying just one of its equivalence classes (e.g., a normal subgroup in a group) is not available in general algebras; semigroups and lattices provide natural counterexamples. Thus, the congruence relations themselves are the analogs, in the general theory, of normal subgroups in group theory. The analogy turns out to be surprisingly far-reaching; for example, every algebra has a canonically defined congruence called its center, a generalization of the center of a group.

To convey some of the flavor of the subject, here are a few representative results. (1) There is a complete characterization of the lattices that can occur as the lattice of subalgebras of an algebra. (Lattices of congruence relations and monoids of endomorphisms have also been treated, singly and in combination.) (2) If the congruences on an algebra commute, under relational composition, then the lattice of congruences is modular. (3) If all the operations of an algebra are at most binary, then the cardinalities of finite minimal sets of generators constitute a set of consecutive integers. (If ternary operations are allowed, there can be gaps, but of length at most one. In general, the length of a gap is bounded by the maximum rank of the operations minus two.)

It would be misleading, however, to think that properties of individual algebras, as in the

preceding examples, are the primary focus of universal algebra. At least as much effort is devoted to the properties of general classes of algebras, such as the class of all groups, or the class of all rings. The most studied classes are the *varieties*, classes of algebras (of a fixed type) definable by identities. For example, groups constitute a variety (defined by $x \times (y \times z) = (x \times y) \times z$, $e \times x = x$, and $x^{-1} \times x = e$), as do abelian groups, rings, Lie rings, etc. One of the fundamental results about varieties is that they are exactly the classes closed under the formation of subalgebras, direct products (of possibly infinitely many factors), and homomorphic images. It may be instructive to note that, if we had taken groups to be algebras with just one operation, $\times$, rather than $\times, e, ^{-1}$, then they would not form a variety because their definition would require existential statements (the identity and inverses exist) rather than just identities; their subalgebras would be not just their subgroups but all their subsemigroups as well. Fields also fail to form a variety because they are not closed under direct products, or because identities between terms cannot take into account the exception, 0, to the existence of inverses.

Many of the basic properties of varieties are consequences of the fact that all varieties contain free algebras on arbitrary sets of generators. One example of such a property is the following, which served as a prototype for a number of similar results. In order for all the algebras in a variety to have commuting congruences (as in (2) above) it is necessary and sufficient that there exist a term $p(x, y, z)$ such that every algebra in the variety satisfies the identities $p(x, y, y) = p(y, y, x) = x$. For example, in group theory, the term $x \times (y^{-1} \times z)$ clearly works, so all groups are congruence-permutable and therefore congruence-modular.

All the results mentioned above (except for the parenthetical comment about characterizing congruence lattices and endomorphism monoids) are proved in Chapter II of Burris and Sankappanavar's book. This chapter, which constitutes about a third of the book and contains the basic core of the subject, is preceded by a chapter on lattice theoretic and set-theoretic preliminaries and followed by three chapters on more specialized topics. Chapter III contains some applications of universal algebra, including a nice approach to constructing orthogonal latin squares. Chapter IV, entitled "Starting from Boolean Algebras", consists of three distinct, though interconnected parts. First, there is the general theory of Boolean algebras, up to and including the Stone duality theorem relating these algebras to totally disconnected compact Hausdorff spaces. Second, there is a study of Boolean products, a generalization of direct products, that can often be used to transfer to arbitrary varieties constructions carried out in the particularly well-understood variety of Boolean algebras. Third, there is a study of various conditions on a variety that cause it to share some of the pleasant structural properties of the variety of Boolean algebras. Finally Chapter V covers two sorts of interactions between universal algebra and mathematical logic: applications of the compactness theorem and questions of algorithmic decidability.

After this listing of what is in the book, it seems appropriate to mention one topic that is absent, the category-theoretic approach to universal algebra. This approach, like many aspects of category theory, offers primarily conceptual insights and simplifications rather than technical results. For example, the presentation of group theory in terms of $\times, e, ^{-1}$ and the alternate presentation in terms of division are viewed as two presentations of a single theory rather than two varieties with a sort of equivalence between them. Again, the category viewpoint gives a neat description of the widespread phenomenon that an algebra in one variety can be viewed as an algebra in another variety: e.g., an abelian group is also a group, an associative ring "is" also a Lie ring (with $[x, y] = x \times y - y \times x$), etc. And it follows, from general category theory, that, in such a situation, every algebra in the second variety freely generates one in the first. This result subsumes the abelianization of groups (dividing by the commutator subgroup), the universal enveloping ring of a Lie ring, and the algebra (in any variety) freely generated by a set. Burris and Sankappanavar's decision to omit this material is justified (perhaps even forced) by the fact that it would require a good deal of preparatory material and that it is already treated in Manes's book, *Algebraic Theories*, in the same Springer series; nevertheless, I would have preferred that they give some indication that the subject exists.

Apart from this one complaint, I was extremely pleased with this book. It is a well-written and carefully planned introduction to a rapidly growing subject, and it takes the reader from the basics up to at least part of the current research frontier.

Algebraic Geometry. An Introduction to Birational Geometry of Algebraic Varieties. By Shigeru Iitaka. Graduate Texts in Mathematics, Vol. 76, Springer-Verlag, New York, 1982. $x + 357$ pp.

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It may be that it is easier to study and practice algebraic geometry now than it has been at any previous time this century. One reason is the stabilization of the foundations and the return of interest to the classical concrete problems. A less fundamental reason, important to the prospective student, is the current luxuriant bloom of new textbooks, at all levels, including the book under review.

Before coming to that book I would like to sketch some history and to mention a few of my favorite texts, as a basis for comparison. It will be a rather idiosyncratic tour. I hope it will be useful to the prospective student. To make this review reasonably short, I will in any case omit mention of related texts on complex analysis, number theory, algebraic groups, and the many local questions like singularity theory and commutative algebra, as well as most of the really elementary books suitable for an undergraduate audience.

A large proportion of the results that we now know as the core of algebraic geometry, those on projective curves and surfaces and the many varieties (Jacobians, Grassmannians,...) associated to them in one way or another, was already known by the 1920's. But there were many gaps in the theory, including a lack of rigorous justification of the powerful "principles" that were used both implicitly and explicitly. Some of the famous men of the time were able to avoid employing the principles in objectionable ways; others used them to perpetrate serious errors.

The inadequacy of the foundations was clearly perceived by Zariski, van der Waerden, and others, who began to extend the algebra pioneered partly by Hilbert and Kronecker to the level necessary to treat the highly developed geometry. The book of van der Waerden [1939], still a useful elementary text, and that of Hodge and Pedoe [1947], with its still-standard account of Grassmann varieties and Schubert cycles, give an idea, on the textbook level, of the direction taken. The central objects of study in this period were projective varieties, always implicitly and often explicitly given through a particular projective embedding, over a particular field, initially the complex numbers.

Weil's classic *Foundations* [1946] represented a transition to a more intrinsic point of view, with varieties represented as unions of affine varieties, without explicit or implicit projective embeddings, and points having coordinates in variable subfields of a fixed "universal domain." The rationale for this transition was the difficulty of embedding certain very natural varieties of higher dimensions, such as the Jacobians of curves. The principal textbook version embodying Weil's viewpoint is surely that of Lang, whose series of books, beginning with [1958], treats many aspects of the theory.

It was not long before the foundations began to shift again, as Serre introduced sheaves, which already played an important role elsewhere in geometry, as the formalism through which the "gluing" of affine varieties was to be done. The fundamental work of Serre [1955] remains today an outstanding introduction to the use of sheaves, accessible with surprisingly little preparation.

Serre's techniques proved well suited to the major foundational overhaul undertaken by Grothendieck, embodied in the shift to *schemes* as the fundamental objects. Very crudely, these are patched together from local pieces with sheaves, just as in Serre, but the local pieces are

allowed to be much more general (spectra of arbitrary commutative rings, where, in Grothendieck's terms, Serre only allows domains finitely generated over an algebraically closed field).

In the period of development that followed, the old theory was very greatly extended; but in the years between, say, 1958 and 1970 there were few textbooks. The enormous work of Dieudonné and Grothendieck [1960–1967] was supposed to become a complete textbook of the theory, though it rapidly became too encyclopedic for any but the most intrepid readers. Moreover, the basic theory was felt to be unsettled. Mumford's useful "Red" notes [1960] was perhaps the leading introductory text of the period, though it actually represents only an aborted attempt at a textbook and Mumford once said to me modestly that its trouble is that it contains "no theorems."

W. H. Auden has said that "a poem is never finished, only abandoned," and so it seems also with foundations in Mathematics. The building period ended with the abandonment by Grothendieck of the project. (Perhaps the complete incorporation of Artin's algebraic spaces, more general then schemes, would have been one of the next steps—see Knutson [1971].) Today, Grothendieck is, to many young algebraic geometers,

> ... No more a person
> now but a whole climate of opinion
> under whom we conduct our different lives:
> Like weather, he can only hinder or help...

(from Auden's poem "In Memory of Sigmund Freud." reprinted in *Auden* [1976]).

After the shift away from Grothendieck's foundational work, interest in algebraic geometry was again focussed on the classical questions. (To be fairer, a large part of Grothendieck's work was aimed at some concrete classical questions: Deligne's brilliant work on the "Weil conjectures," the background to which was recently exposed in textbook form by Milne [1980], represents the success of the ideas in one of these directions.) As the techniques have come to seem less strange, quite a number of people have set to work writing interesting texts.

The first major book to appear was that of Shafarevich [1972]. It is a leisurely, cultured, altogether charming book which aims to do algebraic geometry without—or at least with a minimum of—algebra. Part I (about 200 pages) makes an excellent introduction to quasiprojective varieties. The part on schemes is perhaps too brief to be useful, but the last part, on real and complex varieties, with some analytic theory, is again quite valuable. It is a good book to read first, especially for someone with little background in algebra and other kinds of geometry. Unfortunately, it doesn't include enough, in any direction, I think, to give the reader access to the literature. I think that this is a fault in a serious book of over 400 pages, though at the time the book was written, it seemed very brief by comparison with the Dieudonné-Grothendieck volumes [1960–1967].

Soon after, the far shorter text by Mumford [1976] appeared. Like that of Shafarevich, it had as its goal the exhibition of some of the interesting objects of algebraic geometry without serious use of algebra (sheaves and schemes), but there the resemblance ends. The difference in styles of writing will be apparent to the most casual browser. As for content, Mumford has as his goal to show how many different approaches to complex projective varieties there are, how many different kinds of geometric tools serve in the study. For this reason, and because of its relative brevity, I think Mumford's text is the best starting point for someone who knows only a little commutative algebra, but who is already at home in a basic way with manifolds and geometry in the differentiable setting. The book has no exercises as such, but most people find reading Mumford to be quite an active enterprise.

My own favorite textbook, for those with a suitable algebraic background, is that of Hartshorne [1977]. The first chapter seems to me an extraordinarily good introduction to the subject, culminating with the equivalence of smooth curves and "algebraic function fields in 1 variable." It

should provide motivation for the two rather technical chapters (schemes, sheaves, and cohomology), which go, perhaps, a little far toward abstraction for those without either a previous background or an active guide. (The prospective reader might consider Serre [1955] as an inoculation against "schematitis" before beginning those chapters.) Chapter 4, on curves, is once again an exposition of the greatest simplicity and beauty. Very little of the hard chapters 2 and 3 is required; one could, with only a little help, read this after an elementary book on curves such as Fulton [1969] or even Walker [1950] (the theory of Jacobians is unfortunately omitted here; the reader could turn to Serre's book [1959] (whose first chapter should be read last) to fill the gap). Hartshorne's final chapter, on surfaces, is somewhat less satisfying, if only because the field is so vast, but it is a good beginning. The book contains a very large number of excellent and fascinating exercises, many of which are fairly easy applications of material or techniques in the text. One of the advantages of this book is that it suffices for the understanding of many current papers and seminars in the field.

The last of the modern "general introductions" I wish to mention is that of Griffiths and Harris [1978]. This book—"collection of essays" would be more descriptive, since the chapters are rather independent—is much further toward the analytic side than the others here: schemes are not mentioned, though sheaves, treated in a simple way, abound. The book suffers from an unusually large number of errors, both serious and merely annoying. But it also has far more of the ravishingly beautiful topics from "synthetic" projective geometry than any of the other texts, and it very effectively exploits its restriction to varieties over the complex numbers to come quickly to grips with such central notions as the Jacobian of a curve or the Chern classes of a bundle. It is, I think, the book of choice for an experienced reader looking for that part of the subject which made it exciting 50 or 100 years ago, and keeps it exciting today. My personal advice to prospective readers is to read chapters 2 and 4 first (maybe in the opposite order, skipping bits that seem less interesting) picking up the theory of the Grassmannian (Chapter 1.5) and the rest as necessary.

In addition to the four basic texts above, there are a number of more specialized, "second level" texts which are now or are about to be available, and which bridge, to some extent, the gap between the books already mentioned and the ongoing research of various schools. I can mention here only a few favorites. An early but important one of these is Mumford's book on families of curves on surfaces [1966], which contains a textbook account of Hilbert and Picard schemes. Also very interesting is the same author's book on Abelian varieties [1974] (which, alas, ignores Jacobians).

On the matter of curves, Mumford's little survey [1975] should not be missed by any admirer (either of curves or of Mumford). A large book by Arbarello, Cornalba, Griffiths, and Harris will presumably make available, in a year or so, an account in detail of a number of topics from the classical theory which have seen intense recent development.

As for surfaces, the short volume of Beauville [1978] (soon to be available in English) goes somewhat further than Hartshorne; a larger and long awaited work by Barth, Peters and Van de Ven is now in press.

Two books on other topics which deserve mention are those by Okonek, Spindler and Schneider [1980] (to which, I think, commutative algebraists interested in module theory should pay more attention) and a forthcoming book by Fulton on intersection theory, which will fill a gap in an important area.

Let us turn, at last, to the book of Iitaka under review. It is clear that it faces a lot of competition.

Iitaka's book is a revision of a text in Japanese which, I imagine, must have been aimed at helping students to the expertise necessary to work with the author and his colleagues in the interesting new area around the birational classification and maps of (possibly incomplete) varieties that they have pioneered and developed. At least a good part of it would therefore have fit into the "second level" category above. The author has attempted, in the revision, to make the book accessible to the beginner. Indeed, the jacket blurb announces that "This edition has been

revised so that even the beginner can use it without referring to other books. Knowledge of the definition of Noetherian rings and the statement of the Hilbert Basis Theorem is all that is required."

Unfortunately, I cannot recommend this as a book for beginners. Indeed, the elementary definitions are carefully given, and many interesting results are proved from scratch. But despite the blurb quoted above, many of the central results are simply quoted from elsewhere (for example, from Hartshorne [1971], Dieudonné-Grothendieck [1960–67], or Mumford [1974]). The level of writing is quite uneven, there is no motivational treatment like that of the opening chapters of Hartshorne or Shafarevich to justify the plunge into the formalism of abstract schemes and commutative algebra, and there is a serious shortage of nontrivial (and, especially, nonrational) examples.

Here are some brief comments to justify each of these objections: The results, large and small, for which the reader is referred elsewhere include practically all the basic theorems on cohomology, the Zariski Connectedness Theorem, and the result that the only connected algebraic groups of dimension 1 are $\mathbb{A}^1$ and $\mathbb{G}_m$.

The variability of level is perhaps illustrated by the contrast between the first four examples of spectra—those of the trivial ring, the integers, a field, and the ring of polynomials in one variable —and the proof given for the basic properties of higher direct images, which I quote in full: "The next result is a consequence of the theory of derived functors." (P. 183.) I believe that this is the first mention of derived functors in the text.

The lack of motivation is illustrated by the first mention of the projective line which appears on page 63, *after* treatments of spectra, coherent and quasi-coherent sheaves, schemes, and open and closed immersions.

The lack of examples is typified by the treatment of curves, where one might expect finally to have reasonably many interesting examples. It does not start until page 208, and lasts, in any case, only 17 pages. The examples (broadly interpreted) that it contains are: The projective line (1 brief paragraph), hyperelliptic curves, treated in three brief remarks, and the Fermat plane curves $X^n + Y^n = Z^n$, treated in two paragraphs.

A more trivial, but rather pervasive problem is in style. For example, the verbal expression is often rather loose, and explication of the symbols frequently follows long symbolic expressions. More seriously, there is a heavy dependence on a casually selected notation, for which there is no index. For example, the notation spm (Y) (Y being a variety) appears without reference or hint on pages 314 and 315, in the statement of two propositions, and the proofs given do not make it (for me, anyway) especially easy to guess the meaning of the symbol. I feel that only the most devoted reader will remember the definition, on page 84, where it says that spm (Y) is the set of closed points of Y. I presume that the notation comes from the fact that closed points correspond to maximal ideals, so spm is the Maximal Spectrum (or at least the "spectre maximal"), but Iitaka does not comment on this. If there is a second printing of this work, a notational index would be a great help.

If we ignore the blurb and regard this as a "second level" text for someone, say, who has read Mumford [1976], Shafarevich [1972], or even sections of Griffiths-Harris [1978], and wants a view of schemes, or an introduction to an interesting and active group of topics, the book fares somewhat better. The ideal reader will bring with him his own motivation and examples, and will enjoy the many references to results proved elsewhere for the culture they bring, sighing perhaps with relief at the postponement of some hard work. He will certainly find items not available in other texts to reward him, from the notions of strict birational maps to the automorphisms of varieties of logarithmically general type. But Iitaka remarks in the Preface that he has had to cut the "high end" of the book, the sections on curves, surfaces, and logarithmic Kodaira dimension, to make room for more elementary material, and this hypothetical reader might, as I did, feel that the book stops only a little after the going becomes interesting.

In sum, I feel that Iitaka's book should stand not with the introductions to algebraic geometry, but with the more specialized works; and that by comparison with other works of this type, the

book suffers, both through its style and use of notation, which make it hard to use for reference, and through the relatively small space given to advanced topics. The reader who braves the thicket does get, however, a taste to whet his appetite for more of the striking results from the author's distinguished school.

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LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

Editor,

In his paper, "Generalized Hyperbolic Functions" (this MONTHLY, vol. 89, pp. 688–691) A. Ungar discusses the circulant matrix $H_n(z)$ in which the elements of the first column are

$$\sum_{k=0}^{\infty} \frac{z^{nk+r}}{(nk+r)!} \qquad r = 0,1,\ldots,n-1.$$

I would like to point out that the properties of the matrix $H_n(z)$ can be obtained more easily from the properties of the matrix-valued exponential

$$\exp A = \sum_{k=0}^{\infty} \frac{A^k}{k!}, \quad A \text{ an } n \times n \text{ matrix.}$$

$$\text{Put } A = \begin{vmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & 0 & 0 \\ 0 & 1 & & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & \cdots & 1 & 0 \end{vmatrix}$$

then $H_n(z) = \exp(Az)$. The relations $|H_n(z)| = 1$, $H_n(z_1 + z_2) = H_n(z_1).\ H_n(z_2)$ now follow immediately from the well-known formulas $\det \exp(Az) = \exp \operatorname{tr}(Az) = 1$ and $\exp(Az_1) \cdot \exp(Az_2) = \exp(A(z_1 + z_2))$.

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Editor:

Your readers might be interested in the views of mathematics of three winners of the U.S.A. Mathematical Olympiad who have kept in touch with me concerning their careers. All have bachelor's degrees in mathematics and, unsurprisingly, all did very well on the Putnam Examination. One is now in his fourth year of graduate study in operations research; one is in his third year of graduate study in mathematics, with a pronounced interest and involvement in experimental music; the third has not yet pursued graduate study but is taking time out to do computer work for a year or two at a college research center abroad, with the idea of travelling a bit thereafter and then entering graduate study or getting a job.

The quotes are from letters received over a number of years. To preserve confidentiality, I am not using names.

The Operations Researcher. "As an undergraduate [three years past the Olympiad, sophomore in college], I presently have grave doubts about pursuing a career in mathematics. I feel I have one of the best minds in our society and to use it for mathematics, although an enjoyable experience, would require my making a CONSIDERABLE sacrifice. I am not a slave to the dollar bill, but the thought that, because of society's method of determining salaries, some of my associates with considerably poorer minds would be earning more than I, is quite upsetting. I might just 'cop out' and assume my place among society's best paid.

I still enjoy mathematics [three years later, post M.S. in operations research], but I know I would find it an unsatisfying career; it is fun but it does not seem to be the sort of thing grown men do. Finding the ethereal nature of pure mathematics sterile, I decided to pursue a degree in an applied field, operations research. However, at the institution at which I am studying, the only thing applied is the vocabulary. The professors are interested either in mathematical finesse and manipulation or government intervention. I have little interest in the former, and as a libertarian, I abhor the latter.

My career has taken still another turn away from mathematics; I am now mainly interested in economics, especially monetary theory and libertarianism. Indeed, I increasingly become convinced that many of the problems with economics today result from the overuse of mathematics and the underuse of thought by economists."

The Experimental Musician. "I am probably not at all typical of the class in which I find myself [five years after a second winning of the Olympiad, a graduate student]. In fact, my progress through college has been quite uneventful, except for occasional Putnam citations and the like. By contrast, many of my Olympiad colleagues have published papers while I find myself scanty in scholarly attributes. I am a B student in a B-centered school, and still deficient in basic areas such as differential equations, complex analysis and functional theory.

I have long given up the idea of being a leading mathematician. The talent might still be there, but the motivation is lacking—there is no real appeal for me in a life spent bandying symbols. Most of us from the Olympiad have had incredible training opportunities and are well prepared to go out into the world and do something. There are real questions out there. Can a dozen nations develop nuclear weaponry without destroying each other? Do pesticides actually do more good than harm? Is welfare aiding the nations's poor or destroying them? The prevailing opinion is that these are not math problems, hence beyond the scope of the practicing mathematician. I disagree. Certainly we can contribute something, given a desire to do so, and any such progress is worth fifty theorems in algebraic topology."

The Computer Consultant. "I often [two years past the Olympiad, a junior] find that people assume that I will automatically get a degree in math or some closely allied field like physics and become a professional mathematician, just because I was in the Olympiad program. Personally, I don't like math that much; I detest physics; and I think that engineering is boring. I decided to be a math major partly because it was the expected thing to do; now I think that I may have made a mistake. The Olympiad program gave me an early shove in the direction of mathematics, and this may have been a good thing. On the other hand, most Olympiad winners I know seem quite content to be math or physics majors; so perhaps I am an exceptional case. I don't mean to say that the Olympiad program was bad. On the contrary, I thought it was great. The problem is just that I let it force me into becoming a math major when I should have resisted the pressure."

I am confident that all three young men will end up making contributions, in some way, to mathematics, but that is to be seen.

Nura D. Turner
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ANSWER TO QUESTION ON PAGE 56

The letters partly quoted on p. 56 are copied from *Thomas Jefferson and mathematics*, by David Eugene Smith, Scripta Mathematica, 1 (1932); the first is on p. 88, the second on p. 9.

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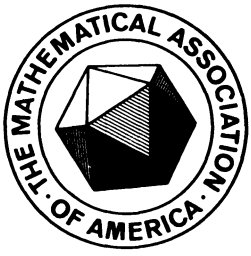
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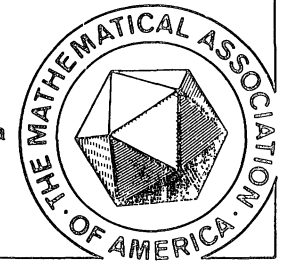
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THE MEASURABILITY OF UNCOUNTABLE UNIONS

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1. Introduction. Standard courses in measure theory have been known to leave their students with one or more of the following impressions.

(1.1) Null sets are unimportant and can be ignored.
(1.2) Nonmeasurable sets play the role of black sheep in an old established family. One grudgingly acknowledges their existence on special occasions, and otherwise one tries not to think about them.
(1.3) Consequently, uncountable Boolean operations on measurable sets are taboo, since they might bring the scandalous black sheep into full view.
(1.4) The Lusin criterion of measurability provides transcendent insight into the true nature of a measurable function, but is otherwise useless.
(1.5) Its manifold applications notwithstanding, measure theory is stone cold dead as a viable area of current research.

Within advanced measure theory these precepts enjoy a status comparable to that of Newton's laws of motion: they have served long and well in many classical problems but break down badly in modern ones.

This paper will try to indicate to the general reader that advanced measure theory is alive and well, and to show, moreover, that at least substantial special cases of some important recent results can be obtained by techniques which are elementary and not overly intricate. Thus to the descriptives "alive" and "well" we may add "accessible."

The results are unified by the theme: how to handle uncountable unions of measurable sets. We suggest that the traditional "taboo" against such unions may have obstructed advancement in some areas. Once they are analyzed and "tamed" to a certain extent, one can proceed with surprising rapidity to a general measurable selection theorem (Theorem 2.3), and to a general separable range theorem for measurable maps (Theorem 4.1). Section 3 presents an elementary exposition of the "taming" which is required by the principal result of Section 4. Only [**22**] is needed to understand the content of Section 3.

Unless otherwise specified, X will denote an arbitrary nonempty set, $\mathscr{A}$ will denote a σ-algebra of subsets of X, and μ will denote a nonnegative, finite, countably additive measure defined on $\mathscr{A}$. (The finiteness assumption can usually be relaxed. We make it to avoid distractions.) If $\mathscr{T}$ is a topology on X, we refer to the elements of the σ-algebra generated by $\mathscr{T}$ as the *Borel sets* of X, and if $\mathscr{T} \subseteq \mathscr{A}$, we refer to the quadruple $(X, \mathscr{A}, \mu, \mathscr{T})$ as a *topological measure space*. It is called a *Radon measure space*, and μ is called a *Radon measure*, if the topology $\mathscr{T}$ is Hausdorff, and if μ is inner regular (or tight), that is, if we have

$$\mu(A) = \sup\{\mu(K) : K \subseteq A;\ K \text{ compact}\}$$

Joseph Kupka was born in Los Angeles, California, in 1942. Shortly thereafter, in 1970, he received a Ph.D. in mathematics from the University of California at Berkeley for work in measure theory. He currently holds the position of Senior Lecturer in Mathematics at Monash University in Melbourne, Australia. He is noted for his ability to achieve instant rapport with small animals and to run faster than a speeding wombat.

Karel Prikry was born in Czechoslovakia and did his undergraduate work in mathematics at the Prague University and the Warsaw University in Poland. In Warsaw he was strongly influenced by Professor A. Mostowski. He wrote his dissertation in set theory under Jack Silver at Berkeley. He is currently at the University of Minnesota. He enjoys long walks in the countryside.

for every set $A \in \mathscr{A}$. In this case, given an $\mathscr{A}$-measurable function $f: X \to \mathbb{R}$, the Lusin theorem [**6**, Theorem 7.4.3, p. 227] implies that:

(1.6) *For every nonnull set $A \in \mathscr{A}$, there exists a nonnull compact subset K of A such that $f|_K$ is continuous.*

Conversely, if an arbitrary function $f: X \to \mathbb{R}$ satisfies (1.6), and if the (finite Radon) measure μ is complete, then f is $\mathscr{A}$-measurable. These two facts together comprise the *Lusin criterion* of measurability. We remark that an infinite Radon measure is required to have one additional property: every point of X must have a neighborhood of finite measure.

2. Uncountable Measurable Unions of Nonnull Sets. Measure theoretic liftings were not invented in order to gain control over uncountable unions of measurable sets. They nevertheless do enable some control to be exercised, and this section will attempt to illustrate the significance of that fact. Throughout the section we shall write $A \approx_\mu B$ when A and B are μ-equivalent (that is, when $\mu(A \Delta B) = 0$), we shall let $[A]_\mu$ denote the μ-equivalence class of a set $A \in \mathscr{A}$, and, for $\mathscr{F} \subseteq \mathscr{A}$, we shall let $[\mathscr{F}]_\mu$ denote the collection of μ-equivalence classes of the members of $\mathscr{F}$.

Let us begin by recounting the definition of a lifting.

2.1. DEFINITION. Let $(X, \mathscr{A}, \mu)$ be a (finite) measure space. A *lifting* for μ is a function $\rho: \mathscr{A} \to \mathscr{A}$ which satisfies the following identities for arbitrary sets $A, B \in \mathscr{A}$:

(i) $\rho(A) \approx_\mu A$.

(ii) If $A \approx_\mu B$, then $\rho(A) = \rho(B)$.

(iii) $\rho(\varnothing) = \varnothing$.

(iv) $\rho(X) = X$.

(v) $\rho(A \cap B) = \rho(A) \cap \rho(B)$.

(vi) $\rho(A \cup B) = \rho(A) \cup \rho(B)$.

If $(X, \mathscr{A}, \mu, \mathscr{T})$ is a topological measure space, a lifting ρ for μ is said to be *strong* if we have $U \subseteq \rho(U)$ for every set $U \in \mathscr{T}$. ■

A more succinct equivalent definition reads as follows: A lifting constitutes a *right inverse* Boolean algebra homomorphism for the canonical homomorphism $A \mapsto [A]_\mu$. There are theorems which assert the existence of liftings and strong liftings under general conditions. With some justice these theorems have been considered to be impenetrable. But they are not. Let it become well known that the totally elementary arguments in [**49**] and in [**29**, §2, pp. 247–252] have defanged this particular tiger.

An early, simple example of von Neumann [**50**, Footnote 1, p. 109] made it clear from the start that a lifting would almost never preserve countable Boolean operations. In the light of this, the following theorem of Maharam about *arbitrary* unions may seem somewhat surprising, but we wish to underscore its complete simplicity.

2.2. THEOREM. *Let $(X, \mathscr{A}, \mu)$ be a complete, finite measure space, and let ρ be a lifting for μ. Then, for every subfamily $\mathscr{F}$ of the range of ρ, we have $\cup\mathscr{F} \in \mathscr{A}$.*

Proof. By property (vi), we may assume without loss of generality that $\mathscr{F}$ is closed under finite unions. Let s be the supremum of the set of numbers $\mu(F)$, for $F \in \mathscr{F}$, and let A be a set of measure s which constitutes the (countable) union of an increasing sequence of sets in $\mathscr{F}$. Then it is clear from the definition of s that for every set $F \in \mathscr{F}$, the set $F \setminus A$ is μ-null. By (i) and (ii), we have $\rho(F) = F$ for every $F \in \mathscr{F}$, and the usual combination of (vi), (iv), (v), and (iii) shows that $\rho(A^c) = \rho(A)^c$. Therefore (v), (ii), and (iii) yield the identities $F \setminus \rho(A) = \rho(F) \setminus \rho(A) = \rho(F \setminus A) = \rho(\varnothing) = \varnothing$. It follows that $A \subseteq \cup\mathscr{F} \subseteq \rho(A)$, and we obtain $\cup\mathscr{F} \in \mathscr{A}$

from this, from (i), and from the completeness of μ. ■

This unprepossessing little theorem originally appeared in [**33**, Theorem 4, p. 993] and has since become the cornerstone of a body of work devoted to "topologies induced by measure." (See [**17**], [**11**], [**41**], [**47**], [**48**], [**23**, Chapter V], and [**36**, Chapter 22]. In particular, the arguments of [**41**, Proposition 1.1, p. 232] can be adapted to yield an interesting alternative proof of Theorem 2.2.) To illustrate how Theorem 2.2 may be used outside of its own area, we present the following special case of a recent result due to Graf [**18**, Theorem 5, p. 348].

2.3. THEOREM. *Let X and Y be compact Hausdorff spaces, and let $\mathscr{A}$ and $\mathscr{B}$ denote the Borel subsets of X and Y, respectively. Let ν be a (finite) Radon measure defined on $\mathscr{B}$, let $\mathscr{B}_\nu$ denote the completion of $\mathscr{B}$ with respect to ν, and assume that there exists a strong lifting $\rho: \mathscr{B}_\nu \to \mathscr{B}_\nu$ for ν. Then every continuous function ψ which maps X onto Y has a $\mathscr{B}_\nu$-$\mathscr{A}$-measurable right inverse θ.*

Proof. We shall first obtain a Radon measure (alias a regular Borel measure) μ on $\mathscr{A}$ such that ν is the image measure of μ under ψ (that is, such that $\nu(B) = \mu(\psi^{-1}(B))$ for every set $B \in \mathscr{B}$.) To this end, let $C(Y)$ be the Banach space of real-valued continuous functions on Y, equipped with the supremum norm. Define $C(X)$ likewise. Notice that, for $f \in C(Y)$, the map $f \mapsto f \circ \psi$ will isometrically embed $C(Y)$ into $C(X)$. The measure ν, which acts on $C(Y)$ by integration, determines a bounded linear functional on $C(Y)$, and hence on a closed subspace of $C(X)$. By the Hahn-Banach theorem, there is a norm-preserving extension of ν to all of $C(X)$, and by the Riesz representation theorem, we may represent this extension by a (necessarily nonnegative) Radon measure μ on $\mathscr{A}$. It is easy to check that ν is the image of μ under ψ.

With μ in hand, we shall next define a Boolean algebra homomorphism $\tau: \mathscr{A} \to \mathscr{B}_\nu$ which will be used to obtain the right inverse θ. For $B \in \mathscr{B}$, the correspondence $[B]_\nu \leftrightarrow [\psi^{-1}(B)]_\mu$ clearly constitutes a σ-algebra isomorphism of $[\mathscr{B}]_\nu$ and the subalgebra $[\psi^{-1}(\mathscr{B})]_\mu$ of $[\mathscr{A}]_\mu$. Define $\eta_0[\psi^{-1}(B)]_\mu = [B]_\nu$. We now invoke the (elementary) theorem of Sikorski [**43**, Theorem 33.1, p. 141] (the "Hahn-Banach theorem" of complete Boolean algebras) to extend η_0 to a Boolean algebra homomorphism η defined on all of $[\mathscr{A}]_\mu$. Finally, treating the lifting ρ as a homomorphism of $[\mathscr{B}]_\nu = [\mathscr{B}_\nu]_\nu$ into $\mathscr{B}_\nu$, we define $\tau(A) = \rho(\eta[A]_\mu)$ for all $A \in \mathscr{A}$.

We are now prepared to obtain the map θ. Given $y \in Y$, let

$$\mathscr{K}_y = \{K \subseteq X: K \text{ compact}; y \in \tau(K)\}.$$

Clearly $X \in \mathscr{K}_y$, and it is easy to check that $\mathscr{K}_y$ has the finite intersection property. Indeed, if $K_1, \ldots, K_n \in \mathscr{K}_y$, then $y \in \cap_{i=1}^n \tau(K_i) = \tau(\cap_{i=1}^n K_i)$, and so $\cap_{i=1}^n K_i$ must be nonnull as well as just nonempty. It follows that $\cap \mathscr{K}_y \neq \varnothing$, and we define $\theta(y)$ to be any point $x \in \cap \mathscr{K}_y$. (In fact there is no choice involved here. An easy argument as in the next paragraph will show that $\cap \mathscr{K}_y$ is always a singleton.)

Both of the desired properties of θ are consequences of one immediately deduced fact:

$$\theta(\tau(K)) \subseteq K \quad \text{for every compact set } K \subseteq X. \tag{2.4}$$

Let us first check the $\mathscr{B}_\nu$-$\mathscr{A}$-measurability of θ. Given $A \in \mathscr{A}$, we are to show that $\theta^{-1}(A) \in \mathscr{B}_\nu$. As usual, it suffices to assume that $A = U$, an open set. Fix any point $y \in \theta^{-1}(U)$, and define $x = \theta(y)$. Let V be an open neighborhood of x such that $\overline{V} \subseteq U$. Apply (2.4) to the compact set V^c to obtain $\theta(\tau(V^c)) \subseteq V^c$. It follows that $y \in \tau(\overline{V})$ (since $Y = \tau(X) = \tau(\overline{V} \cup V^c) = \tau(\overline{V}) \cup \tau(V^c)$). Also apply (2.4) to $\overline{V}$ to obtain $\theta(\tau(\overline{V})) \subseteq \overline{V} \subseteq U$. In this way we recognize $\theta^{-1}(U)$ to be the union of a family of sets (of the form $\tau(\overline{V})$) which belong to the range of the lifting ρ. By Theorem 2.2, $\theta^{-1}(U) \in \mathscr{B}_\nu$.

Finally, we verify the right inverse property: namely, that $\psi(\theta(y)) = y$ for every point $y \in Y$. Let H be a compact neighborhood of y, and define $K = \psi^{-1}(H)$. Since η extends η_0, we have $\tau(K) = \rho(H)$. Since ρ is strong, we also have $y \in \operatorname{int} H \subseteq \rho(\operatorname{int} H) \subseteq \rho(H) = \tau(K)$, and it is therefore clear from (2.4) that $\theta(y) \in K$. But H was arbitrary, and Y is a Hausdorff space, so this is sufficient to establish that $\theta(y) \in \psi^{-1}(\{y\})$, and hence that $\psi(\theta(y)) = y$. ■

Theorem 2.3 provides a typical example of what is called a *measurable selection theorem*. (Thus, for each point $y \in Y$, the map θ measurably *selects* a point x from the compact fibre $\psi^{-1}(\{y\})$.) As it stands, it is already general enough to imply the existence of a measurable cross section in the topological group setting without the traditional assumption that the group be separable [**29**, Theorem 4.4, p. 258]. While the inspirational idea behind the construction of θ appeared fairly early (see [**11**, p. 855] and [**7**, p. 914]), it has only recently been recognized to what a great extent the foregoing arguments may be elaborated. The result of this elaboration appears in [**18**, Theorem 1, p. 343]. Losert [**32**, Theorem 1, p. 154] has pointed out that the existence of the strong lifting ρ is *almost* necessary in order to obtain a measurable selection. (See also [**23**, Theorem 2, p. 169].) Curiously, it is precisely the right inverse property (i) of ρ which is *not* needed, in the proof above and more generally, to obtain the measurable right inverse map θ.

3. Uncountable Measurable Unions of Null Sets. There is one uncountable union of null sets which everyone is familiar with: the union of the null open sets in a Radon measure space. This union is an open set, and it is also a null set. And perhaps the most singular feature of a Radon measure is the utter triviality of this fact. Beyond it lies a dark terrain which has only recently begun to be charted. We shall explore the edges of this terrain in the present section and enjoy the fruits of that exploration in the next.

A family $\mathscr{F}$ of open subsets of X poses no problems of measurability: the union of *every* subfamily of $\mathscr{F}$ is measurable. This statement remains true when $\mathscr{F}$ is a family of inverse images of open sets under a measurable map. Thus many families of this sort are likely to exist. When the elements of $\mathscr{F}$ are both open and null, then $\cup\mathscr{F}$ is null. When the elements of $\mathscr{F}$ are just null, it is perhaps remarkable that anything can be said about $\cup\mathscr{F}$ at all. One might anticipate that $\cup\mathscr{F}$ would be null in the presence of a sufficient amount of overlap among the members of $\mathscr{F}$, but the known results are addressed to precisely the opposite conditions. We shall treat what turns out to be the simplest case: the elements of $\mathscr{F}$ are pairwise disjoint.

3.1. THEOREM. *Let $(X, \mathscr{A}, \mu, \mathscr{T})$ be a finite Radon measure space, and let $\mathscr{F} \subset \mathscr{A}$ be a family of pairwise disjoint μ-null sets such that $\cup\mathscr{E} \in \mathscr{A}$ for every subfamily $\mathscr{E}$ of $\mathscr{F}$. Then $\cup\mathscr{F}$ is μ-null.*

The remainder of the section will be devoted to an elementary exposition of the proof of this theorem, which forms the essential content of [**13**, Lemma 2A, p. 391]. The exposition is based upon ideas in [**27**].

To place the underlying issues into proper focus, let us index the elements of $\mathscr{F}$ by writing $\mathscr{F} = \{N_\xi\}_{\xi \in S}$. Then let us define a (countably additive) "image measure" ν on every subset E of the index set S by the identity

$$\nu(E) = \mu\big(\cup\{N_\xi : \xi \in E\}\big). \tag{3.2}$$

Clearly ν vanishes on finite sets, and we may assume that $\nu(S) > 0$, for there is otherwise nothing to prove. This situation is already curious, and the reader who has a headache may seize the opportunity for a cop out. The existence of ν qualifies the cardinal number of S to be called a *real-valued measurable cardinal*. (See [**45**].) It is consistent with the ordinary (ZFC) axioms of set theory to believe that such objects do not exist. This belief leads to a quick proof of Theorem 3.1. But it is not an attractive proof. One must accept not only an annoying extra axiom but also the concomitant deep mathematical argument which establishes the consistency of that axiom. It is preferable, and in the event more entertaining, to remain agnostic about real-valued measurable cardinals.

So at this point we recognize two possibilities: either ν has an atom, or it does not. (The immediate consequences of these two cases form the content of what is called Ulam's dichotomy [**8**, Theorem 1.3, p. 174].) Let us first treat the latter possibility and review a familiar construction. Partition S into two subsets E_0 and E_1, each of ν-measure $\nu(S)/2$ [**21**, Exercise (2), p. 174]. Partition E_0 into two subsets E_{00} and E_{01}, each of ν-measure $\nu(E_0)/2$, and partition E_1 likewise into E_{10} and E_{11}. Continue in this manner to form an infinite tree, with every set E_γ therein being

partitioned into two subsets $E_{\gamma 0}$ and $E_{\gamma 1}$, each of ν-measure $\nu(E_\gamma)/2$. Let $\delta \in 2^\omega$, that is, let δ be an infinite sequence of 0's and 1's. For $n = 1,2,\ldots$, let δ_n denote the first n entries of δ. (Thus, if $\delta = (1,1,0,1,0,\ldots)$, then $\delta_4 = 1101$.) Define $E_\delta = \cap_{n=1}^{\infty} E_{\delta_n}$. Then $\nu(E_\delta) = 0$ (since $\nu(E_{\delta_n}) = \nu(S)/2^n$ for all n). Moreover, the E_δ form a partition of S. If we now define $A_\delta = \cup\{N_\xi : \xi \in E_\delta\}$, we see that we have obtained a $2^{\aleph_0}$-sized partition of $Y = \cup\mathscr{F}$ into μ-null sets (some of which may be empty). After a short digression, we shall obtain a quick contradiction from this.

We digress to review the notion of a *Bernstein set*, an example of which is obtained in the proposition below. But first we set two counting exercises for the reader.

3.3. *Exercise.* Prove that there are precisely $2^{\aleph_0}$ uncountable compact subsets of the interval $[0,1]$. (See [**30**, Theorem 1, p. 255].) ■

3.4. *Exercise.* Prove that every uncountable compact subset of $[0,1]$ has cardinality $2^{\aleph_0}$. (See [**38**, Exercise (10-12), p. 132] and note also [**30**, Theorem 3, p. 447].) ■

In each case one may pair up sequences $\delta \in 2^\omega$ with the items to be counted by considering an appropriate family of subintervals of $[0,1]$. (The Schröder-Bernstein theorem [**22**, p. 88] is also relevant.) We shall not deprive the reader of the enjoyment of discovering the details for himself. And if he again has a headache, then he is free, in the following proof, to use another extra axiom of set theory in place of Exercises 3.3 and 3.4. (This time it is the continuum hypothesis [**22**, p. 102].)

3.5. PROPOSITION. *There exists a set $B \subset [0,1]$ which has cardinality $2^{\aleph_0}$, and which does not contain any uncountable compact subset of* $[0,1]$.

Proof. Set $\kappa = 2^{\aleph_0}$, and let $\{K_\alpha\}_{\alpha\in\kappa}$ denote the collection of uncountable compact subsets of $[0,1]$. We employ transfinite induction to obtain sequences $\{r_\alpha\}_{\alpha\in\kappa}$ and $\{s_\alpha\}_{\alpha\in\kappa}$ in $[0,1]$ as follows: at the αth stage of the construction, fewer than κ many points r_γ and s_γ (for $\gamma < \alpha$) have been determined. Therefore, since K_α has cardinality κ, it is possible to select a point $s_\alpha \in K_\alpha$ which is distinct from every r_γ, and from every s_γ, for $\gamma < \alpha$. With s_α in hand, we select $r_\alpha \in [0,1]$ likewise, and also such that $r_\alpha \neq s_\alpha$. Define $B = \{r_\alpha : \alpha \in \kappa\}$. Then B clearly has cardinality κ, and K_α is not a subset of B for any α because $s_\alpha \notin B$. ■

Needless to say, the construction of B may be carried out in a more general setting (see [**30**, §40.I, pp. 514–516]). As it stands, the set B can be used to establish the existence of a nonmeasurable subset of *any* nonatomic measurable set in a Radon measure space. We shall essentially be showing this in the next paragraph.

We return to the set $Y = \cup\mathscr{F}$ which we have partitioned into $2^{\aleph_0}$ μ-null sets A_δ. We may now reindex the A_δ by the elements r of the Bernstein set B of Proposition 3.5. Define a function $f: Y \to [0,1]$ by $f(y) = r$, where r is the uniquely determined index such that $y \in A_r$. Then f is measurable; indeed, the f-inverse image of *every* subset of $[0,1]$ constitutes the union of a subfamily of the A_r, and hence of the N_ξ. Therefore:

(3.6)

By condition (1.6), there exists a nonnull compact subset K of Y such that $f|_K$ is continuous.

It follows that $f(K)$ is a compact subset of $[0,1]$. It is also a subset of B. Therefore it is countable. Therefore K is a countable union of μ-null sets. Therefore K is null. This is a contradiction. *Voilà*! (A similar use of the Bernstein set is to be found in [**4**].)

The second possibility to consider is that the measure ν defined by (3.2) does have an atom $T \subseteq S$. This situation is again curious. The existence of ν on the set T qualifies the cardinal number of T to be called a *(zero-one-)measurable cardinal*. It is again consistent with the ordinary axioms of set theory to disbelieve in such objects, and so to shorten the proof of Theorem 3.1. It is again possible to avoid the extra axiom.

The pattern of the previous discussion will be repeated. First we review a familiar construction,

then we digress to cover a well-known consequence of that construction, and finally we exploit the inner regularity of μ to obtain a straightforward contradiction to the assumption that $\nu(S) = \mu(\cup\mathscr{F}) > 0$.

Here is the familiar construction. Define κ to be the smallest cardinal number for which there exists a family $\mathscr{E}$ of ν-null subsets of T such that $\mathscr{E}$ has cardinality κ, and such that $\nu(\cup\mathscr{E}) > 0$. Since all subsets of S are ν-measurable, we may well order $\mathscr{E}$ and disjointize it in *exactly* the way we would do if $\mathscr{E}$ were countable [**21**, Theorem B, p. 37]. Thus we may assume without loss of generality that the sets in $\mathscr{E}$ are pairwise disjoint. Again let us index the elements of $\mathscr{E}$ by writing $\mathscr{E} = \{M_\xi\}_{\xi\in R}$, and let us define an "image measure" ϕ exactly as ν was defined in (3.2). (Thus, the ϕ-measure of a set $D \subseteq R$ is precisely the ν-measure of its "preimage" $\cup\{M_\xi : \xi \in D\}$.) Note first that:

(3.7) The cardinality of R is $\kappa > \aleph_0$.

For otherwise $\mathscr{E}$ would be countable, and then $\cup\mathscr{E}$ would have to be ν-null. Define $\mathscr{U} = \{U \subseteq R : \phi(U) > 0\}$. Then:

(3.8) The cardinality of every set $U \in \mathscr{U}$ is κ.

For otherwise the family $\{M_\xi\}_{\xi\in U}$ would have fewer than κ members, but its union would be ν-nonnull, contrary to the minimality of κ. The principal property of $\mathscr{U}$ is the following:

(3.9) If $\mathscr{V}$ is a subfamily of $\mathscr{U}$ which has cardinality $< \kappa$ (and in particular if $\mathscr{V}$ is a countable subfamily of $\mathscr{U}$), then $\cap\mathscr{V} \in \mathscr{U}$.

For otherwise the "preimages" of the complements of the members of $\mathscr{V}$ would form a collection of fewer than κ many ν-null subsets of T, but the union of this collection would be ν-nonnull, contrary to the minimality of κ. (This property qualifies the measure ϕ to be called *κ-additive*.) Since T, and hence $\cup\mathscr{E}$, are ν-atoms, it follows that R is a ϕ-atom. Therefore:

(3.10) $\mathscr{U}$ is an ultrafilter.

After a short digression, we shall obtain our contradiction from facts (3.7) to (3.10).

We digress for an excursion into *Ramsey theory*. Ramsey-type theorems and related combinatorial results hold a well established place within set theory because of their applications both in set theory and in model theory (see [**44**] and [**5**, §3.3, pp. 141–155]). They have therefore received extensive treatment in the literature (see, for example, [**28**] and [**19**]). Their gradual ingress into analysis began a few years ago and is today gaining momentum (see [**1**], [**35**], and [**9**]). The particular Ramsey-type theorem which we shall need was first obtained by Erdős and Hajnal [**10**, Theorem 9a, pp. 125-126] (see also [**44**, p. 58]). For this theorem, and for the remainder of the section, we introduce the following notation: Let $|R|$ denote the cardinality of R, let $\mathscr{P}_f(R)$ denote the collection of all finite subsets of R, and, for $m = 1, 2, \ldots$, let $\mathscr{P}_m(R)$ denote the collection of all subsets of R of size m.

3.11. THEOREM. *Let R be a set, and let $\mathscr{U}$ be a nonprincipal ultrafilter of subsets of R (that is, $\mathscr{U}$ does not contain any finite sets). Let κ be an infinite cardinal number, and assume that $\mathscr{U}$ is closed under the intersection of any subfamily which has fewer than κ elements. Let C be a set of cardinality $< \kappa$, and let $\Phi : \mathscr{P}_f(R) \to C$ be an arbitrary function. Then, for each $m = 1, 2, \ldots$, there exists a set $Q = Q_m \subseteq R$ such that $|Q| = \kappa$, and such that the restriction of Φ to $\mathscr{P}_m(Q)$ is constant. Moreover (and this is most important), if $\kappa > \aleph_0$, then the set Q may be chosen independently of m.*

Note. It is a common practice to "humanize" this type of theorem in some such manner as the following: let the set C be likened to a palette of colors. The function Φ assigns a color (in C) to each finite subset of R. The choice of color can be arbitrary. There is then a κ-sized set Q all of whose m-sized subsets receive the same color. Moreover, if $\kappa > \aleph_0$, then Q may be chosen so that this fact holds simultaneously for all m, although the actual color associated with $\mathscr{P}_m(Q)$ may

vary with m. (Our application of this result will require only two "colors.")

Proof. The present exposition will aim as far as possible to avoid an impenetrable notation. Thus our strategy will be to set out the proof of a special case of the original "infinite" Ramsey theorem [**39**, Theorem A, p. 264] in such a way as to make the general result clear with little or no additional notation.

To this end, let $D = R$ be a denumerable set, let $F = C$ be a finite set, let $\mathcal{U}$ be any nonprincipal ultrafilter of subsets of D, and let $\Phi : \mathcal{P}_3(D) \to F$. (It does not really simplify matters to assume that $|F| = 2$.) Let us consider in this case the following schema of subsets of D. First: For $i \in F$, and for $x, y \in D$ such that $x \neq y$, define

$$A_i(x, y) = \{z \in D : z \neq x;\ z \neq y;\ \Phi(\{x, y, z\}) = i\}.$$

Note that, with x and y fixed, we have $\cup_{i \in F} A_i(x, y) = D \setminus \{x, y\}$. By the ultrafilter property of $\mathcal{U}$, and also by the fact that $\mathcal{U}$ is nonprincipal, we obtain $A_i(x, y) \in \mathcal{U}$ for precisely one point $i \in F$. Next: For $i \in F$, and for $x \in D$, define

$$A_i(x) = \{y \in D : y \neq x;\ A_i(x, y) \in \mathcal{U}\}.$$

Note, then, that $\cup_{i \in F} A_i(x) = D \setminus \{x\}$, so that, with x fixed, we obtain as before that $A_i(x) \in \mathcal{U}$ for precisely one point $i \in F$. Finally: For $i \in F$, define

$$A_i = \{x \in D : A_i(x) \in \mathcal{U}\}.$$

Then $\cup_{i \in F} A_i = D$, and we now *fix* the unique point $i \in F$ such that $A_i \in \mathcal{U}$. *This* point is the chosen "color."

We complete the proof of this special case by using induction to construct an infinite subset Q of D such that every 3-element subset of Q has the "color" i. To this end, let $x_1 \in A_i$, and notice that this containment qualifies $A_i(x_1)$ to belong to $\mathcal{U}$. It follows that $A_i \cap A_i(x_1) \in \mathcal{U}$, and we are therefore free to let $x_2 \in A_i \cap A_i(x_1)$. Notice that the choice of $x_2 \in A_i(x_1)$ will ensure that x_2 is distinct from x_1. Now comes the key step. For $n \geqslant 3$, let

$$x_n \in A_i \cap \left[\cap_{1 \leqslant j < n} A_i(x_j)\right] \cap \left[\cap_{1 \leqslant j < k < n} A_i(x_j, x_k)\right]. \tag{3.12}$$

To see clearly what is happening in this step will be to understand the proof of the entire theorem, as we hope to show. The critical point to verify is that each of the sets in the intersection above belongs to $\mathcal{U}$. Fix $1 \leqslant j < n$. By the obvious (necessary) induction assumption, we would already have obtained the point $x_j \in A_i$, so that, as above, $A_i(x_j) \in \mathcal{U}$. And then the choice of $x_n \in A_i(x_j)$ will ensure that x_n is distinct from x_j. Now fix $1 \leqslant j < k < n$. By induction, we would already have obtained the point $x_k \in A_i(x_j)$, so that $A_i(x_j, x_k) \in \mathcal{U}$. And then the choice of $x_n \in A_i(x_j, x_k)$ will ensure that $\Phi(\{x_j, x_k, x_n\}) = i$. This takes care of everything. The *finite* intersection in (3.12) belongs to $\mathcal{U}$; thus, a point x_n may be chosen from that intersection; the choice of x_n guarantees that the induction procedure may be continued; and the resulting set $Q = \{x_n : n \geqslant 1\}$ has the desired properties.

At this juncture the reader is invited to study the foregoing proof until it becomes clear, *without further notation*, that an analogous schema of sets can be defined for any $m = 1, 2, \ldots$, and that the classical result can thereby be obtained for a function $\Phi : \mathcal{P}_m(D) \to F$. (The case $m = 1$ is trivial.) For example, in the case $m = 4$, one would define a schema of sets of the form $B_i(x, y, z)$, $B_i(x, y)$, $B_i(x)$, and B_i. The "chosen color" is the unique element $j \in F$ such that $B_j \in \mathcal{U}$. We then let $x_1 \in B_j$, we let

$$x_2 \in B_j \cap B_j(x_1),$$

we let

$$x_3 \in B_j \cap B_j(x_1) \cap B_j(x_2) \cap B_j(x_1, x_2),$$

and, for $n \geqslant 4 = m$, we determine x_n by analogy with (3.12).

One important observation remains: the various schemata can be combined to yield a single construction for more than one value of m. Thus, for example, if $\Phi : \mathscr{P}_3(D) \cup \mathscr{P}_4(D) \to F$, then we can let

$$x_1 \in A_i \cap B_j,$$

we can let

$$x_2 \in A_i \cap A_i(x_1) \cap B_j \cap B_j(x_1),$$

and so on, thereby obtaining an infinite set $Q \subseteq D$ such that *simultaneously* $\Phi(\mathscr{P}_3(Q)) = \{i\}$ and $\Phi(\mathscr{P}_4(Q)) = \{j\}$. The only thing which prevents us (in this line of attack) from obtaining a set Q which would "work" simultaneously for all $m = 1, 2, \ldots$ is the fact that, when $\kappa = \aleph_0$, the ultrafilter $\mathscr{U}$ may not be closed under countable intersections.

And this brings us to the key point of the proof of the general case. It is nothing more and nothing less than the *closure properties* of $\mathscr{U}$ which will permit the extension of the foregoing arguments. (If we were lecturing, we would be shouting.) It remains only to do some counting. Thus, for example, if we had a (disjoint) union $R = \cup_{i \in C} A_i$, then we should like to conclude, as we did before, that $A_i \in \mathscr{U}$ for precisely one point $i \in C$. But this follows from the fact that the sets A_i will be fewer than κ in number (since $|C| < \kappa$), and from the fact that $\mathscr{U}$ is closed under the intersection of fewer than κ of its elements. The set Q is constructed as a transfinite sequence $\{x_\alpha\}_{\alpha \in \kappa}$ of (distinct) elements of R. At the αth stage of the construction, the analogue of (3.12) comprises the intersection of fewer than κ of the elements of $\mathscr{U}$. So it belongs to $\mathscr{U}$. And when $\kappa > \aleph_0$, we can take a countable intersection of the analogues of (3.12) and still have an intersection of fewer than κ elements of $\mathscr{U}$. ■

It is well known that the original Ramsey theorem is amenable to an ultrafilter approach (see [5, Theorem 3.3.7, pp. 145–147]).

We return for the final assault upon Theorem 3.1. Recall the image measure ν of (3.2) and the fact, arising from our first contradiction, that ν must have an atom. This led to the cardinal number κ, the index set R, and the further image measure ϕ, all defined just prior to (3.7). Notice now that properties (3.7) to (3.10) make Theorem 3.11 applicable to R. For $\zeta \in R$, define the "pre-preimage" $A_\zeta = \cup\{N_\xi : \xi \in M_\zeta\}$. It is easily checked from the definitions of ν and ϕ that $\mu(A_\zeta) = 0$, and from the minimality of κ that the union of fewer than κ of the A_ζ is also (measurable and) μ-null, whereas the union of all of them is μ-nonnull. We may now forget about ν and ϕ.

Since we have $|R| = \kappa$ by (3.7), we may reindex the A_ζ by the elements α of the *set* κ (which we regard as an initial ordinal). Define $Z = \cup\{A_\alpha : \alpha \in \kappa\}$, and, for $\alpha \in \kappa$, define $Z_\alpha = \cup\{A_\gamma : \gamma \in \kappa;\ \gamma \leqslant \alpha\}$. Then Z_α is μ-null, whereas $Z \setminus Z_\alpha$ is not. By the inner regularity of μ, there exists a nonnull compact subset K_α of $Z \setminus Z_\alpha$. Notice at once that if the family $\mathscr{K} = \{K_\alpha : \alpha \in \kappa\}$ had the finite intersection property, then it would have nonempty intersection. A point $x \in \cap\mathscr{K}$ would have to belong to Z, but then it could not belong to Z_α for any α, and therefore it could *not* belong to Z. Thus a contradiction is achieved.

We shall now use Theorem 3.11, as it applies to κ, to turn that contradiction into a reality. To this end, define a function $\Phi : \mathscr{P}_f(\kappa) \to \{0, 1\}$ by declaring, for $F \in \mathscr{P}_f(\kappa)$, that $\Phi(F) = 1$ if $\cap\{K_\alpha : \alpha \in F\} \neq \varnothing$, and that $\Phi(F) = 0$ otherwise. Since $\kappa > \aleph_0$ by (3.7), Theorem 3.11 implies the existence of a set $Q \subseteq \kappa$ such that $|Q| = \kappa$, and such that the restriction of Φ to $\mathscr{P}_m(Q)$ is constant for each integer $m = 1, 2, \ldots$. To discover what those constant values are, we set for the reader the following entertaining exercise in elementary measure theory.

3.13. *Exercise.* Let $(X, \mathscr{A}, \mu)$ be a finite measure space, and let $\mathscr{S} \subseteq \mathscr{A}$ be an uncountable family of μ-nonnull sets. Prove that, for every integer $m = 1, 2, \ldots$, there exists at least one point $x \in X$ which belongs to m or more of the sets in $\mathscr{S}$. (Hint: Add up characteristic functions and integrate.) ■

We apply this fact to the family $\mathscr{S} = \{K_\alpha : \alpha \in Q\}$ to deduce at once that $\Phi(\mathscr{P}_m(Q)) = \{1\}$ for all m. Thus $\mathscr{S}$ (at least) has the finite intersection property. But since $|Q| = \kappa$, the set Q must be *unbounded* in κ. And since we have $Z_\alpha \subseteq Z_\beta$ whenever $\alpha \leqslant \beta$, it follows clearly that any point $x \in \cap\mathscr{S}$ must simultaneously belong to Z and *not* belong to Z. This is the final contradiction. It finishes the proof of Theorem 3.1, and it finishes the section.

4. Fremlin's Theorem and Its Significance. The set theory of the last section is for the most part classical and could easily find a place as part of a first course in set theory for analysts. But there is a lot of it. And we suspect that most has not yet found its way into the armamentarium of the working analyst. To justify its inclusion, we need an application whose significance will match the effort expended in mastering it. We are pinning our hopes upon the following result. It essentially amounts to a recent theorem of Fremlin [**13**, Theorem 2B, p. 393], who is also responsible for the principal result of Section 3. Fremlin's treatment exploits much of the material in Section 3. Where we use the Ramsey-type theorem (Theorem 3.11), he requires the lifting theorem, the Maharam characterization of homogeneous measure algebras, an ancillary construction, and the Choksi-Fremlin theorem on completion regularity.

4.1. THEOREM. *Let $(X, \mathscr{A}, \mu, \mathscr{T})$ be a finite Radon measure space, let Y be an arbitrary metric space, let $\mathscr{B}$ denote the Borel subsets of Y, and let $\psi : X \to Y$ be an $\mathscr{A}$-$\mathscr{B}$-measurable function. Then there exists a (closed) separable subspace Z of Y such that $\psi(x) \in Z$ for μ-almost all points $x \in X$.*

Proof. Here is the key argument. Let $\mathscr{V}$ be a collection of pairwise disjoint open subsets of Y. At most countably many of the ψ-inverse images of the elements of $\mathscr{V}$ are nonnull. (This is a corollary of Exercise 3.13.) The remainder satisfy the hypotheses of Theorem 3.1, and so their union is null.

To turn this argument to good account, we need to invoke the following elementary property of a metric space (see [**25**, Theorem 21, p. 129]): Every open cover $\mathscr{U}$ of Y has an open refinement [**25**, p. 128] of the form $\mathscr{V} = \cup_{j=1}^{\infty}\mathscr{V}_j$, where, for each index j, the (open) sets in $\mathscr{V}_j$ are pairwise disjoint. With note of the fact that $\cup\mathscr{V} = Y$, we apply the key argument to each $\mathscr{V}_j$ separately to produce, altogether, a countable subfamily $\mathscr{G}$ of $\mathscr{V}$ such that $\psi(x) \in \cup\mathscr{G}$ for μ-almost all points $x \in X$.

We are now ready to obtain the separable subspace Z. For each integer $n = 1, 2, \ldots$, let $\mathscr{U}_n$ comprise the open subsets of Y which have diameter $< 1/n$. From a suitable refinement of $\mathscr{U}_n$, select a countable subfamily $\mathscr{G}_n$, as in the last paragraph, such that $\psi(x) \in G_n = \cup\mathscr{G}_n$ for μ-almost all points $x \in X$. Note from the definition of refinement that the (open) sets in $\mathscr{G}_n$ all have diameter $< 1/n$. Define Z to be (the closure of) $\cap_{n=1}^{\infty}G_n$. Then it is clear from the countable additivity of μ that $\psi(x) \in Z$ for μ-almost all points $x \in X$. It is also clear that Z is separable. A countable dense subset of Z may be obtained, for example, by picking a point in $G \cap Z$ for each set $G \in \cup_{n=1}^{\infty}\mathscr{G}_n$ whose intersection with Z is nonempty. ■

While Theorem 4.1 is new, the proof above is old. For example, in [**2**, Theorem 2, pp. 235–236], precisely this proof is used to show that, if real-valued measurable cardinals do not exist, then every nonnegative finite measure ν defined on $\mathscr{B}$ is supported on a (closed) separable subspace Z of Y. (And it is clear in this case that Theorem 4.1 would be valid for an *arbitrary* finite measure space $(X, \mathscr{A}, \mu)$.) What is new, and striking, in Theorem 4.1 is that its proof is now valid as an ordinary result in mathematics. It may be used and enjoyed by measurable-cardinal agnostics.

As with Theorem 2.3, we have only provided a taste of what can be achieved with these methods. Some elaboration already appears in Fremlin's original paper [**13**, §3, pp. 396–404], and it has very recently come to light that further elaborations will yield a new line of attack upon measurable selection theorems. (The details are to be found in [**14**] and [**27**].)

We turn to a discussion of the significance of Theorem 4.1 as it stands. Let B be a Banach space, and let $\mathscr{B}$ denote the Borel subsets of B. It is a fundamental fact of the elementary theory of Bochner integration [**6**, Appendix E, pp. 350–357] that an $\mathscr{A}$-$\mathscr{B}$-measurable function $f : X \to B$

cannot be approximated a.e. by simple $\mathscr{A}$-measurable functions, and therefore has no chance of being Bochner integrable, unless f essentially has separable range (that is, unless f satisfies the conclusion of Theorem 4.1) [**6**, Proposition E.2, p. 351]. It is moreover possible, without the separable range condition, for even the *sum* of two $\mathscr{A}$-$\mathscr{B}$-measurable functions to fail to be $\mathscr{A}$-$\mathscr{B}$-measurable (see [**46**]). Fremlin's theorem renders these problems nonexistent, at least when the given measure μ is Radon. (And it is now known that a considerably larger class of measure spaces will satisfy the conclusion of Theorem 3.1, and hence of Theorem 4.1.)

We remark in this connection that when the range of f is separable, a classical theorem of Pettis [**6**, Theorem E.9, pp. 354–355] asserts that the $\mathscr{A}$-$\mathscr{B}$-measurability of f (otherwise known as *strong* measurability) is equivalent to the $\mathscr{A}$-measurability of $\phi \circ f$ for every continuous linear functional ϕ on B (otherwise known as *weak* or *scalar* measurability). It is worth recounting the very well-known example which illustrates the failure of this equivalence when the range of f is not (essentially) separable, for it demonstrates how badly Theorem 4.1 can fail when the topology on the space Y (in this case the weak topology on B) is not metric.

4.2. EXAMPLE. Let the Radon measure be Lebesgue measure on $X = [0,1]$, let B be a Hilbert space of orthogonal dimension $\geqslant 2^{\aleph_0}$, and let $f: [0,1] \to B$ be any one-to-one function whose range constitutes an orthonormal subset of B. Then f is weakly measurable because all but countably many of the Fourier coefficients of each linear functional $\phi \in B^* = B$ vanish. But the range of f is far from being essentially separable, and so f could not be strongly measurable. ■

Perhaps the most important consequence of Theorem 4.1 is the fact that the map ψ will then be *Lusin measurable* (that is, every nonnull measurable subset of X will contain a nonnull compact subset K such that $\psi|_K$ is continuous). This follows by a straightforward adaptation of the proof of the classical Lusin theorem [**6**, Theorem 7.4.3, pp. 227–228]. We do not seriously believe that many students nowadays are left with impression (1.4). If there are nevertheless any doubts remaining about the immense value of the Lusin property, let us now attempt to erase them.

Imagine that you are slogging through a paper of N. J. Kalton on the endomorphisms of L_p [**24**]. The Lusin theorem is used often and is explicitly stated [**24**, Theorem 2.4, p. 356]. On p. 368 you encounter a compact metric space K with its Borel sets $\mathscr{B}$, a probability measure λ defined on $\mathscr{B}$, a $\mathscr{B}$-$\mathscr{B}$-measurable function $\sigma: K \to K$, and a nonnull Borel set $C \subseteq K$ such that σ is one-to-one on C. In order that the important conclusion of Lemma 5.1(c) on that page be conceivable, it is necessary, for the suitable choice of a nonnull compact subset E of C, that the inverse function σ^{-1} be measurable on the image $\sigma(E)$. The Lusin measurability of σ makes this requirement both possible and easy.

Suppose that you have (even just) a compact Hausdorff group G with its Borel sets $\mathscr{B}$ and Haar measure μ. For a given $\mathscr{B}$-$\mathscr{B}$-measurable function $\theta: G \to G$, you wish to make use of the map $T(x) = x\theta(x)$, defined for all $x \in G$. (Maps such as this are needed, for example, in [**20**, p. 92].) Its deceptive simplicity notwithstanding, it seems to be unknown whether or not T is $\mathscr{B}$-$\mathscr{B}$-measurable. But if θ is Lusin measurable, then T is *trivially* also Lusin measurable, and therefore at least $\mathscr{B}_\mu$-$\mathscr{B}$-measurable (where $\mathscr{B}_\mu$ denotes the completion of $\mathscr{B}$ with respect to μ).

It can fairly be said that the generality of the important results in both [**20**] and [**24**] is limited nontrivially by the (apparent) need for the attendant measurable mappings to be Lusin measurable. (And let us not forget how we ourselves took advantage of the Lusin criterion following statement (3.6).)

Apart from its utility in a variety of technical arguments, Lusin measurability serves to preclude what Laurent Schwartz has called the *image measure catastrophe* [**42**, pp. 30–31]. To describe this, let $(X, \mathscr{A}, \mu)$ be a complete, finite measure space, let Y be a set, and let $f: X \to Y$ be a function. Then f determines a natural image σ-algebra $\mathscr{A}_f$ of subsets of Y which is defined by

$$\mathscr{A}_f = \{E \subseteq Y : f^{-1}(E) \in \mathscr{A}\}.$$

On $\mathscr{A}_f$ there is defined a natural image measure $\mu_f: E \mapsto \mu(f^{-1}(E))$. This is not catastrophic. But it usually is the case that Y is already equipped with a natural σ-algebra $\mathscr{B} \subseteq \mathscr{A}_f$ of its own. We

may wish (or we may only be able) to consider the restriction of μ_f to $\mathscr{B}$. (This situation occurs, for example, in the proof of Theorem 2.3.) Even if we form the completion, $\mathscr{B}_f$, of $\mathscr{B}$ with respect to this restriction, it can still happen that $\mathscr{B}_f$ is a proper subset of $\mathscr{A}_f$. We are then, in effect, artificially declaring certain sets in $\mathscr{A}_f$ to be "nonmeasurable." This is potentially catastrophic if μ_f is being employed to gain information about f or about μ.

So far as we know, these considerations are important only when Y is a topological space, and when $\mathscr{B}$ constitutes the Borel subsets of Y. The classical examples arise in probability theory. Therefore, let μ be a probability measure, let $Y = \mathbb{R}$, and let $f: X \to \mathbb{R}$ be an $\mathscr{A}$-measurable function (that is, a random variable). It is a universal practice to study f by means of its *distribution function* F (defined by $F(r) = \mu_f((-\infty, r])$ for all $r \in \mathbb{R}$) [**31**, §10.2, pp. 170–172]. When F is known, it determines μ_f, as a Lebesgue-Stieltjes measure, precisely on $\mathscr{B}_f$. Now suppose, for example, that you were to establish the independence of two random variables f and g by verifying the familiar identity for their distribution functions. Then you may certainly conclude that the σ-subalgebras $f^{-1}(\mathscr{B}_f)$ and $g^{-1}(\mathscr{B}_g)$ of $\mathscr{A}$ are independent [**31**, Theorem B, p. 225]. But you lay yourself open to a catastrophe if you come to imagine that the σ-subalgebras $f^{-1}(\mathscr{A}_f)$ and $g^{-1}(\mathscr{A}_g)$ of $\mathscr{A}$ are independent, for this is simply not true in general. A few more of the "catastrophic" effects of the strict containment of $\mathscr{B}_f$ within $\mathscr{A}_f$ are discussed in [**3**, §1, pp. 1–2] and in [**16**, pp. 248–251].

Let us now see that the Lusin measurability of f will eliminate the image measure catastrophe.

4.3. PROPOSITION. *Let $(X, \mathscr{A}, \mu, \mathscr{T})$ be a complete, finite, Radon measure space, let Y be a Hausdorff space, let $\mathscr{B}$ denote the Borel subsets of Y, and let $f: X \to Y$ be a Lusin measurable function. Then $\mathscr{A}_f = \mathscr{B}_f$.*

Proof. Since μ is complete, the Lusin measurability of f entails its $\mathscr{A}$-$\mathscr{B}$-measurability, and so $\mathscr{B}_f \subseteq \mathscr{A}_f$. Conversely, let $E \in \mathscr{A}_f$, and let $A = f^{-1}(E) \in \mathscr{A}$. By a standard exhaustion argument (or just by Zorn's lemma), there exists a sequence $\{K_n\}_{n=1}^{\infty}$ of compact subsets of A such that the set $N = A \setminus \cup_{n=1}^{\infty} K_n$ is μ-null, and such that $f|_{K_n}$ is continuous for each n. Since Y is Hausdorff, the set $B = \cup_{n=1}^{\infty} f(K_n)$ is a Borel subset of E, and since we also have $f^{-1}(E \setminus B) \subseteq N$, it follows that $E \in \mathscr{B}_f$. ■

In [**16**, §3, pp. 18–19], Gnedenko and Kolmogorov introduced a class of probability measure spaces wherein the image measure catastrophe for random variables is avoided by fiat. Thus, a (probability) measure space $(X, \mathscr{A}, \mu)$ is said to be *perfect* if, for every $\mathscr{A}$-measurable function $f: X \to \mathbb{R}$, we have $\mathscr{A}_f = \mathscr{B}_f$ (where $\mathscr{B}$ denotes the Borel subsets of $\mathbb{R}$). In view of [**40**, Theorem 1, p. 233], a perfect measure may with some justice be regarded as the abstract analogue of a Radon measure. In terms of its restriction of generality, the assumption that a measure be perfect compares roughtly to the assumption that a topological space be completely regular. Recently some astonishing discoveries about perfect measures have been made (see, for example, [**12**]), and research into perfect measure spaces continues to be very active (see [**40**], [**12**], [**37**], [**34**], [**26**], and [**15**]).

At this point we seem to have strayed a considerable distance from our original theme: uncountable measurable unions. But they are there all the same. The maximal control which was exerted, in Theorem 3.1, over the uncountable unions of certain families of null sets led, in Theorem 4.1, to the demonstration that a substantial class of measurable mappings will have essentially separable range, and will therefore be Lusin measurable. These properties in turn are useful, if not vital, to a number of technical arguments; they eliminate certain intractable, and irritating, measurability problems in Banach spaces and topological groups; and they rescue us from the dire sequelae of the image measure catastrophe.

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A PERIODICITY PROBLEM IN PLANE GEOMETRY

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Given a triangle Z with vertices z_0, z_1, z_2 and given a number r in the interval $(0,1)$, define a new triangle $Z' = T_r(Z)$ with vertices z_0', z_1', z_2', where

$$z_0' = (1 - r)z_0 + rz_1,$$
$$z_1' = (1 - r)z_1 + rz_2,$$
$$z_2' = (1 - r)z_2 + rz_0.$$

Then Z' is formed by joining consecutive r-division points of the sides of Z. That is, each vertex of Z' lies on a side of Z and divides it in the ratio $r : (1 - r)$. (See Fig. 1.)

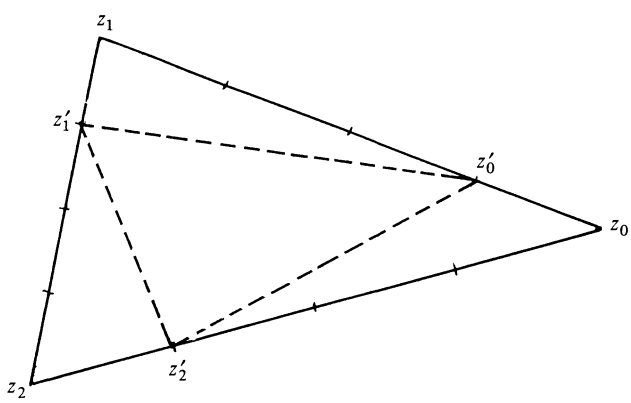


FIG. 1. ($r = 1/4$).

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For example, when $r = \frac{1}{2}$, then Z' is the midpoint triangle, and it is easy to see that Z' is similar to Z. In fact Z' is $\frac{1}{2}$ the size of Z, and the sides of Z' are parallel to the sides of Z.

The case $r = \frac{1}{3}$ also leads to a similar triangle, provided the operation is repeated. For a triangle Z let $Z' = T_{1/3}(Z)$ and $Z'' = T_{1/3}(Z')$. Then Z'' is $\frac{1}{3}$ the size of Z and the sides of Z'' are parallel to the sides of Z (see Fig. 2a). This property can be verified by considering a suitable underlying triangular grid, as in Fig. 2b.

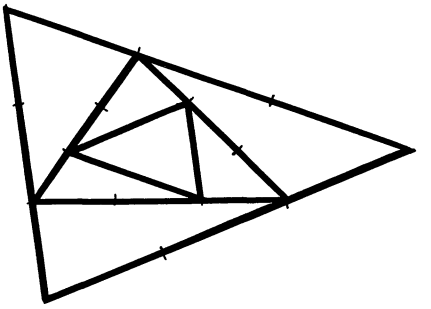


FIG. 2a.

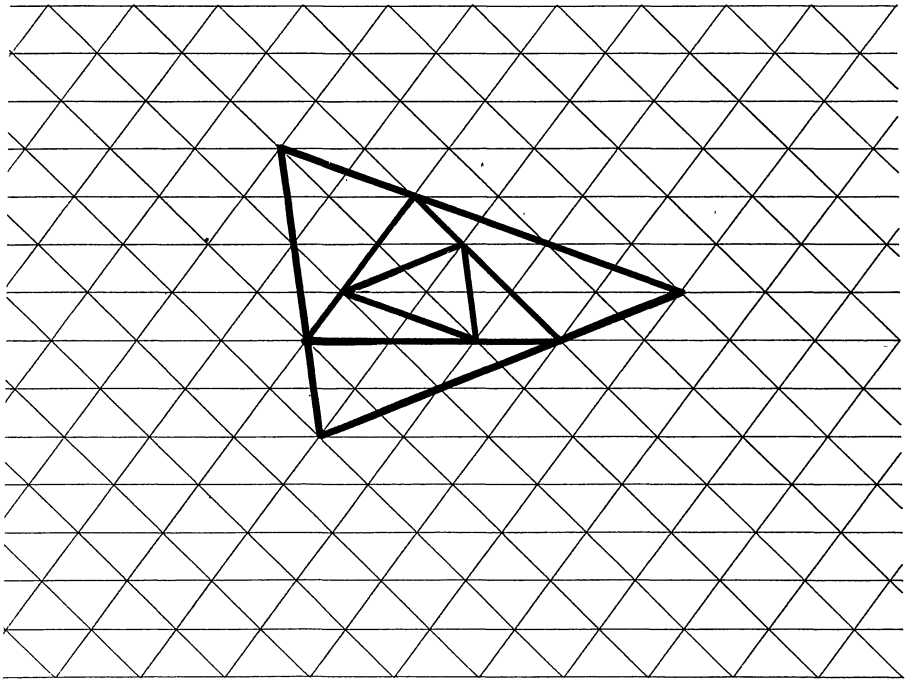


FIG. 2b.

Let us say that the ratio r has *period* m if, for any triangle Z, the mth derived triangle $T_r^m(Z)$ is similar to Z. Then $\frac{1}{2}$ has period 1, and $\frac{1}{3}$ has period 2. Also $\frac{2}{3}$ has period 2, by symmetry.

Question 1. Given $m \geqslant 1$, are there values $r \in (0,1)$ having minimal period m?

Question 2. Which *rational* numbers $r \in (0,1)$ have finite period?

This article provides a complete answer to both of these questions. We consider, more generally, the behavior of the r-division point operations on n-gons (polygons with n vertices). If $n \geqslant 4$, then T_r cannot be periodic on all n-gons. However, it might be periodic or "eventually

periodic" on a particular n-gon Z. This eventual periodicity on Z means roughly that for some $m > 1$, the sequence of derived n-gons $T_r^{mj}(Z)$ approaches a limiting shape as j approaches infinity. The eventual periodicity of the operator T_r on n-gons can be investigated empirically by displaying the derived n-gons on a computer screen, after blowing up each image to a standard size. Such observations were the motivation for investigating the two questions above. Interested readers may contact the author for more information about the Applesoft Basic program used to display these polygons.

For example, the midpoint operation $T_{1/2}$ is eventually periodic with period 2 on every n-gon. This fact has been observed independently by a number of authors, as noted in the appendix of [**FRS**]. The major goal of this article is to determine all the rational numbers r for which T_r is periodic or eventually periodic on n-gons. It turns out that all the cases have been mentioned above.

MAIN THEOREM. *Let $n \geq 3$ be an integer and r a rational number with $0 < r < 1$. The following statements are equivalent.*

(i) *The operator T_r is eventually periodic on almost all n-gons.*
(ii) *The operator T_r is periodic on some nondegenerate, nonregular n-gon Z.*
(iii) *Either $r = \frac{1}{2}$, or $n = 3$ and $r = \frac{1}{3}$ or $\frac{2}{3}$.*

Many results concerning transformations like T_r on n-gons have appeared and reappeared in the literature. References include [**K**], [**Ne**], [**S**], [**BGS**], [**Ca**], [**BS**], [**D1**], [**D2**], [**D3**], [**Cl**], [**F**], [**Wo**], [**FRS**]. For the convenience of the reader, several well-known results are stated and proved in Section 1 and 2 below. There is an especially large overlap with the excellent paper [**BGS**].

1. Polygons and Circulant Matrices. Let $n \geq 3$ be a fixed integer. An *oriented n-gon* $\mathscr{Z}$ in the plane is specified by its n vertices $z_0, z_1, \ldots, z_{n-1}$ connected in order: z_0 to z_1 to z_2 to $\ldots$ to z_{n-1} to z_0. Viewing these points as elements of the complex plane $\mathbb{C}$, we can represent $\mathscr{Z}$ as a column vector

$$Z = \begin{pmatrix} z_0 \\ z_1 \\ \vdots \\ z_{n-1} \end{pmatrix} \text{ in } \mathbb{C}^n.$$

Since the geometric n-gon $\mathscr{Z}$ does not prescribe which vertex should be listed first, there are other vectors in $\mathbb{C}^n$ representing $\mathscr{Z}$. These are obtained by cyclically permuting the entries of the vector Z. These permutations are derived from Z by multiplications with the shift operator Π, having matrix

$$\Pi = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 1 & 0 & 0 & \ldots & 0 \end{pmatrix}.$$

So the vectors representing $\mathscr{Z}$ are $Z, \Pi Z, \Pi^2 Z, \ldots, \Pi^{n-1} Z$.

Some geometric transformations of an n-gon $\mathscr{Z}$ can be realized as linear maps of a representative vector Z. An $n \times n$ matrix C provides such a transformation if $C\Pi = \Pi C$, for then the n-gon $\mathscr{Z}'$ with representative CZ is independent of the choice of the representative Z for $\mathscr{Z}$. One can prove that if $C\Pi = \Pi C$, then C can be expressed as a polynomial in Π, that is, $C = c_0 I + c_1 \Pi + c_2 \Pi^2 + \cdots + c_{n-1} \Pi^{n-1}$, for some $c_i \in \mathbb{C}$. (This can be seen directly, or by using the fact that $\mathbb{C}^n$ is a cyclic $\mathbb{C}[\Pi]$ module.) Such a matrix C is called a *circulant* matrix and looks like

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ c_2 & c_3 & c_4 & \cdots & c_1 \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}.$$

In this case we write $C = \operatorname{circ}(c_0, c_1, \ldots, c_{n-1})$ and define the *associated polynomial* to be $p(x) = c_0 + c_1 x + \cdots + c_{n-1}x^{n-1}$.

For example the midpoint operation on n-gons is given by the matrix $T_{1/2} = \operatorname{circ}(\frac{1}{2}, \frac{1}{2}, 0, \ldots, 0)$, and more generally for $r \in (0,1)$ the r-division point operation has matrix $T_r = \operatorname{circ}(1 - r, r, 0, \ldots, 0)$.

We are interested in iterating the operation T_r, that is, in investigating the powers T_r^m. To do this we first display the eigenvalues and eigenvectors of Π, and hence of all circulant matrices. If μ is any nth root of 1 in $\mathbb{C}$, define $W(\mu) = (1, \mu, \mu^2, \ldots, \mu^{n-1})^t$. Here the superscript t denotes the transpose, so $W(\mu)$ is a column vector. It follows that $\Pi \cdot W(\mu) = \mu W(\mu)$, so that $W(\mu)$ is a μ-eigenvector. Letting $\zeta = \exp(2\pi i/n)$, we have n distinct eigenvectors $W_k = W(\zeta^k)$ for $0 \leqslant k < n$.

If C is a circulant matrix with associated polynomial $p(x)$, then $C = p(\Pi)$ and we see that W_k is an eigenvector for C with eigenvalue $p(\zeta^k)$. When $C = T_r$, the polynomial is $p(x) = (1 - r) + rx$ and we can estimate the sizes of the eigenvalues.

1. LEMMA. *For $r \in (0,1)$ the matrix T_r has eigenvectors W_k with eigenvalue $\lambda_k = (1 - r) + r\zeta^k$, for $0 \leqslant k < n$. Then $\lambda_{n-k} = \bar{\lambda}_k$ and*

$$1 = \lambda_0 > |\lambda_1| > |\lambda_2| > \cdots > |\lambda_q|,$$

where $q = [n/2]$.

Proof. Here $[m]$ denotes the greatest integer in m. These eigenvalues lie on the circle $f(t) = (1 - r) + re^{it}$ with center $1 - r$ and radius r. (See Fig. 3.) The inequalities follow from this observation. □

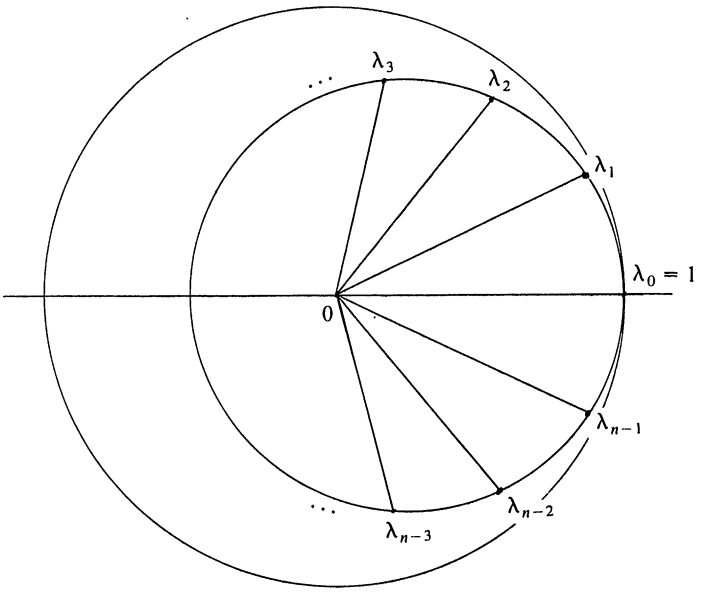


FIG. 3.

Viewed geometrically, the eigenvector W_k is a regular n-gon inscribed in the unit circle. For example, $W_1 = (1, \zeta, \zeta^2, \ldots, \zeta^{n-1})^t$ is the convex regular n-gon traced counterclockwise, while $W_{n-1} = \overline{W}_1$ is the same polygon traced clockwise. Generally W_k and $W_{n-k} = \overline{W}_k$ are the same n-gon except for the direction of traversal. If $1 < k < n$ the n-gon W_k might be self-intersecting (a "star"), multiply traversed, or degenerate. For example, $W_0 = (1, 1, \ldots, 1)^t$ is the degenerate n-gon with all vertices at 1. Pictures of some eigen-8-gons are given in Fig. 4.

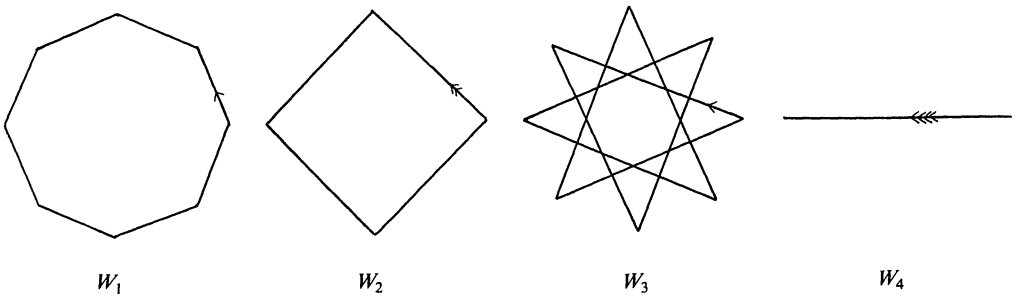


FIG. 4. Some eigen-8-gons.

Since the eigenvectors W_k correspond to n distinct eigenvalues, they form a basis of $\mathbb{C}^n$. In fact the W_k are orthogonal and have length $\sqrt{n}$, with respect to the usual inner product on $\mathbb{C}^n$. With some risk of confusion, we will identify the geometric n-gon $\mathscr{Z}$ with a representative vector Z. Then every n-gon Z can be expressed uniquely as a combination of the eigen-n-gons:

$$Z = a_0 W_0 + a_1 W_1 + \cdots + a_{n-1} W_{n-1}.$$

These scalars a_k are sometimes called the finite Fourier coefficients of Z. (Compare [S].)

The coefficient a_0 is the centroid of Z. To see this, recall that the n-gon $Z = (z_0, z_1, \ldots, z_{n-1})^t$ has centroid $(1/n)\Sigma z_i$. Then W_0 has centroid 1 while all the other W_k have centroid 0, and the claim follows.

Repeatedly applying the operator T_r to the n-gon Z, we get the mth descendant

$$T_r^m(Z) = a_0 W_0 + \lambda_1^m a_1 W_1 + \cdots + \lambda_{n-1}^m a_{n-1} W_{n-1}.$$

Since $|\lambda_j| < 1$ for $1 \leqslant j \leqslant n-1$, the descendants approach the limit $a_0 W_0$, which is the degenerate n-gon with all vertices at the centroid a_0.

If Z is a convex n-gon, it is clear geometrically that all its descendants will be convex. Moreover, many nonconvex polygons Z have convex descendants. The situation is clarified by the following theorem from [**BGS**]. Related results appear in [**Ne**] pp. 233–234 and [**Cl**]. Further references are mentioned in the appendix of [**FRS**].

2. THEOREM. *Suppose $Z = \Sigma a_k W_k$ as above. If $|a_1| \neq |a_{n-1}|$, then Z has a convex descendant. Hence for almost all n-gons Z, the descendants eventually become convex.*

The proof is presented below. We use here a somewhat more restrictive definition of convexity than usual. All self-intersecting, multiply traversed and degenerate polygons are considered nonconvex. Here a *degenerate* polygon is one having some three consecutive vertices collinear. Then the only eigen-n-gons W_k which are convex are W_1 and W_{n-1}. Furthermore, if Z is convex and Z' is very close to Z in $\mathbb{C}^n$, then Z' is also convex. (The convex n-gons form an open set.)

To investigate the shapes of the descendants of Z, we first translate Z to assume its centroid is the origin, so that $a_0 = 0$, then we expand each image by a constant factor: replace T_r by $\hat{T}_r = (1/|\lambda_1|)T_r$. Suppose $Z = a_1 W_1 + \cdots + a_{n-1} W_{n-1}$, where a_1, a_{n-1} are not both zero. From Lemma 1 we find that for sufficiently large m, the "normalized descendant" is

$$\hat{T}_r^m(Z) = \alpha^m a_1 W_1 + \bar{\alpha}^m a_{n-1} \overline{W}_1 + (\text{very small components}).$$

Here $\alpha = \lambda_1/|\lambda_1|$ is a value of modulus 1. Therefore, the normalized descendants become very close to n-gons of the type $bW_1 + c\overline{W}_1$. We will see that these are the 1-affine-regular polygons.

Define an n-gon U to be *p-affine-regular* if it is an affine image of the regular n-gon W_p.

3. LEMMA. *Suppose U is an n-gon centered at the origin. Then U is p-affine-regular if and only if $U = bW_p + c\overline{W}_p$ for some $b, c \in \mathbb{C}$. In this case, U is inscribed in the ellipse $f(t) = be^{it} + ce^{-it}$ at the points $f(2\pi pj/n)$. This ellipse is degenerate if and only if $|b| = |c|$.*

Proof. (Compare **[BGS]**, **[S]**, **[G]** and **[FRS** (2.4b)].) Suppose $U = (u_0, u_1, \ldots, u_{n-1})^t$ is of the form $bW_p + c\overline{W}_p$. Then $u_j = b\zeta^{pj} + c\zeta^{-pj} = f(2\pi pj/n)$, as claimed. Expressing $b = b_1 + ib_2$, $c = c_1 + ic_2$ and $f(t) = f_1(t) + if_2(t)$, a calculation shows that

$$\begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} = M \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} b_1 + c_1 & -b_2 + c_2 \\ b_2 + c_2 & b_1 - c_1 \end{pmatrix}.$$

Therefore U is the image of W_p under the affine transformation M. Conversely, if U is an affine image of W_p, then this transformation is realized by a real 2×2 matrix M. By writing M in the form above, we find $b, c \in \mathbb{C}$ fulfilling the requirements. The final assertion follows since the minimum value of $|f(t)|$ for real t is $\|b| - |c\|$. □

If $U = bW_1 + c\overline{W}_1$ is 1-affine-regular and $|b| \neq |c|$, the lemma shows that U must be convex. Furthermore, all the normalized descendants $\hat{T}_r^m(U) = b\alpha^m W_1 + c\bar{\alpha}^m \overline{W}_1$ are also 1-affine-regular and are inscribed in that same ellipse $f(t)$.

The proof of Theorem 2 is now easily completed. Given Z as in the theorem, let $U = a_1W_1 + a_{n-1}\overline{W}_1$. Since $|a_1| \neq |a_{n-1}|$, U is convex and all its normalized descendants are inscribed in one ellipse $f(t)$. For sufficiently large m, $\hat{T}_r^m(Z)$ is so close to $\hat{T}_r^m(U)$ that it must also be convex. □

As pointed out by the referee, the converse of Theorem 2 has not been adequately handled in the literature. (Compare **[BGS]**, **[Cl]**.) We sketch the proof, leaving details and further generalizations to the reader.

4. PROPOSITION. *Suppose $Z = \Sigma a_k W_k$ is convex. Then $|a_1| \neq |a_{n-1}|$.*

Proof Sketch: Assume Z is convex (in our restrictive sense) and that $|a_1| = |a_{n-1}|$. We may also assume $a_0 = 0$ and $n \geqslant 4$. Define $\Delta_j(Z) = \{tz_j + (1-t)z_{j+1} : 0 \leqslant t < 1\}$ to be the jth edge of Z. Then every line l through the origin meets exactly 2 of the edges $\Delta_j(Z)$. Let $U = a_1W_1 + a_{n-1}W_{n-1}$ and $V = a_2W_2 + \cdots + a_{n-2}W_{n-2}$. Since $|a_1| = |a_{n-1}|$, Lemma 3 implies that the vertices of U all lie on one line l.

Whenever $e \in \mathbb{C}$ is small enough, the n-gon $Z + eW_2$ is also convex. Replace Z by $Z + eW_2$ for suitable e, to assume $|a_2| \neq |a_{n-2}|$. Then replace Z by a remote descendant under T_r to arrange that V is close to the 2-affine-regular n-gon $a_2W_2 + a_{n-2}W_{n-2}$. Then the line l meets exactly 4 of the edges $\Delta_j(V)$. Alter e a little if necessary to assume that none of the vertices of V lies on l. For definiteness, rotate the picture to assume $l = \mathbb{R}$.

With $V = (v_0, v_1, \ldots, v_{n-1})^t$, suppose $v_p, v_{q+1}, v_r, v_{s+1}$ lie above $\mathbb{R}$ while $v_{p+1}, v_q, v_{r+1}, v_s$ lie below $\mathbb{R}$. Since $z_j = u_j + v_j$ and $u_j \in \mathbb{R}$, we conclude that the line $\mathbb{R}$ meets the edges $\Delta_p(Z)$, $\Delta_q(Z)$, $\Delta_r(Z)$ and $\Delta_s(Z)$, contrary to convexity. □

2. Periodicity. The operator T_r is said to be *periodic* on the n-gon Z if, for some $m > 0$, $T_r^m(Z)$ is similar to Z. This m is a *period* of T_r on Z. We will determine the algebraic conditions for such periodicity to occur.

5. LEMMA. *If T_r is periodic on Z, then Z is p-affine-regular, for some p.*

Proof. We may assume that $Z \neq 0$ and that Z is centered at the origin. Any similarity transformation in $\mathbb{C}$ which fixes the origin is realized as multiplication by some nonzero $\gamma \in \mathbb{C}$. Then for some γ, $T_r^m(Z)$ and γZ represent the same polygon. That is, $T_r^m(Z) = \gamma \Pi^k Z$ for some integer k. Expressing $Z = a_1W_1 + \cdots + a_{n-1}W_{n-1}$, this equation becomes: $\Sigma \lambda_j^m a_j W_j =$

$\Sigma\gamma\zeta^{jk}a_jW_j$. Since the eigenvectors W_j are independent, we have $\lambda_j^m = \gamma\zeta^{jk}$ whenever $a_j \neq 0$. Then if $a_i \neq 0$ and $a_j \neq 0$, we have $|\lambda_j^m| = |\gamma| = |\lambda_i^m|$ and hence $|\lambda_i| = |\lambda_j|$. The result follows from Lemmas 1 and 3. □

Suppose p and n have a common factor, say $p = dp_1$ and $n = dn_1$. Then a p-affine-regular n-gon is the same as a p_1-affine-regular n_1-gon which is traced d times. In studying the behavior of T_r, we may then assume that p and n are coprime. The degenerate cases when $n \leqslant 2$ are omitted.

6. PROPOSITION. *Suppose m, n, p are fixed positive integers with p and n coprime and $1 \leqslant p < n/2$, and suppose $r \in (0,1)$. The following statements are equivalent.*

(1) *$T_r^m(U)$ and U have parallel sides, for every p-affine-regular n-gon U.*
(2) *T_r is periodic with period m on every p-affine-regular n-gon.*
(3) *For some p-affine-regular n-gon U which is not regular (and not reduced to a single point), $T_r^m(U)$ is similar to U.*
(4) *$\alpha_p^m = \pm\zeta^j$ for some integer j, where $\alpha_p = \lambda_p/|\lambda_p|$.*

Proof. Here, as before, $\zeta = \exp(2\pi i/n)$ and $\lambda_p = (1 - r) + r\zeta^p$. The implications (1) → (2) and (2) → (3) are easy. We prove (3) → (4): We are given $T_r^m(U)$ is similar to U, for some $U = bW_p + c\overline{W}_p$, where b and c are nonzero. As in the preceding proof, we get $T_r^m(U) = \gamma\Pi^k U$ for some $\gamma \in \mathbb{C}$ and integer k. Since W_p and $\overline{W}_p$ are independent and b, c are nonzero, we find $\lambda_p^m = \gamma\zeta^{pk}$ and $\bar{\lambda}_p^m = \gamma\bar{\zeta}^{pk}$. Then γ is real and $\lambda_p^m \in \mathbb{R}\zeta^{pk}$, and hence $\alpha_p^m \in \mathbb{R}\zeta^{pk}$. Since $|\alpha_p| = 1$, we conclude $\alpha_p^m = \pm\zeta^{pk}$. (4) → (1): Since p and n are coprime, ζ^p is also a primitive nth root of unity. Then we have $\alpha_p^m = \pm\zeta^{pk}$ for some k, and the preceding argument can be reversed. □

REMARK. If n is odd, the condition in (4) is equivalent to: $\alpha_p^{2mn} = 1$. If n is even, it becomes: $\alpha_p^{mn} = 1$. In particular, α_p must be a root of unity.

It now follows that the midpoint operation $T_{1/2}$ is periodic with period 2 on every affine-regular polygon. For in this case $\alpha_p = \exp(p\pi i/n)$ so that $\alpha_p^{2n} = 1$. In fact, if n is odd, the minimal period is 1.

At this point we can answer Question 1 posed in the Introduction. Given $m \geqslant 1$, there do exist values $r \in (0,1)$ for which T_r is periodic on triangles with minimal period m. More generally, given m, n, p there exist $r \in (0,1)$ where T_r has minimal period m on p-affine-regular n-gons. We need only choose $\alpha \in \mathbb{C}$ with $\alpha^m = \pm\zeta^j$, where such an equation fails for exponents smaller than m. For instance, if n is even, let $\alpha = \exp(2\pi i/mn)$. If n is odd, let $\alpha = \exp(\pi i/mn)$, so that $\alpha^m = \exp(\pi i/n) = -\zeta^{(n+1)/2}$. Having such α, we can solve for $r \in (0,1)$. This calculation is done in the next lemma, in the case $0 < r < \frac{1}{2}$.

7. LEMMA. *Let $\omega = \exp(i\phi)$ where $0 < \phi < \pi$, and let $\lambda = (1 - r) + r\omega$ where $0 < r < \frac{1}{2}$. If $\alpha = \lambda/|\lambda| = \exp(i\theta)$, then*

$$r = \frac{1}{2}\left(1 - \frac{\tan\left(\frac{\phi}{2} - \theta\right)}{\tan\left(\frac{\phi}{2}\right)}\right).$$

Proof. See Fig. 5 for a picture of this situation. Let $\mu = \frac{1}{2}(1 + \omega)$ be the midpoint of the line segment $\overline{1\omega}$. Note that λ is the r-division point of $\overline{1\omega}$. From the right triangles $\Delta 0\mu\lambda$ and $\Delta 0\mu 1$, we see that $\tan((\phi/2) - \theta) = |\mu - \lambda|/|\mu|$ and $\tan(\phi/2) = |\mu - 1|/|\mu|$. Now $r = |1 - \lambda|/|1 - \omega| = (|1 - \mu| - |\mu - \lambda|)/2|1 - \mu| = \frac{1}{2}(1 - |\mu - \lambda|/|\mu - 1|)$ and the result follows. □

For example, to find a value of r where T_r has period 3 for all triangles, we choose $\alpha = \exp(\pi i/9)$. Applying the Lemma with $\phi = 2\pi/3$ and $\theta = \pi/9$, we calculate that $r = .25777$, approximately. In the picture of these triangles in Fig. 6, note that the inner and outer triangles have parallel sides.

More general than periodicity is the notion of "eventual periodicity." The operator T_r is said to

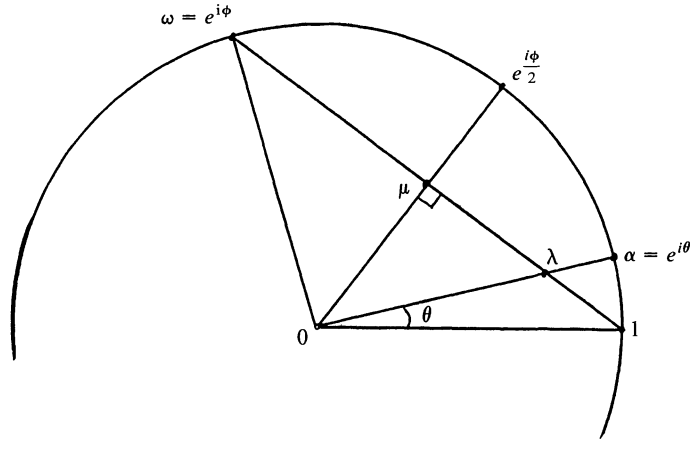


FIG. 5.

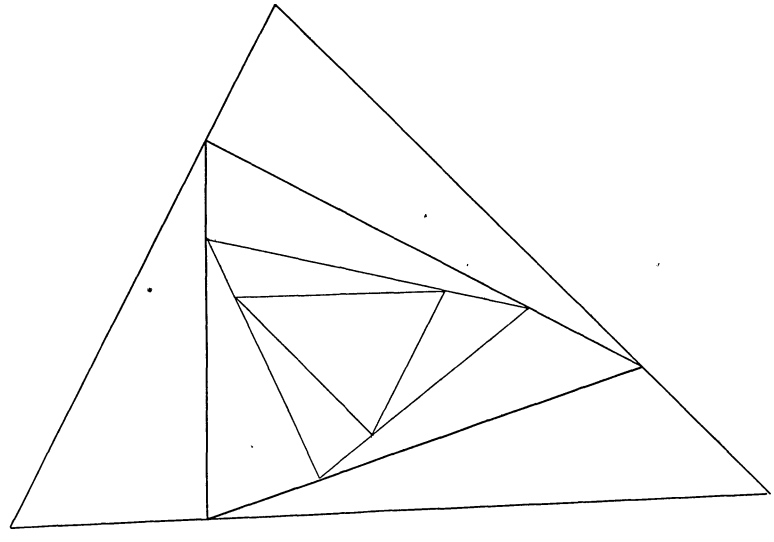


FIG. 6. $r \approx .258$.

be *eventually periodic* with period m on an n-gon Z if the sequence $T_r^{mj}(Z)$ approaches a limiting shape as j gets large. More precisely (assuming Z is not reduced to a point) this says: there exist n-gons Y_j similar to $T_r^{mj}(Z)$ such that $\lim_{j\to\infty} Y_j = Y$ exists in $\mathbb{C}^n$ and Y is not reduced to a point. Not surprisingly, eventual periodicity on Z is related to periodicity on the "dominant term" in the expansion $Z = \Sigma a_k W_k$.

8. PROPOSITION. *Let* $Z = a_p W_p + a_{p+1} W_{p+1} + \cdots + a_{n-p} W_{n-p}$, *where* a_p, a_{n-p} *are nonzero. Let* $\hat{T}_r = (1/|\lambda_p|)T_r$. *The following statements are equivalent.*

(1) T_r *is eventually periodic with period* m *on* Z.
(2) *The normalized descendants* $\hat{T}_r^{mj}(Z)$ *approach a limit* n*-gon as* $j \to \infty$.
(3) T_r *is periodic with period* m *on every* p*-affine-regular* n*-gon.*

Proof Sketch: (2) → (1) follows by the definitions.

(3) → (2): Express $Z = U + V$ where $U = a_p W_p + a_{n-p} W_{n-p}$. As in the proof of Proposition 6, we get $\hat{T}_r^m(U) = \varepsilon \Pi^k U$, where $\varepsilon = \pm 1$ and k is some integer. (Here $\varepsilon = \gamma/|\lambda_p|^m$.) Thus the

geometric n-gons U and $\hat{T}_r^m(U)$ either coincide or differ by a half-turn. It also follows that $\hat{T}_r^l(V) \to 0$ as $l \to \infty$, and we conclude that the normalized descendants $\hat{T}_r^{mj}(Z)$ approach the n-gon U (up to congruence).

(1) $\to$ (3). For convenience let $f = \hat{T}_r^m$. For each $j = 1, 2, 3, \ldots$ we are given an n-gon Y_j centered at 0 where Y_j is similar to $f^j(Z)$ and $\lim Y_j = Y$ exists and $Y \neq 0$. We can write $Y_j = \Sigma b_k^{(j)} W_k$ and $Y = \Sigma b_k W_k$, where the sums range over k with $p \leqslant k \leqslant n - p$. Then $b_k = \lim b_k^{(j)}$. Since $f(Y_j)$ is similar to Y_{j+1}, it is easy to see that we should have $f(Y)$ similar to Y. The details of this step are left to the reader. If we can prove that Y is p-affine-regular and not regular, then Proposition 6 applies and the proof is complete. By Lemma 5 we know Y must be q-affine regular for some q, so it remains to prove: $b_p \neq 0$ with $b_{n-p} \neq 0$.

We know $f(W_k) = \tau_k W_k$ where $\tau_k = (\lambda_k/|\lambda_p|)^m$, so that $|\tau_k| \leqslant 1$ for $p \leqslant k \leqslant n - p$, with equality if and only if $k = p$ or $n - p$. Since Y_j is similar to $f^j(Z)$, we have $Y_j = \gamma_j \Pi^{l_j} f^j(Z)$ for some $\gamma_j \in \mathbb{C}$ and $l_j \in \mathbb{Z}$. Comparing the coefficients of W_k we find: $b_k^{(j)} = \gamma_j \zeta^{kl_j} \tau_k^j a_k$. Consequently $|b_k^{(j)}| \leqslant |\gamma_j a_k|$ for every j, k, with equality when $k = p$ or $n - p$. Now $|b_p| = \lim |b_p^{(j)}| = \lim |\gamma_j a_p|$ so that $\gamma = \lim |\gamma_j|$ exists. If $\gamma = 0$, then $|b_k| = \lim |b_k^{(j)}| \leqslant \gamma |a_k| = 0$ for every k, contrary to the hypothesis $Y \neq 0$. Then $\gamma \neq 0$ and hence $|b_p| = \gamma |a_p| \neq 0$. Similarly $b_{n-p} \neq 0$. □

The set of n-gons $Z = \Sigma a_k W_k$ with $a_1 = a_{n-1} = 0$ forms a subspace of $\mathbb{C}^n$ of codimension 2. Then, almost all n-gons Z have $a_1 \neq 0$ and $a_{n-1} \neq 0$. From Proposition 8 we conclude that: T_r is eventually periodic on almost all n-gons if and only if T_r is periodic on the 1-affine-regular n-gons. To require that T_r be eventually periodic on all n-gons is more restrictive: it occurs if and only if T_r is periodic on the p-affine-regular n-gons, for each $p = 1, 2, \ldots$.

As mentioned after Proposition 6, the midpoint operation $T_{1/2}$ does have this strong property. The eventual periodicity of the midpoint operation on all n-gons has been observed independently by a number of authors, including [**Ca**], [**BGS**, Comment 1], and [**Cl**]. Further comments and a list of references are given following Theorem B in the appendix of [**FRS**]. The converse problem is presented in Proposition 15 below.

3. Rationality Questions. Before beginning the proof of the Main Theorem stated in the Introduction, we mention two lemmas from number theory.

9. LEMMA. *Let ζ be a root of unity. Then the only roots of unity in the field $\mathbb{Q}(\zeta)$ are the elements $\pm\zeta^j$, for integers j.*

One proof uses mainly the fact that finite multiplicative subgroups of fields are cyclic. For details, see e.g. [**We**, p. 267, Prop. 7-6-1].

10. LEMMA. *If $\cos(\pi/n)$ is rational for a positive integer n, then $n = 1$, 2 or 3.*

A clever elementary proof due to H. W. Richmond is reported in [**Co**, p. 443] (in the solution to 13.4 #15). More generally, if $m \geqslant 3$ is an integer, then $\cos(2\pi/m)$ is an algebraic number of degree $\frac{1}{2}\phi(m)$, where ϕ is the Euler function. A nice exposition of this fact is given in [**Ni**, Theorem 3.9].

Proof of the Main Theorem (*i*) $\to$ (*iii*): As noted after Proposition 8, the hypothesis (i) here says that T_r is periodic with period m on all 1-affine-regular n-gons. As before, let $\zeta = \exp(2\pi i/n)$, $\lambda_1 = (1 - r) + r\zeta$ and $\alpha_1 = \lambda_1/|\lambda_1|$. From Proposition 6, we know that α_1 is a root of unity. Also $\alpha_1^2 \in \mathbb{Q}(\zeta)$ since $\alpha_1^2 = \lambda_1/\bar{\lambda}_1$, and hence $\alpha_1^2 = \pm\zeta^k$ for some k, by Lemma 9. Set $\omega = \zeta$ in Fig. 5 for a picture of this situation.

By the symmetry between r and $1 - r$, we may assume $0 < r < \frac{1}{2}$. Then α_1 lies between 1 and $\exp(\pi i/n)$ so that α_1^2 lies between 1 and ζ. But the only term $\pm\zeta^j$ that can lie between 1 and ζ is $-\zeta^{(n+1)/2} = \exp(\pi i/n)$, when n is odd. Hence n is odd and $\alpha_1 = \exp(\pi i/2n)$.

Using Lemma 7 with $\phi = 2\pi/n$ and $\theta = \pi/2n$, we find $r = \frac{1}{2}(1 - (\tan(\pi/2n)/\tan(\pi/n)))$. Since $1 - \tan(\theta)/\tan(2\theta) = 1/(1 + \cos(2\theta))$, we conclude that $\cos(\pi/n) = 1 - 1/2r$ is a rational number. Since $n \geqslant 3$ by hypothesis, Lemma 10 implies $n = 3$. In this case $r = \frac{1}{3}$ and the

proof is complete. □

We can strengthen the result just proved by using some Galois theory.

11. THEOREM. *Suppose n, p are coprime integers with $1 \leqslant p < n/2$, and r is a rational number with $0 < r < 1$. The operator T_r is periodic on the p-affine-regular n-gons if and only if either $r = \frac{1}{2}$ or $n = 3, p = 1$ and $r = \frac{1}{3}$ or $\frac{2}{3}$.*

Proof. Suppose T_r has period m on the p-affine-regular n-gons. With $\alpha_p = \lambda_p/|\lambda_p|$ where $\lambda_p = (1 - r) + r\zeta^p$, we know that $\alpha_p^2 = \lambda_p/\overline{\lambda}_p$ lies in $\mathbb{Q}(\zeta)$. Proposition 6 implies that α_p is a root of unity. Then $K = \mathbb{Q}(\zeta, \alpha_p)$ is a cyclotomic field. Since n, p are coprime, there exists an automorphism $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$ with $\sigma(\zeta^p) = \zeta$, and hence $\sigma(\lambda_p) = \lambda_1$. Since complex conjugation induces an element of $\mathrm{Gal}(K/\mathbb{Q})$, it commutes with σ, and $\alpha_1^2 = \lambda_1/\overline{\lambda}_1 = \sigma(\lambda_p/\overline{\lambda}_p) = \sigma(\alpha_p^2)$. Then $\alpha_1 = \pm\sigma(\alpha_p)$ is also a root of unity. Now the proof of the Main Theorem (i) → (iii) with its "betweenness" argument applies, and the conclusion follows. □

Proof of the Main Theorem (ii) → (iii): We are given a rational number $r \in (0,1)$ such that T_r is periodic on Z, where Z is a nondegenerate, nonregular n-gon. By Lemma 5, Z is p-affine-regular for some p with $1 \leqslant p \leqslant n/2$. Since Z is nondegenerate, n and p must be coprime. Since Z is nonregular, Proposition 6 implies that T_r is periodic on every p-affine-regular n-gon, and the result follows by Theorem 11. □

Theorem 11 can be restated in terms of roots of unity as follows.

12. COROLLARY. *Let α and ω be any roots of unity in $\mathbb{C}$ with $0 < \arg(\alpha) < \arg(\omega) < \pi$. Let λ be the intersection point of the line segments $\overline{1\omega}$ and $\overline{0\alpha}$. If λ divides the segment $\overline{1\omega}$ rationally, then either λ is the midpoint of the segment, or $\omega = \exp(2\pi i/3)$ and $\alpha = \exp(\pi i/6)$ or $\exp(\pi i/2)$.*

Proof. (See Fig. 5.) Suppose ω is a primitive nth root of unity and express $\omega = \zeta^p$, where $\zeta = \exp(2\pi i/n)$. Then $1 \leqslant p < n/2$ and p, n are coprime. The argument proving Theorem 11 shows that either $r = \frac{1}{2}$ and λ is the midpoint, or $n = 3$, $p = 1$ and $r = \frac{1}{3}$ or $\frac{2}{3}$. Then $\omega = \zeta = \exp(2\pi i/3)$ and the claim follows. □

13. COROLLARY. *Suppose a, b are rational numbers with $0 < a < b < \frac{1}{2}$.*

(i) $\tan(a\pi)/\tan(b\pi)$ *is rational if and only if* $a = \frac{1}{6}$ *and* $b = \frac{1}{3}$.

(ii) $\tan(a\pi)\tan(b\pi)$ *is rational if and only if* $a + b = \frac{1}{2}$.

Proof. (i) Let $\omega = \exp(2b\pi i)$ and $\alpha = \exp((b - a)\pi i)$, and define λ and r so that $\lambda = (1 - r) + r\omega$ and $\alpha = \lambda/|\lambda|$. If $\tan(a\pi)/\tan(b\pi)$ is rational, then Lemma 7 says r is rational, and the result is a consequence of Corollary 12. Part (ii) follows from (i) using the identity $\tan((\pi/2) - \phi) = 1/\tan(\phi)$. □

14. REMARK. The rationality results above can also be deduced from a theorem of Conway and Jones. Suppose n, p, r are given as usual. If T_r is periodic on p-affine-regular n-gons, then Proposition 6 implies that α is a root of unity, where $\alpha = \lambda/|\lambda|$ and $\lambda = (1 - r) + r\zeta^p$. Then $\alpha^2\overline{\lambda} = \lambda$ and we find:

$$(1 - r)\alpha^2 + r\alpha^2\zeta^{-p} - (1 - r) - r\zeta^p = 0.$$

These 4 roots of unity (and their negatives) are distinct, except in the familiar cases $r = \frac{1}{2}$ or $n = 3$ and $r = \frac{1}{3}$ or $\frac{2}{3}$. If r is rational, then all other cases are eliminated by Theorem 6 of [**CJ**], and our Theorem 11 follows. A similar technique is used to prove the next result.

15. PROPOSITION. *Suppose $n \geqslant 5$ and a real number $r \in (0,1)$ are fixed. Then T_r is eventually periodic on every n-gon if and only if $r = \frac{1}{2}$.*

Proof Sketch: Suppose T_r has the stated property, so that T_r is periodic on every p-affine-regular n-gon for $p = 1, 2, \ldots$. Proposition 6 implies that α_p is a root of unity, so that $\alpha_p^2 = ((1 - r) + r\zeta^p)/((1 - r) + r\zeta^{-p})$ is a root of unity for each p. From this equation for $p = 1$ and $p = 2$, we

can solve for r, equate the results and simplify. The outcome is a vanishing sum of 8 roots of unity. If $r \neq \frac{1}{2}$ these roots (and their negatives) are distinct, and Theorem 6 of [CJ] can be applied. □

16. REMARK. We close with some comments on two generalizations which provide some interesting pictures.

(1) Dropping the restriction that $0 < r < 1$, we can consider T_r when $r > 1$ or $r < 0$. The generalized Theorem 11 contains only one further case: T_2 and T_{-1} are periodic with period 2 on 1-affine-regular hexagons. (See Fig. 7, where the inner hexagon is Z.)

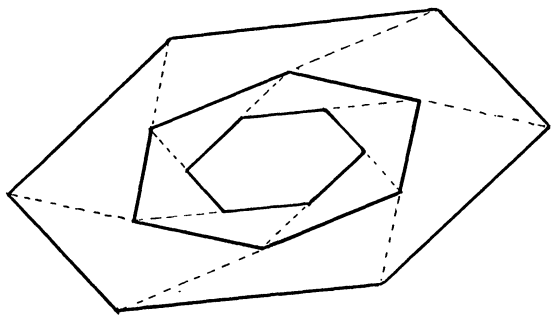
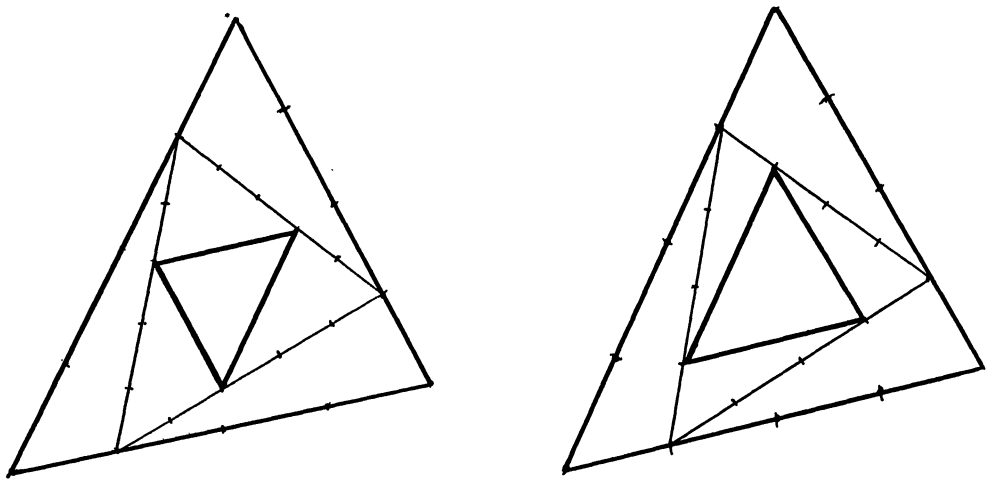


FIG. 7. T_2 on 1-affine-regular hexagons.

(2) If $r, s \in (0,1)$, when is $T_r \circ T_s(Z)$ similar to Z for every triangle Z? If $r + s = 1$, then this is always true, and we are familiar with the case $r = s = \frac{1}{3}$. This property holds for some other values as well, for instance $r = \frac{2}{5}$ and $s = \frac{1}{4}$. (See Fig. 8.) Finding all such pairs (r, s) is a nice exercise.



$T_{2/5} \circ T_{1/4}$ $T_{3/4} \circ T_{1/4}$

FIG. 8.

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ARCHIMEDES AND THE COMPLEX PLANE

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1. Introduction. Let us consider the double recurrence relation

$$1/a_{n+1} = \frac{1}{2}(1/a_n + 1/b_n), \tag{1a}$$

$$b_{n+1} = (a_{n+1}b_n)^{1/2}, \tag{1b}$$

where $a_0, b_0 > 0$ are given. If $a_0 = 3\sqrt{3}$, $b_0 = \frac{1}{2}3\sqrt{3}$, it may be shown by elementary trigonometry that a_n then denotes half the perimeter of the regular polygon of 3×2^n sides which circumscribes the circle of unit radius and b_n denotes half the perimeter of the corresponding regular polygon which is inscribed in the unit circle. (In this case, equations (1a) and (1b) become relations between sines and tangents.) This geometrical description makes it clear that, for the initial values of a_0 and b_0 above, the sequence (a_n) is monotonic decreasing and (b_n) is monotonic increasing. The two sequences converge to the common limit π. These are the sequences which were used by Archimedes to estimate π. From his computation of a_5 and b_5, corresponding to polygons with 96 sides, he obtained his famous inequalities

$$3\frac{10}{71} < b_5 < \pi < a_5 < 3\frac{1}{7}$$

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although, according to Dijksterhuis [2, p. 228], Archimedes did not, in fact, use (1) to compute the sequences (a_n) and (b_n).

If we write $c_n = 1/a_n$, $d_n = 1/b_n$, (1) becomes

$$c_{n+1} = \frac{1}{2}(c_n + d_n), \tag{2a}$$

$$d_{n+1} = (c_{n+1}d_n)^{1/2}. \tag{2b}$$

The recurrence relations (2) are associated with the names of Borchardt, Gauss, Pfaff and Schwab. (In this connection, Gauss is better known for his analysis of the variant of (2) where c_{n+1} is replaced by c_n in (2b). This is the arithmetic-geometric mean process which may be used to compute elliptic integrals.) See, for example, Carlson [1], Miel [4], Schoenberg [6] and Todd [8]. Schoenberg credits the first thorough investigation of (2) to Schwab [7] for the case $0 < c_0 < d_0$ and to Pfaff for the case $0 < d_0 < c_0$. An application of (2) in the complex plane is provided by Miel [3] who gives an algorithm for the evaluation of $\log z$.

As we have seen, the double recurrence relations (1) and (2) are equivalent. In this paper we will be concerned with the behaviour of (1) in the complex plane although, obviously, to any statement which we make about (1) there is a corresponding statement applicable to (2).

First we note the monotonic behaviour of the sequences (a_n) and (b_n) for arbitrary $a_0, b_0 > 0$ which we have already remarked upon for the special case where $a_0 = 3\sqrt{3}$, $b_0 = \frac{1}{2}3\sqrt{3}$. It is easily verified that, if $0 < b_0 < a_0$, then (a_n) is a decreasing sequence, (b_n) is an increasing sequence and the two sequences have a common limit. The reader may also wish to verify that, if $0 < a_0 < b_0$, the two sequences are again monotonic and have a common limit. In both of these cases, the common limit lies between a_n and b_n, for each value of n. Note that we do not need to use the explicit representations for a_n and b_n (to which we will refer presently) in order to verify the monotonicity of these sequences.

We will find that this monotonicity extends to the complex case in that the sequences (a_n) and (b_n) are monotonic in argument for all n and also in modulus for all n sufficiently large. The monotonicity of modulus is not an obvious result as the reader will appreciate if he interprets (1a) on the Argand diagram, for example taking $|a_n| = |b_n|$ and using (2a). In order to pursue this, we will obtain in §2 explicit forms for a_n and b_n, given initial complex values a_0 and b_0. These are based on the real case with $0 < a_0 < b_0$ (see, for example Phillips [5]), where we may write

$$b_0/a_0 = \cosh\theta. \tag{3}$$

If we restrict $\theta > 0$, this defines a unique value of θ. Let us now write

$$a_0 = \lambda\tanh\theta, \quad b_0 = \lambda\sinh\theta$$

and use the identity $\coth^2\theta = 1 + \operatorname{cosech}^2\theta$ to give

$$\lambda = \left(1/a_0^2 - 1/b_0^2\right)^{-1/2}.$$

We deduce from (1) that

$$a_n = 2^n\lambda\tanh(\theta/2^n), \quad b_n = 2^n\lambda\sinh(\theta/2^n) \tag{4}$$

and the common limit of the sequences (a_n) and (b_n) for the case where $0 < a_0 < b_0$ is

$$\lambda\theta = \left(1/a_0^2 - 1/b_0^2\right)^{-1/2}\cosh^{-1}(b_0/a_0).$$

Although the above is all we require here for the generalization to complex a_0 and b_0, the reader may wish to derive expressions for a_n and b_n analogous to (4) for the case where $0 < b_0 < a_0$, using circular functions in place of hyperbolic functions above.

2. Convergence of the Complex Case. Let us rewrite (1a) for $n = 0$ as

$$a_1 = 2a_0b_0/(a_0 + b_0).$$

Then we see that a_1 is not defined if $a_0 + b_0 = 0$ and that we ought also to exclude the cases $a_0 = 0$ and $b_0 = 0$. Therefore, the case $a_0 = b_0$ being trivial, we shall assume in what follows that $a_0 b_0 (a_0^2 - b_0^2) \neq 0$ and that, otherwise, a_0 and b_0 are arbitrary complex numbers. Following (3), we put

$$b_0/a_0 = u + iv = \cosh z. \tag{5}$$

The real and imaginary parts of b_0/a_0 are thus

$$u = \cosh x \cos y, \quad v = \sinh x \sin y. \tag{6}$$

Since $u^2 + v^2 + 1 = \cosh^2 x + \cos^2 y$, we see that $t_1 = \cosh^2 x$ and $t_2 = \cos^2 y$ are the roots of the quadratic equation

$$p(t) = t^2 - (1 + u^2 + v^2)t + u^2 = 0.$$

We note that $p(0) = p(1 + u^2 + v^2) = u^2$, $p(1) = -v^2$, so that $1 \leqslant t_1 \leqslant 1 + u^2 + v^2$ and $0 \leqslant t_2 \leqslant 1$. We will choose the unique $x \geqslant 0$ such that $\cosh^2 x = t_1$ and the unique y, $-\pi < y \leqslant \pi$, such that $\cos^2 y = t_2$ and both equations in (6) are satisfied. (Observe that the required value of y has to give the correct signs of both u and v.)

Having thus determined $z = x + iy$ with $x \geqslant 0$ and $-\pi < y \leqslant \pi$ so that $b_0/a_0 = \cosh z$, we now set

$$a_0 = \lambda \tanh z, \quad b_0 = \lambda \sinh z, \tag{7}$$

where λ is uniquely prescribed, and we find that

$$\lambda^2 = \left(1/a_0^2 - 1/b_0^2\right)^{-1}.$$

It then follows from (1) and (7) that

$$a_1 = 2\lambda \tanh \frac{1}{2}z, \quad b_1^2 = 4\lambda^2 \sinh^2 \frac{1}{2}z. \tag{8}$$

Let us choose the square root in (8) so that $b_1 = 2\lambda \sinh \frac{1}{2}z$. (In §3 we will see that this choice of square root also makes sense geometrically.) In general we may choose the value of the square root in (1) at each stage so that

$$a_n = 2^n \lambda \tanh(z/2^n), \quad b_n = 2^n \lambda \sinh(z/2^n)$$

and this shows that the complex sequences (a_n) and (b_n) converge to the common limit λz.

It is interesting to see how the positive real case, which we discussed in §1, is subsumed in the complex case. If a_0 and b_0 are real and positive, it follows from (5) and (6) that $u > 0$ and $v = 0$. Then (6) shows that $x = 0$ or $y = 0$. We see that $x = 0$ gives

$$u = \cos y = b_0/a_0,$$

which corresponds to the case $0 < b_0 < a_0$, and $y = 0$ gives

$$u = \cosh x = b_0/a_0,$$

which corresponds to the case where $0 < a_0 < b_0$. It is left to the reader to investigate what happens when b_0/a_0 is real and negative.

3. Monotonicity of Argument. Let us now think of a_n and b_n as "vectors" in the Argand diagram and transform (1a) into (2a) by means of $c_n = 1/a_n$, $d_n = 1/b_n$. It is then clear that the vector c_{n+1} lies between c_n and d_n (that is, it lies within the smaller angle between c_n and d_n) and thus the vector a_{n+1} lies between a_n and b_n. If only we could now infer from (1b) that the vector b_{n+1} lies (midway) between a_{n+1} and b_n, then we could deduce immediately that indeed the arguments of the vectors a_n and b_n are monotonic, in the sense that the sequence of vectors (a_n) rotates in one direction and the sequence (b_n) rotates in the opposite direction. The two sequences rotate towards the common limiting vector.

It remains to check that the choice of square root in (1b) which we made in §2 so that each b_n is given as in (9), results in b_{n+1} lying within the smaller angle between a_{n+1} and b_n and not in the opposite direction. To do this, we note from (9) that

$$b_n/b_{n+1} = b_{n+1}/a_{n+1} = \cosh z_{n+1}, \tag{10}$$

where we have written

$$z_n = z/2^n = x_n + iy_n,$$

say. From (5) and (6) we may write

$$\arg(\cosh z_{n+1}) = \tan^{-1}(\tanh x_{n+1} \tan y_{n+1}). \tag{11}$$

Note that $-\frac{1}{2}\pi < y_{n+1} \leqslant \frac{1}{2}\pi$ for all n, since $-\pi < y \leqslant \pi$. We then see from (11) that the sign of $\arg(\cosh z_{n+1})$ is the same as the sign of $x_{n+1}y_{n+1}$, which is that of xy. Thus the sign of $\arg(\cosh z_{n+1})$ is constant and, since $-1 < \tanh x_{n+1} < 1$, it follows that

$$|\arg(\cosh z_{n+1})| \leqslant |y_{n+1}| = |y|/2^{n+1} \leqslant \frac{1}{2}\pi. \tag{12}$$

It is now clear from (10) and (12) that b_{n+1} does indeed bisect the smaller angle between b_n and a_{n+1} and the foregoing account of the monotonicity of the arguments of the vectors a_n and b_n is justified.

To conclude this section, note that we can easily obtain a bound on the size of the angle between a_n and b_n at any stage. It is left to the reader to verify that, for all n,

$$|\arg(a_n/b_n)| \leqslant \pi/2^n.$$

4. Monotonicity of Modulus. From (9) and (6) we obtain

$$|b_n/a_n|^2 = |\cosh z_n|^2 = 1 + \sinh^2 x_n - \sin^2 y_n. \tag{13}$$

It also follows from (9) that

$$b_n/b_{n+1} = \cosh z_{n+1} \tag{14}$$

and we deduce from (1) that

$$a_n/a_{n+1} = \frac{1}{2}(1 + a_n/b_n). \tag{15}$$

Having determined $z = x + iy$ so that $b_0/a_0 = \cosh z$, we find it convenient to consider two separate cases.

Case 1. $|x| \geqslant |y|$.

It follows from $|x| \geqslant |y|$ that $|x_n| \geqslant |y_n|$ and

$$\sinh^2 x_n \geqslant \sinh^2 y_n \geqslant \sin^2 y_n$$

for all n. We can, in fact, write the strict inequality

$$\sinh^2 x_n > \sin^2 y_n,$$

since equality holds only for $x = y = 0$, which corresponds to one of the excluded cases, namely the trivial case a_0 and b_0. We deduce from (13) and (14) that $|b_n| > |a_n|$ and $|b_n| > |b_{n+1}|$. Then from (15) we see that $|a_n| < |a_{n+1}|$. We conclude that, if $|x| \geqslant |y|$, the sequence $(|b_n|)$ is monotonic decreasing and the sequence $(|a_n|)$ is monotonic increasing.

Case 2. $|x| < |y|$.

We find that this case is the more awkward and the more interesting of the two, in that the monotonicity of modulus holds only for all n sufficiently large. First, let us consider values of x and y such that $0 \leqslant x < y$. Then there is some $\varepsilon > 0$ such that

$$0 \leqslant x < (1 - \varepsilon)y$$

and thus

$$0 \leqslant x_n < (1 - \varepsilon) y_n$$

for all n, where ε is independent of n. Hence

$$x_n + \frac{1}{2}x_n^2 < y_n - \frac{1}{2}y_n^2 + \left(\frac{1}{2}x_n^2 + \frac{1}{2}y_n^2 - \varepsilon y_n\right).$$

Using $x_n^2 < y_n^2$ in the latter inequality, we deduce that

$$x_n + \frac{1}{2}x_n^2 < y_n - \frac{1}{2}y_n^2 + y_n(y_n - \varepsilon),$$

and this yields

$$x_n + \frac{1}{2}x_n^2 < y_n - \frac{1}{2}y_n^2 \tag{16}$$

if n is chosen sufficiently large so that $0 < y_n < \varepsilon$. It is not essential to use precisely the form of inequality which we have adopted in (16). However, (16) together with the Maclaurin series for $\sinh x_n$ and $\sin y_n$ enables us to show that, for $0 \leqslant x < y$,

$$\sinh x_n \leqslant x_n + \frac{1}{2}x_n^2 < y_n - \frac{1}{2}y_n^2 < \sin y_n$$

for all n sufficiently large. Hence we obtain

$$\sinh^2 x_n < \sin^2 y_n$$

for $|x| < |y|$ and for all n sufficiently large. This last inequality, together with (13) and (14), implies that $|b_n| < |a_n|$ and $|b_n| < |b_{n+1}|$ for all n sufficiently large.

To complete our examination of the case where $|x| < |y|$, we go on to show that $|a_{n+1}| < |a_n|$ for all n sufficiently large. We extend the notation of (5) by writing

$$b_n/a_n = u_n + iv_n = \zeta_n = \cosh z_n. \tag{17}$$

Then, from (15) and (17) we obtain

$$\frac{a_{n+1}}{a_n} = \frac{2\zeta_n}{1 + \zeta_n}, \quad n \geqslant 1. \tag{18}$$

Note that we need to disallow $n = 0$ in (18), since this is the only value of n where it is possible that $\zeta_n = -1$. For, as we see from (17), (5) and (6), $\zeta = -1$ implies that $x_n = 0$ and $y_n = \pi$. From (18), the condition $|a_{n+1}| < |a_n|$ is equivalent to $2|\zeta_n| < |1 + \zeta_n|$ or

$$\left|\zeta_n - \frac{1}{3}\right| < \frac{2}{3}. \tag{19}$$

If we now consider the Argand diagram for the complex variable ζ_n, we see that the region described by the inequality (19) is the interior of the circle with center at $\zeta_n = \frac{1}{3}$ and radius $\frac{2}{3}$. It is clear from the Argand diagram that the inequality $|\zeta_n| < 1$ does not imply that (19) holds; that is, the condition $|b_n| < |a_n|$ does not necessarily entail that $|a_{n+1}| < |a_n|$. However, let us write

$$\zeta_n = \cosh z_n = 2\cosh^2 z_{n+1} - 1 = 2\zeta_{n+1}^2 - 1.$$

Then $|\zeta_n| < 1$ implies that

$$\left|\zeta_{n+1}^2 - \frac{1}{2}\right| < \frac{1}{2}. \tag{20}$$

We now state and prove the following:

LEMMA. *If* $u = R(\zeta) \geqslant 0$ *and* $|\zeta^2 - \frac{1}{2}| < \frac{1}{2}$, *then* $|\zeta - \frac{1}{3}| < \frac{2}{3}$.

Proof. Let us write $\zeta = u + iv = \rho e^{i\phi}$. Then the curve $|\zeta^2 - \frac{1}{2}| = \frac{1}{2}$ can be expressed in the form

$$(u^2 + v^2)^2 = u^2 - v^2$$

and we obviously need to restrict u and v so that $u^2 \geqslant v^2$. This, together with the restriction $u \geqslant 0$ imposed in the statement of the lemma, shows that the portion of the curve which concerns us may be written in polar form as

$$\rho^2 = \cos 2\phi, \quad 0 \leqslant |\phi| \leqslant \frac{\pi}{4},$$

and we note that $\rho = 0$ for $\pi/4 < |\phi| \leqslant \frac{1}{2}\pi$. To verify the lemma, we need to show that, for $R(\zeta) \geqslant 0$, the curve lies within the circle $|\zeta - \frac{1}{3}| = \frac{2}{3}$ which, in polar form, is

$$\rho = \tfrac{1}{3}\left(\cos\phi + (3 + \cos^2\phi)^{1/2}\right), \quad 0 \leqslant |\phi| \leqslant \pi.$$

The reader may find it helpful to sketch the curve and circle on the Argand diagram. Since $\rho = 0$ for the curve over the range $\pi/4 < |\phi| \leqslant \frac{1}{2}\pi$, it remains to verify that

$$\cos\phi + (3 + \cos^2\phi)^{1/2} - 3\cos 2\phi \geqslant 0, \quad 0 \leqslant |\phi| \leqslant \frac{\pi}{4}.$$

On putting $t = \cos^2\phi$, this is equivalent to establishing that

$$F(t) = t^{1/2} + (3 + t)^{1/2} - 3(2t - 1) \geqslant 0, \quad \frac{1}{2} \leqslant t \leqslant 1.$$

Now

$$F'(t) = \frac{1}{2}t^{-1/2} + \frac{1}{2}(3 + t)^{-1/2} - 6 \tag{21}$$

and, since the first two terms on the right of (21) are monotonic decreasing for positive t, we readily verify that $F'(t) < 0$ for $t \geqslant \frac{1}{2}$. Hence, for $\frac{1}{2} \leqslant t \leqslant 1$,

$$F(t) \geqslant F(1) = 0$$

and the lemma is established.

We have already shown that $|b_n| < |a_n|$ for sufficiently large n, so that $|\zeta_n| < 1$ and hence (20) holds. Since

$$R(\zeta_{n+1}) = R(\cosh z_{n+1}) = \cosh x_{n+1}\cos y_{n+1}$$

and $|y_{n+1}| \leqslant \frac{1}{2}\pi$ for all n, we see that $R(\zeta_{n+1}) \geqslant 0$ for all n. We now apply the lemma with $\zeta = \zeta_{n+1}$ to give

$$\left|\zeta_{n+1} - \frac{1}{3}\right| < \frac{2}{3}$$

whence, from (19), we deduce that $|a_{n+2}| < |a_{n+1}|$ for all n sufficiently large.

Having completed the analysis above, we conclude with a numerical illustration of Case 2 where the monotonicity of modulus does fail for small n. We choose

$$a_0 = \tanh(1 + i\pi/3), \quad b_0 = \sinh(1 + i\pi/3),$$

so that $z = 1 + i\pi/3$. In this case we find that $|a_0| < |a_1| < |a_2|$ and the sequence $(|a_n|)$ is decreasing for $n \geqslant 2$. We also find that $|b_0| > |b_1|$ and the sequence $(|b_n|)$ is increasing for $n \geqslant 1$. The values of the first few ratios $|b_n/b_{n+1}|$ and $|a_{n+1}/a_n|$, computed from (14) and (18) respectively, are given in Table 1. The denominator of (18) may be expressed in the surprisingly simple form

$$|1 + \cosh z_n| = \cosh x_n + \cos y_n,$$

as the reader may care to verify. From this, (18) and (13) we then obtain

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2(\cosh^2 x_n + \cos^2 y_n - 1)^{1/2}}{\cosh x_n + \cos y_n}, \tag{22}$$

and we deduce from (14) and (13) that

$$|b_n/b_{n+1}|^2 = \frac{1}{2}(\cosh x_n + \cos y_n). \tag{23}$$

We see from (22) and (23) that the numbers required in Table 1 are simply computed from the values $\cosh x_n$ and $\cos y_n$.

Table 1. Numerical illustration of Case 2.

n	0	1	2	3	4	5
$\lvert a_{n+1}/a_n\rvert$	1.25021	1.01393	0.99974	0.99970	0.99991	0.99998
$\lvert b_n/b_{n+1}\rvert$	1.01071	0.99841	0.99933	0.99982	0.99995	0.99999

Acknowledgments. The idea of looking at this process in the complex plane came from my friend Professor Philip J. Davis. It is a pleasure to record my thanks to him. I am equally grateful to Professor Richard Bieberich whose valuable comments have, in my view, substantially improved the presentation of this paper and to Professor I. J. Schoenberg for providing me with some useful references.

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ON SOME CHARACTERIZATIONS OF POLYNOMIALS

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One of the simplest ways to characterize a function f of a single real variable as a polynomial is that of $D^m f = 0$ for some positive integer m where D is the derivative. Another way is by

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difference operators with $\Delta_h^m f = 0$ for all real h where $\Delta_h f\colon x \to f(x+h) - f(x)$, $\Delta_h^1 = \Delta_h$ and $\Delta_h^m = \Delta_h^{m-1}\Delta_h$ for $m = 2,3,\ldots$

For functions of several real variables, similar conditions can be given. Let $C(R^n)$ denote the set of all complex-valued functions defined on real n dimensional space R^n, and $D_j = \partial/\partial x_j$. For $f \in C(R^n)$, the solution to an equation $D_j^m f = 0$ is understood to be a weak solution when f is merely continuous and so locally integrable. Also, for $f \in C(R^n)$ let $\Delta_{l,h} f\colon x \to f(x + he_l) - f(x)$ for real h, where $\{e_1, e_2, \ldots, e_n\}$ denotes the standard basis for R^n. The following theorem can then be shown.

THEOREM 1. *Let $f \in C(R^n)$. The following conditions are equivalent:*

A. *f is a polynomial,*

B. *for some positive integer m, $D_j^m f = 0$ for $j = 1,2,\ldots,n$, and*

C. *for some positive integer m, $\Delta_{l,h}^m f = 0$ for all real h and $l = 1,2,\ldots,n$.*

We also note that by using Taylor's Theorem for several variables, it can be shown that if m is a positive integer, and if $D_1^{p_1} D_2^{p_2} \cdots D_n^{p_n} f = 0$ for all nonnegative integers, $p_1, p_2, \ldots, p_n$ such that $p_1 + p_2 + \cdots + p_n = m$, then f is a polynomial of degree less than m.

A more unusual way of characterizing $C(R^n)$ functions as polynomials is now given by way of translation and dilation invariant subspaces. For $a \in R^n$ and $\alpha \in R$, let $T_a f$ and $I_\alpha f$ be defined by $T_a f\colon x \to f(x-a)$ and $I_\alpha f\colon x \to f(\alpha x)$ for each $f \in C(R^n)$. Also let X_f denote the subspace of $C(R^n)$ spanned by all translates $T_a f$ and dilations $I_\alpha f$ of f as a ranges over R^n and α over R.

THEOREM 2. *Let $f \in C(R^n)$. A necessary and sufficient condition that f be a polynomial is that X_f be finite dimensional.*

Proof. If f is a polynomial, then $T_a f$ and $I_\alpha f$ are also polynomials and X_f is of finite dimension.

Conversely, let X_f be of finite dimension, say m with a basis $g_1, g_2, \ldots, g_m$ and set $g = (g_1, g_2, \ldots, g_m)$. Let $A(a)$ and $B(\alpha)$ be the $m \times m$ complex matrices for T_a and I_α with respect to this basis so that $T_a g = A(a)g$ and $I_\alpha g = B(\alpha)g$. From $T_a T_b = T_{a+b}$, $I_\alpha I_\beta = I_{\alpha\beta}$ and $I_\alpha T_\alpha = T_{\alpha a} I_\alpha$, it follows that

$$A(a)A(b) = A(a+b), \tag{1}$$

and

$$B(\alpha)B(\beta) = B(\alpha\beta),$$

$$B(\alpha)A(a) = A(\alpha a)B(\alpha) \tag{2}$$

for all $a, b \in R^n$ and $\alpha, \beta \in R \setminus \{0\}$. Thus $A(0) = B(1) = I$, the identity matrix and the matrices $A(a)$, $B(\alpha)$ are nonsingular. From equation (1), for fixed a, $A(pa) = (A(a))^p$ for all integers p, and from equation (2), $A(pa)$ is similar to $A(a)$. Hence if $A(a)$ has eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$, the sets $\{\lambda_1^p, \lambda_2^p, \ldots, \lambda_m^p\}$ are identical for all integers p. Hence all eigenvalues of $A(a)$ are unity.

Accordingly, by the Cayley-Hamilton theorem, $(A(a) - I)^m = 0$. Expanding this by the binomial theorem with $a = he_l$, we obtain

$$0 = (A(he_l) - I)^m = \sum_{k=0}^{m} \binom{m}{k} A^k(he_l) = \sum_{k=0}^{m} \binom{m}{k} A(khe_l) = \Delta_{l,h}^m A(0).$$

for $l = 1,2,\ldots,n$. From equation (1) again, $\Delta_{l,h}^m A(x) = A(x)\Delta_{l,h}^m A(0) = 0$ for $l = 1,2,\ldots,n$ and all real h. So, by Theorem 1, the elements of the matrix A are polynomials. From $g(x) = T_x g(0) = A(x)g(0)$, the components of g are polynomials. Hence f is a polynomial when X_f is finite dimensional.

An alternative proof of this fact is possible. Firstly, we define an exponential polynomial to be a finite linear combination of terms $x_1^{p_1} x_2^{p_2} \ldots x_n^{p_n} \exp(a \cdot x)$, where $a \cdot x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$, $p_1, p_2, \ldots, p_n$ are nonnegative integers, and $a_1, a_2, \ldots, a_n$ are any complex numbers. Next, we set for $f \in C(R^n)$, U_f as the subspace formed by f and all of its translates $T_b f$:

$x \to f(x - b)$ as b ranges over R^n. It is not too difficult to show (see, for example, [1], or [2]), that f is an exponential polynomial if and only if U_f is finite dimensional. So, if X_f is finite dimensional, as $U_f \subset X_f$, f is an exponential polynomial. Now if f contains any exponential $\exp(a \cdot x)$ with $a \neq 0$ in R^n, we would find that since X_f contains dilations as well as translations, X_f would be of infinite dimension. Hence $a = 0$ and so f is a polynomial.

We conclude by noting a related result of Loewner [2] that if $n > 1$ and if $f \in C(R^n)$ is such that its transforms under all isometries of R^n span a finite dimensional space, then f is a polynomial. Thus, if R_f denotes the subspace of $C(R^n)$ spanned by all translations of f and all orthogonal transformations $O_p f$: $x \to f(Px)$, where P is any orthogonal real matrix, the following holds.

THEOREM 3. *Let $n > 1$ and $f \in C(R^n)$. A necessary and sufficient condition that f be a polynomial is that R_f be finite dimensional.*

This theorem was first shown by Loewner [2] using exponential polynomials. Whether it can also be shown by methods similar to those used in the main proof of our Theorem 2 is an interesting conjecture to us.

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POISSON APPROXIMATIONS AND THE DEFINITION OF THE POISSON PROCESS

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1. Introduction. This article has two purposes. The first is to expose two interesting, connected facts about the Poisson distribution, both of which deserve to be better known. The first of these is that it is almost as easy to provide error bounds for certain Poisson distribution approximations as it is to derive these approximations via convergence arguments. The second one is that these bounds make it extremely easy to prove an elegant qualitative characterization of the Poisson process, due to Prékopà [20].

The second aim of the article is to advertise two important general ideas in probability theory. Both ideas have been extremely useful in recent research, but authors of elementary textbooks usually consider these ideas to be too advanced for inclusion. On the other hand, the exposition here only requires that the reader know some undergraduate real analysis and the most trivial facts about probability. Thus, I hope that the *simple* examples here might make the ideas more widely known.

Coupling, a technique introduced by Doeblin [5], is one basic idea in this article. By this we mean the study of one or two specific distributions, via the construction of two random variables (or vectors or processes). The *coupled* random variables are designed so that not only do they have

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specified (*marginal*) distributional properties, but also it is possible to analyse their joint (*point-by-point*) behaviour. We shall use coupling in Section 2 to provide the error bounds. A (random!) sample of other areas in which coupling has proved fruitful are: the convergence of transition probabilities of Markov chains [5], [19], the renewal theorem [15], interacting particle systems [23], the F. K. G. inequality of statistical physics [10], simulation algorithms for spatial processes [16], random walks on finite groups, rapidly mixing Markov chains and the number of shuffles needed to thoroughly mix a pack of cards [2], and the tail behaviour of birth-and-death and stochastically monotone processes [1]. For an excellent survey of coupling methods for Markov processes, the reader may like to consult [8].

The other central idea here is that the path properties of stochastic processes are crucial in their definition. As an example, let us consider Brownian motion. In the simplest case, this is a model for the position of a particle executing a "random" motion in one dimension. In introductory courses, the model is often introduced as a stochastic process with certain joint normal distributions for the positions of the particle at different times. However, it seems initially that this definition hardly corresponds at all to the physical idea of a particles motion. On the other hand, the following model assumptions are easily interpretable:

(a) each path of the particle is continuous,
(b) the path of the particle in the time interval $[0, t]$ is independent of the path in $[t, t + s]$, apart from the common position at time t, and
(c) the mean position at time t is 0, while the variance of the position is t.

It is a consequence of a celebrated result of Lévy [14, p. 78] that precisely formulated versions of (a), (b), and (c) are enough to guarantee that the model *is* Brownian motion. More recently, the profound work of Stroock and Varadhan [22] has showed that characterizations like Lévy's can be extended to a large class of processes satisfying (a). As a result, some open questions on the existence and uniqueness of solutions of partial differential equations were settled. In Section 3 we show that it is very simple to prove the characterization of the ordinary Poisson process that is analogous to (a), (b) and (c). That proof is then easily modified to yield the much more general characterization of Prékopà [20].

2. Poisson Approximations. In this section, we give some results about the error in approximating the distribution of certain random variables by Poisson distributions. Let us first recall that the Poisson distribution with parameter $\lambda \geqslant 0$ attributes probability

$$P(J; \lambda) = \sum_{j \in J} e^{-\lambda} \lambda^j / j!$$

to the set $J \subseteq \{0, 1, 2, \ldots\}$.

Suppose that $A_1, \ldots, A_n$ are *independent* events in a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. For example, in a random sample taken with replacement, A_i might represent the event that the ith person sampled has a certain disease. The random variable S, which gives the number of $A_1, \ldots, A_n$ that occur, is often of interest; in the example, this would be the number in the sample who have the disease. Formally,

$$S = \sum_{i=1}^{n} IA_i \tag{1}$$

where IA_i is the indicator (or characteristic) random variable for the event A_i (that is, $IA_i(\omega) = 1$ if $\omega \in A_i$, and $IA_i(\omega) = 0$ otherwise). If the probabilities

$$p_i = \mathbf{P}(A_i), \quad i = 1, \ldots, n,$$

are all the same, then S has a binomial distribution. On the other hand, if the p's vary, then the distribution is much more complicated. However, in both cases the distribution of S can be

approximated by a Poisson distribution in certain circumstances; intuitively these are when each event is "rare," i.e., each p_i is small. More precisely, we have the following theorem due to Le Cam [13].

THEOREM 1. *For any set* $J \subseteq \{0,1,2,\dots\}$,

$$|\mathbf{P}(S \in J) - P(J;\Sigma p_i)| \leqslant \Sigma p_i^2$$

where here, and for the rest of this section, unmarked sums are over $i = 1,\dots,n$.

We present a modification of a proof of Theorem 1 due to Serfling [21]. Here is the key tool:

THE COUPLING LEMMA. *If* X *and* Y *are any random variables and* J *is any (Borel) set of* $\mathbb{R}$, *then*

$$|\mathbf{P}(X \in J) - \mathbf{P}(Y \in J)| \leqslant \mathbf{P}(X \neq Y).$$

The Coupling Lemma is in itself trivial, as you can see from the proof. However, the quantities in Theorem 1 are *very* difficult to bound analytically and the power of the Coupling Lemma is that it turns this analytic problem into one of probabilistic *construction*. For, provided X is *any* random variable with the same distribution as S, and Y is *any* random variable with Poisson (Σp_i) distribution, the Coupling Lemma tells us that $\mathbf{P}(X \neq Y)$ is a bound for the left side in Theorem 1. The key point here is that the *joint* law of X and Y can be arbitrary, subject only to their having the correct marginals. As in Euclidean geometry, clever construction is just what the doctor ordered! An advantage of the proof of Theorem 1 as an introduction to coupling is that, in contrast to many other applications of coupling, the required coupling is quite straightforward.

Proof of Coupling Lemma. Without loss of generality $\mathbf{P}(X \in J) > \mathbf{P}(Y \in J)$. In this case, the required modulus is bounded by

$$\mathbf{P}(X \in J) - \mathbf{P}(X \in J \quad \text{and} \quad Y \in J) = \mathbf{P}(X \in J \quad \text{and} \quad Y \notin J),$$

and the latter is clearly bounded by $\mathbf{P}(X \neq Y)$.

The coupling we present relies on a fundamental construction in probability. This is the construction showing that a probability space which supports a uniform random variable also has one of any other distribution. It is also important for the computer simulation of random variables; usually the computer has an algorithm which produces uniform random variables and the construction can (at least in principle) then turn these into any desired distribution. We describe the construction only in the simpler discrete case and refer the interested reader to [3, p. 159] for the general case. Suppose that U is a uniform random variable over $(0,1)$ (i.e., its distribution is Lebesgue measure on $(0,1)$). Suppose also that the desired distribution Q attributes probability $q(i)$ to $\{i\}$ $(\in \{0,1,2,\dots\})$. For $x \in (0,1)$, let

$$\hat{Q}(x) = i$$

for the unique $i \in \{0,1,2,\dots\}$ such that

$$\sum_{j=0}^{i-1} q(j) < x \leqslant \sum_{j=0}^{i} q(j),$$

where the sum on the left with $i = 0$ is by convention 0. The constructed random variable is then $\hat{Q}(U)$. It is immediate that $\hat{Q}(U)$ *does* have distribution Q, because for $i \in \{0,1,2,\dots\}$,

$$\mathbf{P}(\hat{Q}(U) = i) = \mathbf{P}\left(U \in \left(\sum_{j=0}^{i-1} q(j), \quad \sum_{j=0}^{i} q(j)\right]\right).$$

Recall that we wish to construct random variables X and Y with prescribed distributions, say Q and R, so that $\mathbf{P}(X \neq Y)$ is as small as possible. The construction above gives us an obvious attempt for such similar random variables. We simply take a *single* uniform random variable U

and set $X = \hat{Q}(U)$ and $Y = \hat{R}(U)$. Unfortunately, $\mathbf{P}(X \neq Y)$ is generally still hard to calculate if we use such X and Y in Theorem 1. However, for $n = 1$ the calculation is easy. In this case, let B be the distribution of S; B is said to be a *Bernoulli* distribution with parameter $p = p_1$. Let P denote the Poisson distribution with parameter p. Then, $\mathbf{P}(\hat{B}(U) \neq \hat{P}(U)$ is the length of the dotted intervals in Fig. 1, and this is

$$(e^{-p} - (1-p)) + (1 - e^{-p}(1+p)) \leqslant p^2,$$

as required, since

$$1 - p \leqslant e^{-p}, \tag{2}$$

an inequality which also justifies the crucial ordering of the points marked in Fig. 1.

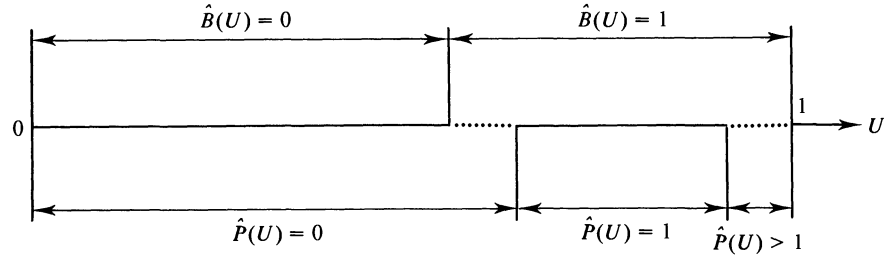


FIG. 1

Parenthetically, we remark that $(\hat{B}(U), \hat{P}(U))$ is called a *maximal coupling*, because $\mathbf{P}(\hat{B}(U) \neq \hat{P}(U))$ actually achieves equality in the Coupling Lemma for $J = \{0, 2, 3, \ldots\}$. As a consequence, $p(1 - e^{-p})$ is actually the best possible bound in Theorem 1 for the case $n = 1$. While there is a method to construct a maximal coupling for *any* distributions Q and R [**12**], even for discrete distributions it is not true that $(\hat{Q}(U), \hat{R}(U))$ is always a maximal coupling. For an example, consider R equal to the Bernoulli(1/2) distribution and Q given by $Q\{0\} = Q\{1\} = \frac{1}{4}$, $Q\{2\} = \frac{1}{2}$; here $\mathbf{P}(\hat{R}(U) \neq \hat{Q}(U)) = \frac{3}{4}$, while the maximum of $|R(J) - Q(J)|$ is $\frac{1}{2}$. Nevertheless, $(\hat{Q}(U), \hat{R}(U))$ does always have a weaker maximal similarity property, namely that it maximises the *correlation* between all pairs of random variables with distributions R and Q (see, for example, [**24**], Theorem 2.5).

Proof of Theorem 1. Let $U_1, \ldots, U_n$ be independent uniform random variables over $(0, 1)$. Let B_i and $P_i (i = 1, \ldots, n)$ denote the Bernoulli and Poisson distributions both with parameters p_i, and denote $\hat{B}_i(U_i)$ by X_i and $\hat{P}_i(U_i)$ by Y_i. Clearly, (X_i, Y_i), $i = 1, \ldots, n$, are independent random vectors. Therefore, ΣX_i has the same distribution as S. Moreover, by a standard result, ΣY_i has the $P(.; \Sigma p_i)$ distribution [**6**, p. 237]. At the same time, $\Sigma X_i \neq \Sigma Y_i$ implies $X_i \neq Y_i$ for some i in $\{1, \ldots, n\}$. Combining these facts with the Coupling Lemma produces

$$\begin{aligned} |\mathbf{P}(S \in J) - P(J; \Sigma p_i)| &\leqslant \mathbf{P}\left(\bigcup_{i=1}^{n} \{X_i \neq Y_i\} \right) \\ &\leqslant \Sigma \mathbf{P}(X_i \neq Y_i). \end{aligned}$$

Applying the argument for $n = 1$ to each term in the sum completes the proof.

Often it is not convenient to have the parameter of the approximating Poisson distribution equal to Σp_i.

COROLLARY 2. *For any* $J \subseteq \{0, 1, 2, \ldots\}$ *and any* $\lambda > 0$,

$$|\mathbf{P}(S \in J) - P(J; \lambda)| \leqslant |\lambda - \Sigma p_i| + \left(\max_i p_i\right) \Sigma p_i.$$

The corollary follows from the triangle inequality, Theorem 1, and the inequalities

$$\Sigma p_i^2 \leqslant \left(\max_i p_i\right)\Sigma p_i$$

and

$$|P(J;\lambda) - P(J;\Sigma p_i)| \leqslant |\lambda - \Sigma p_i|. \tag{3}$$

To see the truth of (3), we use an argument of Freedman [7]. Suppose that $\Sigma p_i > \lambda$. Let Z_1 and Z_2 be independent Poisson random variables with parameters λ and $\Sigma p_i - \lambda$. By the Coupling Lemma, the left side of (3) is bounded by $\mathbf{P}(Z_1 \neq Z_1 + Z_2)$, which in turn is bounded by

$$\mathbf{P}(Z_2 \neq 0) \leqslant 1 - e^{-(\Sigma p_i - \lambda)}.$$

The inequality (2) now gives (3). A symmetrical argument gives (3) if $\Sigma p_i < \lambda$.

To close this section we mention that the coupling technique can also produce bounds similar to Theorem 1 for dependent events and for multinomial distributions. The interested reader is referred to [**21**], [**4**], and [**18**].

3. Defining the Poisson Process. The bounds of the previous section are now applied to see that the general Poisson process arises from purely qualitative assumptions. But first we show how the ideas of that section provide a natural way to introduce the Poisson process in an elementary course (i.e., without $(\Omega, \mathscr{F}, \mathbf{P})$!).

Feller [**6**, p. 159] cites many examples where the Poisson distribution provides an adequate fit to experimental data. We shall consider the first of Feller's examples and construct a mathematical model for it. In this example, a radioactive substance emits α-particles; the number N_t of emissions reaching a given space up to time $t > 0$ is recorded. The following assumptions appear not unreasonable:

(1) that $\{N_t\}$ is a *point process on* $\mathbb{R}^+$, by which we mean that each N_t is a random variable and that a graph of N_t versus t is always an increasing step function with jumps of size 1 and $N_0 = 0$ (the assumption that the jumps in the graph are of size 1 corresponds to the idea that no two emissions occur at *exactly* the same time),

(2) that $\{N_t\}$ is *completely random*, by which we mean, informally, that information concerning the emissions in one interval of time tells us nothing about the emissions in a disjoint interval of time or, formally, that for each integer n and $0 < t_1 < \cdots < t_n$,

$$N_{t_1}, N_{t_2} - N_{t_1}, \ldots, N_{t_n} - N_{t_{n-1}}$$

are independent (lack of complete randomness corresponds to the idea that past information can tell us something about future emissions),

(3) that $\{N_t\}$ is *stationary (in increments)*, by which we mean that $N_t - N_s$ has the same distribution as $N_v - N_u$ provided $t - s = v - u$ (of course, in the long term, this cannot be true as the radioactive substance decays, but it will be *approximately* true over intervals short compared to the half life).

THEOREM 3. *If $\{N_t\}$ is a stationary, completely-random point process on $\mathbb{R}^+$, then each N_t has a Poisson distribution.*

Before proving this theorem, it is instructive to compare Theorem 3 with treatments of the Poisson process in textbooks. Firstly, it seems almost universal that in the first treatment of the Poisson process, assumption (1) is replaced by a stronger and practically less interpretable assumption concerning the probability of two or more emissions in a short time interval. Complete randomness is usually assumed, but the simple assumption (3) is usually replaced by the assumption that there exists λ so that for all t,

$$\mathbf{P}(N_{t+h} - N_t = 1) = \lambda h + o(h)$$

as $h \to 0$. It initially puzzled me as a student that a model for radioactive emissions should be formulated thus; perhaps I was not alone. In any event, with these usual assumptions there are two ways to show that N_t has a Poisson distribution. Either some differential equations are formulated and solved (in which case the proof is longer than the proof of Theorem 3) or the Poisson approximation to the binomial is used (in which case the proof's structure is much the same as that below, but the stronger assumptions do not make the proof very much easier). Finally, in revising the manuscript, I discovered that Feller [6, p. 157] *sketches* a similar proof to the one below; it is hoped that the detailed treatment here will convince others that this is *the* natural way to introduce the Poisson process to students.

Proof. We will define for each $n = 1, 2, \ldots$ a random variable S_n of the form of S in (1). It is then shown that

$$\mathbf{P}(N_t \neq S_n) \to 0 \tag{4}$$

as $n \to \infty$, and also that there exists λ such that for each m in $\{0, 1, 2, \ldots\}$

$$\mathbf{P}(S_n = m) \to e^{-\lambda}\lambda^m/m! \tag{5}$$

as $n \to \infty$ along a subsequence of $\mathbb{N}$. The inequality

$$|\mathbf{P}(N_t = m) - e^{-\lambda}\lambda^m/m!| \leqslant |\mathbf{P}(N_t = m) - \mathbf{P}(S_n = m)| + |\mathbf{P}(S_n = m) - e^{-\lambda}\lambda^m/m!|$$

and the Coupling Lemma then finish the proof.

To define S_n we consider the partition $\mathscr{P}_n$ of $(0, t]$ into n (left-open, right-closed) intervals of equal length. For $i = 1, \ldots, n$, we let A_{ni} be the event that the ith interval of $\mathscr{P}_n$ has *at least one* emission in it and let S_n be the number of $A_{n1}, \ldots, A_{nn}$ which occur.

Let D be the random variable which gives the minimum time between emissions in $(0, t]$. If $N_t \neq S_n$, then at least one interval of $\mathscr{P}_n$ has two or more emissions in it, so that $D \leqslant 1/n$. Hence, $\mathbf{P}(N_t \neq S_n)$ is bounded above by $\mathbf{P}(D \leqslant 1/n)$. Because $\{N_t\}$ is a point process, $D > 0$. This implies that the distribution function of D is 0 at 0 and, since it is also right continuous, $\mathbf{P}(D \leqslant 1/n) \to 0$ and (4) follows.

By complete randomness, the events $A_{n1}, \ldots, A_{nn}$ are independent. To establish (5), we may thus apply Corollary 2 with $S = S_n$ and $J = \{m\}$. By stationarity, $\mathbf{P}(A_{n1}) = \cdots = \mathbf{P}(A_{nn}) = p_n$, say, and thus the bound of Corollary 2 is $|np_n - \lambda| + (np_n)^2/n$. Hence, it suffices to show that $\{np_n\}$ is a bounded sequence, for, in that case, the Bolzano-Weierstrass theorem gives us a subsequence along which np_n converges.

The random variable N_t is zero if, and only if, *each* A_{ni} does *not* occur. Thus, by complete randomness and stationarity,

$$\mathbf{P}(N_t = 0) = (1 - p_n)^n. \tag{6}$$

A consequence of this is that if $\mathbf{P}(N_t = 0)$ were 0, then it would be certain that each A_{ni} would occur. But then, since $S_n \leqslant N_t$ for all n, we would have $N_t = \infty$, contradicting the assumption that $\{N_t\}$ is a point process. Thus we may take negative logs in (6) to conclude that $n\{-\ln(1 - p_n)\}$ equals the finite number $L = -\ln \mathbf{P}(N_t = 0)$. The inequality $x \leqslant -\ln(1 - x)$ $(x \leqslant 1)$ now tells us that L is an upper bound for np_n. Since clearly $np_n \geqslant 0$, the required boundness is now established.

A couple of remarks on this proof seem desirable. There are two methods to avoid the use of subsequences in the proof. One way (Feller [5, p. 157]) uses the fact that $-n\ln(1 - p_n)$ converges only if np_n converges. The other way would be to define $\mathscr{P}_n$ to have 2^n intervals, in which case it is easy to show that np_n increases with n. Strictly speaking, we only need the usual binomial *convergence* to Poisson in the proof and not the bound of Corollary 2; but the bound is the easiest

way to establish the convergence needed for the next theorem.

The parameter λ introduced in the proof clearly depends on t, and so we denote it as $\lambda(t)$. It is now fairly easy to show that $\lambda(t)$ is linear in t. First, the fact that N_t has a Poisson $\lambda(t)$ distribution implies that

$$\lambda(t) = -\ln \mathbf{P}(N(t) = 0) = n\lambda(t/n) \tag{7}$$

from (6). Let i and j be arbitrary integers > 0. Setting $t = 1$ and $n = j$ in (7) gives $\lambda(1) = j\lambda(1/j)$. If we set $t = i/j$ and $n = i$, the same equation gives $\lambda(i/j) = i\lambda(1/j)$, and from this and the previous equation we deduce that $\lambda(t) = t\lambda(1)$, for all rational t. To extend the latter to real t, we consider the distribution function, F, of the time, T_1, to the first emission. Since $\{T_1 > t\} = \{N_t = 0\}$, (7) gives $\lambda(t) = -\ln(1 - F(t))$ and the right continuity of F yields the required extension.

We now generalise Theorem 3 to the setting of point processes on $\mathbb{R}^p (p \geqslant 1)$ and drop the assumption of stationarity (with the requisite topological facts it is straightforward to replace $\mathbb{R}^p$ by a locally compact, second countable, Hausdorff space as in [**11**, p. 48]). Since $\mathbb{R}^p$ has a more complicated order structure, we need to describe our process differently and an easy way to do this is by specifying an appropriate sample space, Ω.

The samples we are trying to model are random sets of points in $\mathbb{R}^p$ (in the previous example, each point is an emission). Two sensible restrictions are that each set ω is countable and that each ω has finite intersection with every bounded set. Accordingly, we take Ω to be the set of all such ω and for any bounded set B define $N(B, \omega)$ to be the cardinality of $\omega \cap B$ (in the previous example $N_t(\omega) = N((0, t], \omega)$). For B a bounded Borel set, we want $N(B) : \omega \to N(B, \omega)$ to be a random variable (so that $\{N_t\}$ is a point process on $\mathbb{R}^+$ in the example). Accordingly, we call a probability measure $\mathbf{P}$ *a point process on* $\mathbb{R}^p$ if $\mathbf{P}$ has domain a σ-algebra $\mathcal{F}$ on Ω, which is large enough so that for any $B \subseteq \mathbb{R}^p$, $N(B)$ is a random variable (hereafter, all subsets of $\mathbb{R}^p$ are assumed bounded Borel). Informally, $\mathbf{P}$ is the law that dictates how to choose the points of the process. The point process is *completely random* if $N(B_1), \ldots, N(B_n)$ are independent whenever $B_1, \ldots, B_n$ are disjoint subsets of $\mathbb{R}^p$ (that, in the previous example, this definition is no stronger than the one given is a standard measure theoretic exercise). The point process is called a *Poisson process* if each $N(B)$ has a Poisson distribution. In this case, since for $x \in \mathbb{R}^p$, $N\{x\}$ only takes the values 0 or 1, the parameter of the distribution of $N\{x\}$ must always be 0. Thus, a Poisson process has *no fixed atoms*, meaning that for each $x \in \mathbb{R}^p$, $\mathbf{P}(N\{x\} = 0) = 1$. This qualitative property appears quite reasonable in many applications; an *exact* time point at which there was a positive probability of observing a radioactive emission would be very special indeed. The remarkable fact along these lines is due essentially to Prékopà [**20**].

THEOREM 4. *A completely random point process is a Poisson process if, and only if, it has no fixed atoms.*

A proof of Theorem 4 is presented by detailing some necessary changes to that of Theorem 3; the result seems much simpler than existing proofs of Theorem 4. Since we now have defined an underlying probability space, we also show how to eliminate the nonmathematical language in the previous proof.

Proof. We have already noted that it is necessary that $\mathbf{P}$ has no fixed atoms.

We fix a subset B of $\mathbb{R}^p$ and replace every occurrence of N_t in the previous proof by $N(B)$. The major change is in the definition of the partition $\mathcal{P}_n$ which will no longer have n sets in it, nor will these sets be intervals of equal size. Let $x \in \overline{B}$ and let $D_x(\omega)$ be the minimum of $|y - x|$ for the finitely many y in $\omega \cap \overline{B}$. The assumption of no fixed atoms means that each D_x has a distribution function which is 0 at 0. Using right continuity again, we may thus choose a strictly positive $d_x \leqslant 1/n$ such that $\mathbf{P}(D_x \leqslant d_x) \leqslant 1/n$. Thus, if C_x is the open ball with centre x and radius d_x,

$$\mathbf{P}(N(C_x) \geqslant 1) \leqslant 1/n.$$

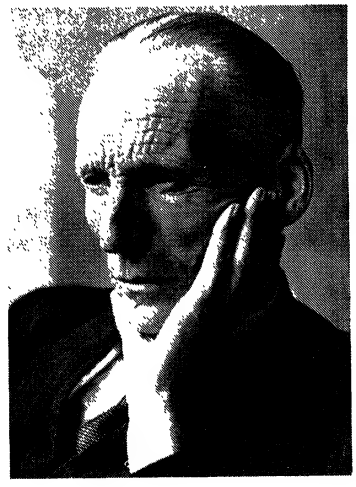
The Heine-Borel Theorem then allows us to choose a finite subcover of $\bar{B}$ from the sets $\{C_x\}_{x\in\bar{B}}$. By taking suitable intersections [9, p. 32], we can use this subcover to form a partition $\mathscr{P}_n = \{B_{n1},\ldots,B_{nk}\}$ of B (where of course k actually depends on n). We now let $A_{ni} = \{N(B_{ni}) \geq 1\}$ $(i = 1,\ldots,k)$ and S_n be S in (1) except that now and hereafter the sum is over $i \in \{1,\ldots,k\}$.

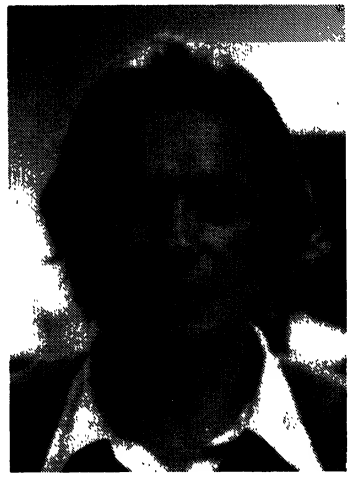
To demonstrate (4), we proceed as before after making the formal definition that $D(\omega)$ is the minimum of $|y - z|$ such that y and z are both in $\omega \cap B$, and noting that $N(B,\omega) \neq S_n(\omega)$ implies there exist $\omega_i \neq \omega_j \in \omega$ such that ω_i and ω_j both lie in the same set of $\mathscr{P}_n$. To demonstrate (5), we note that even in the absence of stationarity, it still suffices to show that $\{\Sigma p_{ni}\}_n$ is bounded. This is because each B_i in $\mathscr{P}_n$ is contained in some C_x and so the maximum of $p_{n1},\ldots,p_{nk}$ is bounded above by $1/n$. Finally, we note that $\mathbf{P}(N(B) = 0)$ now becomes the product of $(1 - p_{n1}),\ldots,(1 - p_{nk})$ and by the last statement this exceeds $(1 - 1/n)^n > 0$ for $n \geq 2$. Thus, we still get the finite number $-\ln \mathbf{P}(N(B) = 0)$ as an upper bound for Σp_{ni}.

4. Acknowledgement. The author is grateful to J. G. Kupka for many valuable conversations on this material.

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Two masters, one a disciple of the other. (See p. 138.)

restriction? Why not take as a basis all the sets of the form $\Pi_\alpha U_\alpha$ where each U_α is open in X_α (as is done in the case of finite product)? This latter is a basis for a topology often called the box topology; see [1, p. 113]. While the box topology agrees with the product topology when the number of spaces is finite, it is quite different for infinite products.

In the product topology, the product of connected spaces is connected. This is not true in the box topology. Consider, for example, $\mathbb{R}^\omega$—the countable product of real lines. The set

$$A = \{(x_1, x_2, \ldots) | \{x_i\} \text{ is a bounded sequence}\}$$

is both open and closed in the box topology, and thus $\mathbb{R}^\omega$ is not connected. (See [1, p. 152].)

In the product topology the product of compact spaces is compact. This, too, fails in the box topology. Consider I^ω—the countable product of copies of the unit interval, I. If $A_0 = [0, 1)$ and $A_1 = (0, 1]$, then the collection of all open sets of the form $A_{\varepsilon_1} \times A_{\varepsilon_2} \times \cdots$, where $\varepsilon_i = 0$ or 1, is an uncountable open cover of I^ω with no proper subcover. For if $A_{\varepsilon_1} \times A_{\varepsilon_2} \times \cdots$ is excluded from the cover, the point $(\varepsilon_1, \varepsilon_2, \ldots)$ is not covered.

Continuous functions $f\colon X \to \Pi_\alpha X_\alpha$ in the product topology are characterized as being those whose coordinate functions $f_\alpha (= \pi_\alpha f)\colon X \to X_\alpha$ are each continuous. To see that this characterization does not hold when the box topology is used, consider the function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ defined by $f(x) = (x, x, x, \ldots)$. An easy argument shows that f is not continuous (see [1, p. 115]).

The following provides a characterization of continuous functions into product spaces with the box topology.

THEOREM. *Let X and X_α ($\alpha \in \Lambda$) be metric spaces. A function $f\colon X \to \Pi_{\alpha\in\Lambda} X_\alpha$ is continuous in the box topology if and only if each coordinate function $f_\alpha (= \pi_\alpha f)\colon X \to X_\alpha$ is continuous and each $x \in X$ has a neighborhood on which all but a finite number of f_α's are constant.*

Proof. Assume that f is continuous but does not satisfy the stated conditions. Choose $x \in X$ for which these conditions fail. Then we can construct an infinite sequence $\alpha_1, \alpha_2, \ldots$ in Λ and a sequence of positive reals $\varepsilon_1, \varepsilon_2, \ldots$ such that for each i one can find $x_i \in N_{1/i}(x)$ so that

$$d\big(f_{\alpha_i}(x_i), f_{\alpha_i}(x)\big) > \varepsilon_i.$$

It follows that for *no* open set U about x is

$$f(U) \subset \prod_{i=1}^{\infty} N_{\varepsilon_i}\big(f_{\alpha_i}(x)\big) \times \Pi_{\alpha\neq\alpha_i} X_\alpha.$$

Conversely, suppose U_x is an open set about x such that f is constant on U_x for all $\alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If $\Pi_\alpha V_\alpha = V$ is a basic open set in the box topology about $f(x)$, then

$$U = f_{\alpha_1}^{-1}(V_{\alpha_1}) \cap f_{\alpha_2}^{-1}(V_{\alpha_2}) \cap \cdots \cap f_{\alpha_n}^{-1}(V_{\alpha_n}) \cap U_x$$

is an open set about x such that $f(U) \subset V$.

As an immediate corollary we obtain the following:

COROLLARY. *Let X and X_α ($\alpha \in \Lambda$) be metric spaces, X compact. If $f\colon X \to \Pi_\alpha X_\alpha$, let $f_\alpha = \pi_\alpha f$ be the coordinate function of f. Then f is continuous in the box topology if and only if all f_α's are continuous and only a finite number of them are not constant.*

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ANSWERS TO PHOTOS ON PAGE 124

L. E. J. Brouwer (1881–1966) and Errett Bishop (1928–1983), the founding fathers of intuitionism and constructive analysis, respectively.

NOTES

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A RULED MOEBIUS BAND WHICH SELF INTERSECTS IN A STRAIGHT LINE

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A familiar object in elementary Differential Geometry is the Moebius band which is useful in illustrating a nonorientable surface. One can easily make a Moebius band out of a flat oblong strip of paper by giving it a half twist and gluing its ends together. A square piece, however, would refuse to take shape unless it were cut because such a surface self intersects. Any horizontal plane will cut this surface in the pattern of Fig. 2 showing a double point D. This note examines these double points for each arbitrary horizontal plane. It is seen that all these double points of this Moebius surface fall on a straight line.

The Moebius band described above is constructed as a ruled surface by using the straight line generator PQP' shown in Fig. 1. A point Q moves along the circumference of a unit circle centered at O in the xy plane making angle θ with the x axis. A straight line PQP' makes an angle $A = \frac{1}{2}\theta$ with the z axis. As Q traverses one complete revolution around the circumference, θ has changed 360° while A has changed 180°, so that line PQP' reverses direction to $P'QP$. The ruled surface generated by the ruling PQP' experiences a half twist resulting in a Moebius strip. Such a surface is one-sided since the normal at any point Q_0 changes its sense during a complete revolution of Q, i.e., the inside of the surface has become the outside.

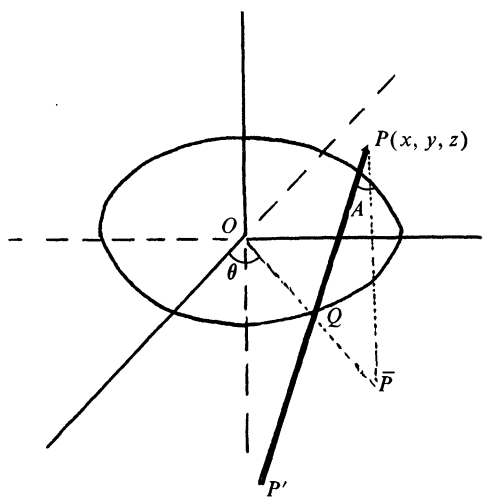


FIG. 1. Generator for PQP'.

For a point P on the surface thus generated, let $\bar{P}$ denote the projection of P onto the xy-plane and $r = |\bar{P}|$. Angle $QP\bar{P}$ equals angle $A = \frac{1}{2}\theta$ so that the point P is located in cylindrical coordinates by the equation

$$\text{(1)} \qquad \operatorname{Tan}\frac{1}{2}\theta = \frac{r-1}{z}$$

which is the equation of the surface. In cartesian coordinates it is

$$(2) \qquad (x^2 + y^2 + z^2)y - 2z(x^2 + y^2) + 2xz - y = 0.$$

An arbitrary horizontal plane $z = z_0$ cuts the surface in a cross sectional curve as seen in Fig. 2.

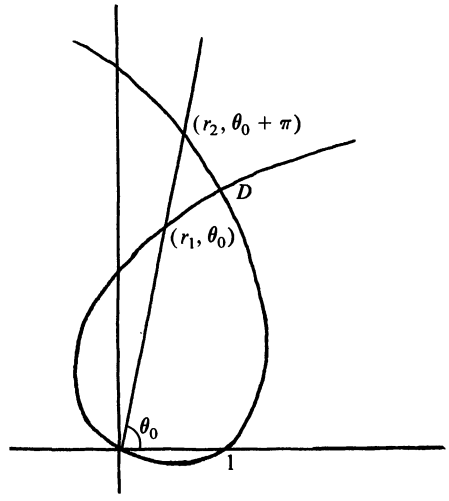


FIG. 2. Cross sectional plane $z = z_0$.

Each ray $\theta = \theta_0$ intersects this locus at the points determined by the two r-values,

$$r_1 = 1 + z \operatorname{Tan} \tfrac{1}{2}\theta$$

and

$$r_2 = 1 + z \operatorname{Tan} \tfrac{1}{2}(\theta_0 + \pi).$$

A double point D occurs when $r_1 = -r_2 \neq 0$ and the curve along which the surface self intersects is clearly the collection of these double points. By using the usual polar-cartesian transformations as well as some elementary algebraic and trigonometric relations, it is now shown that this curve is the straight line $y = z$ and $x = 1$.

The ray $\theta = \theta_0$ along which the double point occurs for an arbitrary horizontal plane $z = z_0$ is found by equating r_1 and $-r_2$:

$$\begin{aligned} 1 + z_0 \operatorname{Tan} \tfrac{1}{2}\theta &= -1 - z_0 \operatorname{Tan} \tfrac{1}{2}(\theta + \pi) \\ &= -1 + \frac{z_0}{\operatorname{Tan} \frac{1}{2}\theta} \end{aligned}$$

which combines to

$$z_0 \operatorname{Tan}^2 \tfrac{1}{2}\theta + 2 \operatorname{Tan} \tfrac{1}{2}\theta - z_0 = 0.$$

Use of the quadratic formula and the tangent of a double angle results in

$$\operatorname{Tan} \theta = \frac{2\left(-1 \pm \sqrt{1 + z_0^2}\right)/z_0}{1 - \left(\dfrac{-1 \pm \sqrt{1 + z_0^2}}{z_0}\right)^2}$$

which reduces to $\operatorname{Tan}\theta = z_0$. But $\operatorname{Tan}\theta$ is also y/x so that $y = z_0 x$.

Now substitution of these values $z = z_0$ and $y = z_0 x$ changes the equation of the surface (2) to

$$z_0 x^3 + z_0^3 x^3 + z_0^3 x - 2z_0 x^2 - 2z_0^3 x^2 + 2z_0 x - z_0 x = 0;$$

separating and factoring

$$z_0 x\left(1 + z_0^2\right)\left(x^2 - 2x + 1\right) = 0$$

whose solution is $x = 1$ (and the trivial case $x = 0$).

Thus $x = 1$ and $y = z_0$ for each arbitrarily chosen horizontal plane $z = z_0$. So all double points of this Moebius surface fall on the line $y = z$ and $x = 1$, which is to say that this Moebius band self intersects along a straight line.

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ANALYTIC CURVES

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By a closed analytic curve, we mean the image of the unit circle, ∂D, under a function f that is analytic in a neighborhood of ∂D. One's intuition is that the complement of such a curve has only a finite number of components. Furthermore, the proof of this fact should be accessible to any student with a semester course in complex variables.

We have not been able to find this proposition stated in the literature. The purpose of this note is to give an argument that falls within the guidelines set forth. We extend our thanks to a number of people who have discussed this problem with us; in particular, John Conway, Carl Cowen, Ken Stephenson, and Jim Thomson.

Before we begin (or before the reader ignores us and embarks on proving the proposition for himself (herself)) let us make a few remarks about the hypotheses of this fact. Can one relax either the hypothesis that ∂D is a circle or the hypothesis that f is analytic in a neighborhood of ∂D? The following two examples show what can go wrong.

Consider the C^∞ curve

$$\gamma(\theta) = \begin{cases} e^{i\theta} & \text{if } \dfrac{\pi}{2} \leqslant \theta \leqslant 2\pi, \\ e^{i\theta}\left(1 + e^{(\theta(\theta-\pi/2))^{-1}\sin\left(\theta\left(\theta - \frac{\pi}{2}\right)\right)^{-1}}\right) & \text{if } 0 < \theta < \dfrac{\pi}{2}. \end{cases}$$

If $f(z) = z^2$ and J denotes the image of this curve, then clearly the complement of $f(J)$ has infinitely many components.

Furthermore, if ϕ is a conformal map of the open unit disc onto int γ, then ϕ extends to a homeomorphism of the closed disc onto the closure of int γ. Thus, ϕ^2 is a disc-algebra function (definition: continuous on the closed disc and analytic on the interior) such that $\phi^2(\partial D)$ is a rectifiable curve, but the complement of $\phi^2(\partial D)$ has infinitely many components.

LEMMA 1. *Let C be a circle in the plane and suppose that ϕ is analytic at $a \in C$. If $\{a_n\}$ is an infinite sequence in C converging to a and such that $\phi(a_n) \in C$ for each n, then there is an $\varepsilon > 0$ such that*

$$\phi\left(B(a, \varepsilon) \cap C\right) \subset C.$$

Proof. No harm is done in assuming that $C = \partial D$. Choose $r > 0$ such that ϕ is analytic on the open ball, $B(a, r)$, with center at a and radius r. Let V be defined by

$$V = B(a, r) \cap \left\{ \lambda : \frac{1}{\bar{\lambda}} \in B(a, r) \right\}.$$

We define an analytic function ψ on the region V by setting

$$\psi(\lambda) = \overline{\phi\left(\frac{1}{\bar{\lambda}}\right)}.$$

Notice that for $\lambda \in V \cap \partial D$, $\psi(\lambda)\phi(\lambda) = |\phi(\lambda)|^2$, and, therefore, $\psi(a_n)\phi(a_n) = 1$ for each n. Thus $\psi\phi = 1$ on V and, in particular, we have $|\phi|^2 = 1$ on $V \cap \partial D$.

The following theorem is the main result concerning the geometry of $f(\partial D)$. We shall denote by $\mathcal{S}$ the set of all points $\lambda \in \partial D$ such that $f'(\lambda) = 0$ or $f|_{\partial D}$ (the restriction of f to ∂D) is not open at λ. Recall that if X and Y are topological spaces, then a function $g\colon X \to Y$ is open at a point x provided that $\{g(U_i)\}$ is a neighborhood base for Y at $g(x)$ whenever $\{U_i\}$ is a neighborhood base for X at x. In this context (referring to the statement $f|_{\partial D}$ is not open at λ) one should take note that $X = \partial D$ and $Y = f(\partial D)$ with the relative topology from $\mathbb{C}$. (So $Y \neq \mathbb{C}$!) The significance of the condition $f|_{\partial D}$ not being open at λ to the context of this paper will become apparent to the reader if one draws a few pictures where $\mathbb{C} \setminus f(\partial D)$ has more than two components. (Loosely speaking, it's our way of getting at the "self crossings" of the curve $f(\partial D)$.)

THEOREM 2. *$\mathcal{S}$ is a finite set.*

Proof. If $f|_{\partial D}$ is not open at λ, then there exist a $\delta > 0$ and a sequence $\{w_n\}$ of points in ∂D such that

$$f(w_n) \to f(\lambda)$$

but

$$f(w_n) \notin f(B(\lambda, \delta) \cap \partial D)$$

for all n. By dropping to a subsequence if need be, we may assume $\{w_n\}$ converges to w, a point in ∂D. To prove the theorem we assume its negation and reach a contradiction.

Suppose then that $\{a_n\}$ is an infinite sequence of distinct points in ∂D such that $f|_{\partial D}$ is not open at each a_n. Without loss of generality, we may assume that $f(a_n) \neq f(a_m)$ whenever $n \neq m$ and that, for all n, we have $f'(a_n) \neq 0$. (The result is clearly true for constant functions.) For each n there exist a $\delta_n > 0$ and a sequence $\{b_{n,k}\}$ of points in ∂D such that

$$\begin{aligned} &f|_{B(a_n, \delta_n)} \text{ is one-to-one,} \\ &f(b_{n,k}) \to f(a_n) \text{ as } k \to \infty, \\ &f(b_{n,k}) \in f(B(a_n, \delta_n) \setminus \partial D), \end{aligned} \tag{1}$$

and there exists b_n so that

$$b_{n,k} \to b_n \text{ as } k \to \infty.$$

By dropping to subsequences if need be, we can assume that the sequences $\{a_n\}$ and $\{b_n\}$ converge to a and b, respectively.

The end of the argument is close at hand if either $f'(a) \neq 0$ or $f'(b) \neq 0$. To see this, we first make the simple observation that if I_i is a homeomorphic image of the unit interval $[0, 1]$ under the function g_i for $i = 1$ and 2 and $I_2 \subset I_1$, then $g_2(0, 1)$ is contained in the interior of I_1. Suppose now that $f'(b) \neq 0$. Let $\varepsilon > 0$ be chosen so that the function $f|_{\overline{B(b, \varepsilon)}}$ is one-to-one. (The set $\overline{B(b, \varepsilon)}$ is the closure of the set $B(b, \varepsilon)$.) By Lemma 1 there exists a positive δ such that

$$f(B(a, \delta) \cap \partial D) \subset f(B(b, \varepsilon) \cap \partial D).$$

For n sufficiently large we may suppose $\overline{B(a_n, \delta_n)} \subset B(a, \delta)$ because we can shrink δ_n if

necessary. Hence, by the observation above, we have that $f(B(a_n, \delta_n) \cap \partial D)$ is an open connected subset of the interior of $f(B(b, \varepsilon) \cap \partial D)$. Observing that $f(b_{n,k}) \to f(b_n)$ as $k \to \infty$, we see that, for large k and large n,

$$f(b_{n,k}) \in f(B(a_n, \delta_n) \cap \partial D),$$

a contradiction. The case that $f'(a) \neq 0$ is even easier and we leave the details to the reader.

We return our attention back to main street. We may assume that the principal branch of the argument, denoted Arg where $-\pi < \operatorname{Arg}(z) \leqslant \pi$, is continuous on neighborhoods of a and b (we rotate the circle if need be). For convenience we assume that the sequences $\{\operatorname{Arg} a_n\}$ and $\{\operatorname{Arg} b_n\}$ are increasing (the proofs of the other cases are similar). In addition, we may normalize f so that $f(a) = 0$.

Let T_i, for $i = 1, 2$, be the Möbius transformations that map the real line, R, onto ∂D such that

$$T_1^{-1}(a) = T_2^{-1}(b) = 0$$

and $T_1^{-1}(a_n)$ and $T_2^{-1}(b_n)$ are positive for all n. Let p and q denote the orders of the zeros of the functions $f \circ T_1$ and $f \circ T_2$ at zero, respectively, and, define

$$g_1 = f \circ T_1 \circ z^q$$

and

$$g_2 = f \circ T_1 \circ z^p.$$

Both functions, g_1 and g_2, have a zero at zero of order $m = pq$. Consequently, there exist $r > 0$ and analytic functions, h_1 and h_2, such that, for each i,

$$h_i(0) \neq 0$$

and, if $|\lambda| < r$, then

$$g_i(\lambda) = \lambda^m h_i(\lambda).$$

By multiplying f by a constant of modulus one, we may assume that the principal branch of the logarithm, denoted Log, is analytic in a neighborhood of $h_1(B(0, \varepsilon)) \cup h_2(B(0, \varepsilon))$ for some positive ε with $\varepsilon \leqslant r$.

Define $\zeta_i(z) = z \exp\left(\frac{1}{m} \operatorname{Log} h_i(z)\right)$ on $B(0, \varepsilon)$, and notice that we may assume that ζ_i is one-to-one on $B(0, \varepsilon)$ since $\zeta_i'(0) \neq 0$. We observe that for $i = 1, 2$

$$\lim_{\substack{z \to 0 \\ z > 0}} \operatorname{Arg} g_i(z) = \operatorname{Arg} h_i(0),$$

and

$$\lim_{\substack{z \to 0 \\ z > 0}} \operatorname{Arg} \zeta_i(z) = \frac{1}{m} \operatorname{Arg} h_i(0).$$

Let $\alpha_n = T_1^{-1}(a_n)$ and $\beta_n = T_2^{-1}(b_n)$ and let $\alpha_n^{1/q}$ and $\beta_n^{1/p}$ be the positive qth and pth roots of α_n and β_n, respectively. It follows then that

$$\begin{aligned}
\operatorname{Arg} h_1(0) &= \lim_{n \to \infty} \operatorname{Arg} g_1\left(\alpha_n^{1/q}\right) \\
&= \lim_{n \to \infty} \operatorname{Arg} f(a_n) \\
&= \lim_{n \to \infty} \operatorname{Arg} f(b_n) \\
&= \lim_{n \to \infty} \operatorname{Arg} g_2\left(\beta_n^{1/p}\right) \\
&= \operatorname{Arg} h_2(0).
\end{aligned}$$

Clearly $\zeta_i^m = g_i$. A moment's thought about the arguments of the ζ_i's then shows that, for n sufficiently large, we have

$$\zeta_1\left(\alpha_n^{1/p}\right) = \zeta_2\left(\beta_n^{1/q}\right). \tag{2}$$

Appealing to the proof of Lemma 1, we can find a positive δ such that

$$\zeta_1[0, \delta] \subset \zeta_2[0, \varepsilon].$$

Hence,

$$g_1[0, \delta] \subset g_2[0, \varepsilon].$$

Choosing ε even smaller (δ will change too, of course), we see that the function $g_2|_{[0,\varepsilon]}$ is one-to-one by the same reasoning used in establishing equality (2).

It now follows, as in the case that $f'(b) \neq 0$, that

$$f(b_{n,k}) \in f\left(B(a_n, \delta_n) \cap \partial D\right)$$

for all large k and n, a contradiction.

COROLLARY 3. *The complement of the closed analytic curve $f(\partial D)$ consists of a finite number of components.*

Proof. Fix one of these components, and denote it by U. Clearly the boundary of U is contained in $f(\partial D)$. Let $a \in (f|_{\partial D})^{-1}(\partial U) \setminus \mathcal{S}$ and choose $\delta_a > 0$ such that the function $f|_{B(a,\delta_a)}$ is one-to-one, $f|_{B(a,\delta_a)\cap\partial D}$ is an open map into $f(\partial D)$, and

$$f(\partial D) \cap f(B(b, \delta_a)) = f\left(B(b, \delta_a) \cap \partial D\right).$$

Since $f(a) \in \partial U$, it follows that either

$$f\left(B(a, \delta_a) \cap D\right) \cap U \neq \varnothing$$

or

$$f\left(B(a, \delta_a) \setminus \overline{D}\right) \cap U \neq \varnothing.$$

If the former case happens, then

$$f\left(B(a, \delta_a) \cap D\right) = f\left(B(a, \delta_a) \cap D\right) \cap U$$

because $f(B(a, \delta_a) \cap D)$ is a connected open subset of the complement of $f(\partial D)$ and U is a component. Hence

$$f\left(B(a, \delta_a) \cap \partial D\right) \subset f(\partial D) \cap \overline{U} = \partial U. \tag{3}$$

Similar reasoning shows that (3) is valid in the case $f(B(a, \delta_a) \setminus \overline{D}) \cap U \neq \varnothing$.

Let I be the open subarc of $\partial D \setminus \mathcal{S}$ that contains the point a. Pick any point $b \in I$ and choose $\delta_b > 0$ such that the function $f|_{B(b,\delta_b)}$ is one-to-one, the function $f|_{B(b,\delta_b)\cap\partial D}$ is an open map into $f(\partial D)$, and

$$f(\partial D) \cap f(B(b, \delta_b)) = f\left(B(b, \delta_b) \cap \partial D\right).$$

Using (3) and a simple compactness argument, one can easily demonstrate that $f(b) \in \partial U$. Hence, either $f(B(b, \delta_b) \cap D) \cap U \neq \varnothing$ or $f(B(b, \delta_b) \setminus \overline{D}) \cap U \neq \varnothing$. But δ_b is chosen so that $f(B(b, \delta_b) \cap D)$ and $f(B(b, \delta_b) \setminus \overline{D})$ are connected open subsets of the complement of $f(\partial D)$. The conclusion of the corollary now follows because there are only a finite number of subarcs of $\partial D \setminus \mathcal{S}$.

We finish by asking the following question: if p is a polynomial (in the variable z) of degree n, then what is a (sharp) bound on the number of components of $\mathbb{C} \setminus p(\partial D)$ in terms of n?

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C E N T E R S E C T I O N
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Telegraphic Reviews

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General, S(13). Eureka! Math Fun From Many Angles. David B. Lewis. Putnam Pub, 1983, 203 pp, $6.95 (P). [ISBN: 0-399-50710-8] A collection of problems and puzzles, most of them fairly well-known. The approach is usually naive. The writing style is amusing. CEC

General, S(17-18), P. Jahrbuch Überblicke Mathematik 1983. S.D. Chatterji, et al. Bibliographisches Institut, 1983, 223 pp, $19.95 (P). [ISBN: 3-411-01650-7] The sixteenth in a series, published annually and devoted to expository papers on various fields of modern mathematics and to the history of mathematics. JD-B

Mathematics Appreciation, S(13-15), L*. Mathematical Time Exposures. I.J. Schoenberg. MAA, 1982, ix + 270 pp, $18; $30 (P). [ISBN: 0-88385-438-4] A slow motion variation--"more time on fewer topics"--on Hugo Steinhaus' classic Mathematical Snapshots: eighteen mostly independent chapters, only half of which use calculus, on topics such as convex sets, skew polygons, billiard ball paths inside a cube, finite Fourier series. Would make an ideal resource for a freshman honors seminar or for Math Club study. LAS

Precalculus, T(13: 1), S, L. College Mathematics. R. Chris Lacher, John L. Bryant. Prindle, Weber & Schmidt, 1982, ix + 544 pp. [ISBN: 0-87150-321-2] Standard pre-calculus topics, including some use of calculators. Ends with some nice applications in linear programming and mathematics of finance. BK

Education, P, L*. Proceedings of the Fourth International Congress on Mathematical Education. Ed: Marilyn Zweng, et al. Birkhauser Boston, 1983, xv + 725 pp, $70. [ISBN: 3-7643-3082-1] Over 100 papers presented at ICME IV (Berkeley, August 1980) representing issues of mathematics education of all types and at all levels: pedagogy, curriculum, applications, educational research, universal education, problem solving, etc. An attractive albeit bulky volume carefully edited and typeset from word-processor equipment. LAS

History, L. Astronomical Cuneiform Texts. Ed: O. Neugebauer. Sources in the History of Math. & Physical Sci., No. 5. Springer-Verlag, 1982, $88 set [ISBN: 0-387-90812-9]. V. I: Introduction; The Moon. xviii + 278 pp; V. II: The Planets; Indices. xii + 232 pp; V. III: Plates, 255 pp. A comprehensive presentation of Babylonian "ephemerides," lists of positions of the sun, moon and five visible planets computed over regular time intervals. Transcriptions and photographs of the ephemerides are in Volume III; critical commentaries are in Volumes I and II. Reprinted with minor additions from a 1955 edition published by Lund Humphries, London. LAS

History, P, L. Collected Papers of J.E. Littlewood. J.E. Littlewood. Oxford U Pr, 1982, $115 [ISBN: 0-19-853353-5]. V. I, xxxviii + 790 pp; V. II, xxxviii + 886 pp. Introduced by a biographical memoir written for the Royal Society by J.C. Burkill, these two volumes contain all of Littlewood's papers arranged in five groups: differential equations, real analysis, number theory, complex analysis, probabilistic analysis and miscellaneous. Most papers are followed by brief comments by the editors providing a modern perspective. LAS

History, S(17-18), P. Leonhard Euler 1707-1783. Birkhauser Verlag, 1983, 555 pp, $29.95. [ISBN: 3-7643-1343-9] Twenty-nine papers on his life and work, and on the history of the attempts to publish his collected scholarly writings. Contains an enormous bibliography of publications about him. JD-B

History, S*(15-16), P, L*. Astronomy and History: Selected Essays. O. Neugebauer. Springer-Verlag, 1983, xii + 538 pp, $19.80 (P). [ISBN: 0-387-90844-7] Forty-three essays reprinted from a wide variety of journals on the methodology and cultural background of ancient and medieval astronomy. Illustrates the role of astronomy in the development of ancient oriental and western science. Most of the essays are accessible to the layman. JK

Foundations, P. Degrees of Unsolvability: Local and Global Theory. Manuel Lerman. Perspectives in Math. Logic. Springer-Verlag, 1983, xiii + 307 pp, $49.50. [ISBN: 0-387-12155-2] If A and B are two unsolvable problems such that any oracle for solving A could also be used to solve B, but not vice versa, then A has a higher degree of unsolvability than B. This treatise on degree theory emphasizes methods of proof, as well as results, and tries to give a feeling for the usefulness of local results in determining global properties. Presupposes graduate level background in recursion theory. GHM

Foundations, P. Abstract AUTOMATH. A. Rezus. Math. Centre Tracts, No. 160. Math Centrum, 1983, vi + 188 pp, Dfl. 25,30 (P). [ISBN: 90-6196-256-0] AUTOMATH is a program for formalizing actual mathematics based on typed lambda-calculus. Though not a programming language, AUTOMATH is motivated by the desire to perform computer-assisted checking of real proofs. This research monograph examines the technical relationship of a certain "pure" subsystem of AUTOMATH to the system as a whole. GHM

Foundations, P. Remarks on the Foundations of Mathematics, Revised Edition. Ludwig Wittgenstein. MIT Pr, 1983, 444 pp, $10.95 (P). [ISBN: 0-262-73067-7] Paperback issue of the revised 1978 issue (TR, May 1979). Contains (translations of) extensive selections which did not appear in the first edition of 1956. GHM

Combinatorics, P. Combinatorics ´81 in Honour of Beniamino Segre. Ed: A. Barlotti, P.V. Ceccherini, G. Tallini. Math. Stud., V. 78. Elsevier North-Holland, 1983, xii + 823 pp, $89.25 (P). [ISBN: 0-444-86546-2] An outgrowth of a conference in honor of B. Segre held in Rome (1980); contents are 76 articles on various themes in combinatorial mathematics including finite geometries, design theory, finite groups, coding, and graph theory. JS

Number Theory, P. Lecture Notes in Mathematics-987: Gauss Sums and p-adic Division Algebras. Colin J. Bushnell, Albrecht Fröhlich. Springer-Verlag, 1983, xi + 187 pp, $10.50 (P). [ISBN: 0-387-12290-7] A study of the theory of congruence Gauss sums attached to irreducible admissable representations of the multiplicative group of a central division algebra over a finite extension of a p-adic rational field. RM

Group Theory, P. Unitary Representations and Coadjoint Orbits of Low-Dimensional Nilpotent Lie Groups. Ole A. Nielsen. Papers in Pure & Appl. Math., No. 63. Queen´s U, 1983, xiii + 117 pp, (P). Coding and cataloging (for characteristic 0) the nilpotent Lie algebras and groups of dimension 6 or less. Includes representation as differential operators, exponential map, adjoint and coadjoint representation, irreducible unitary representation. JS

Algebra, P. Lecture Notes in Mathematics-998: Recent Developments in the Algebraic, Analytical, and Topological Theory of Semigroups. Ed: K.H. Hofmann, H. Jürgensen, H.J. Weinert. Springer-Verlag, 1983, vi + 486 pp, $22 (P). [ISBN: 0-387-12321-0] Proceedings of a May 1981 Oberwolfach conference. LAS

Algebra, P. Angeordnete Strukturen: Gruppen, Körper, projektive Ebenen. S. Priess-Crampe. Ergebnisse der Math., No. 98. Springer-Verlag, 1983, xi + 286 pp, $71. [ISBN: 0-387-11646-X] Study of ordered structures. Only linearly ordered structures (not partially ordered) are considered. Various embedding results. Applications to ordered projective planes and homomorphisms of projective planes. Good bibliography. Expensive. DD

Finite Mathematics, T(14-15: 1), L. Finite Mathematics with Applications for Business and Social Sciences, Fourth Edition. Abe Mizrahi, Michael Sullivan. Wiley, 1983, xiii + 637 pp, $26.95. [ISBN: 0-471-05398-8] Changes from earlier editions are relatively minor. (First Edition, TR, October 1973; Third Edition, TR, June-July 1979.) JS

Calculus, T(14-15: 1, 2), S, L. Mathematical Methods in the Physical Sciences, Second Edition. Mary L. Boas. Wiley, 1983, xx + 793 pp, $32.95. [ISBN: 0-471-04409-1] Changes in the Second Edition include a rewritten chapter on linear algebra, a doubling of the number of problems, high-lighting of important notions and results. Intended for students with one year of calculus who wish to develop a basic competence in several areas of mathematics. Flexible chapter arrangement. Usable in a variety of courses or for review by advanced students. Answers to selected exercises. Solutions manual available. (First Edition, TR, June-July 1967.) JK

Complex Analysis, P. Lecture Notes in Mathematics-981: Value Distribution Theory. Ed: I. Laine, S. Rickman. Springer-Verlag, 1983, 245 pp, $14.50 (P). [ISBN: 0-387-12003-3] Six sets of notes of invited lectures from the Nordic Summer School in Mathematics at Joensuu, Finland, June 1981. The longer papers are devoted to value distribution theory, mostly in several complex variables. PZ

Differential Equations, P. Uniqueness and Non-Uniqueness in the Cauchy Problem. Claude Zuily. Prog. in Math., V. 33. Birkhauser Boston, 1983, xi + 168 pp, $14. [ISBN: 3-7643-3121-6] Surveys (for the professional in partial differential equations) the uniqueness question for the noncharacteristic smooth Cauchy problem for not necessarily hyperbolic differential operators. Chapters cover first order operators on R^n, Calderon´s theorem and its extensions, and uniqueness and pseudoconvexity. Substantial bibliography. PZ

Differential Equations, P. Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, Volume IV. Ed: H. Brezis, J.L. Lions. Research Notes in Math., No. 84. Pitman

Pub, 1983, 379 pp, $27.50 (P). [ISBN: 0-273-08592-1] A collection of 13 papers, 3 of them in French. AWR

Differential Equations, T(16-17: 1), S. Nonlinear Differential Equations. Raimond A. Struble. Krieger Pub, 1983, x + 267 pp, $19.50. [ISBN: 0-89874-056-8] Reprint of a 1962 edition published by McGraw-Hill. For advanced undergraduate or beginning graduate courses. Prepared for the needs of the applied mathematician, engineer and physicist. Not wordy, but sound treatment. Lots of examples. Non-routine exercises extend the text. JK

Differential Equations, T(16-17: 2). Partial Differential Equations of Applied Mathematics. Erich Zauderer. Wiley, 1983, xiii + 779 pp, $49.95. [ISBN: 0-471-87517-1] For advanced undergraduates or beginning graduate students in applied mathematics, the sciences and engineering. Analyzes construction of models of physical processes. Perturbation and asymptotic methods for partial differential equations. Generalized solutions and functions sans heavy dose of functional analysis. Prerequisites include an exposure to the theory of partial differential equations at some elementary level. Many examples. Excellent exercises. Useful as a reference for engineers and scientists. JK

Numerical Analysis, P. Preconditioning Methods: Analysis and Applications. Ed: David J. Evans. Topics in Computer Math., V. 1. Gordon & Breach, 1983, xi + 556 pp, $74.50. [ISBN: 0-677-16320-7] A collection of twenty articles on the analysis and application of preconditioning strategies in the numerical solution of partial differential equations. AO

Functional Analysis. Lecture Notes in Mathematics-991: Banach Space Theory and its Applications. Ed: A. Pietsch, N. Popa, I. Singer. Springer-Verlag, 1983, x + 302 pp, $15.50 (P). [ISBN: 0-387-12298-2] Proceedings of a September 1981 conference in Bucharest. LAS

Functional Analysis, S(18), P. Abschätzungen für Differentialoperatoren im Halbraum. I.W. Gelman, W.G. Mazja. Birkhauser Verlag, 1982, 221 pp, $36.95. [ISBN: 3-7643-1275-0] A monograph for specialists on inequalities in a half-space for differential operators with constant coefficients. JD-B

Functional Analysis, S(18), P. Weak Convergence of Measures. Harald Bergström. Prob. & Math. Stat. Acad Pr, 1982, x + 245 pp, $49.50. [ISBN: 0-12-091080-2] Founded on work of A.D. Alexander, this is a systematic and rather austere development of weak convergence of measures (bounded linear functionals) in various settings: normal spaces, $R^{(k)}$, C- and D-function spaces, and separable Hilbert space. Notes and comments, bibliography, no exercises. JS

Functional Analysis, P. Hamilton-Jacobi Equations in Hilbert Spaces. V. Barbu, G. Da Prato. Research Notes in Math., No. 86. Pitman Pub, 1983, 172 pp, $17.95 (P). [ISBN: 0-273-08597-2] A unified treatment relating to existence and approximation of solutions to Hamilton-Jacobi equations. Presents two interrelated methods: a constructive approximation and dynamic programming. Bibliography. JS

Analysis, T(15-17: 1-3). Advanced Engineering Mathematics, Fifth Edition. Erwin Kreyszig. Wiley, 1983, xvii + 1080 pp, $36.95. [ISBN: 0-471-86251-7] The fifth edition of a popular textbook. This edition places a greater emphasis on modeling and numerical methods. (First Edition, TR, December 1967; Third Edition, TR, October 1972, Extended Review, June-July 1974; Fourth Edition, TR, June-July 1979.) AO

Analysis, P. Lecture Notes in Mathematics-992: Harmonic Analysis. Ed: Giancarlo Mauceri, Fulvio Ricci, Guido Weiss. Springer-Verlag, 1983, x + 449 pp, $22 (P). [ISBN: 0-387-12299-0] Proceedings of a July 1982 conference held in Cortona, Italy, the third in a series of Italian-American conferences on harmonic analysis. (See Lecture Notes in Mathematics-908 for proceedings of the first two conferences.) LAS

Analysis. Harmonic Analysis on Free Groups. Alessandro Figà-Talamanca, Massimo A. Picardello. Lect. Notes in Pure & Appl. Math., V. 87. Dekker, 1983, viii + 145 pp, $35 (P). [ISBN: 0-8247-7042-0] An overview of recent work on the theory of representations and harmonic analysis on free groups. End of chapter notes provide extensive references to current literature. BK

Algebraic Geometry, S, P. Zur algebraischen Geometrie, Selected Papers. B.L. van der Waerden. Springer-Verlag, 1983, vii + 479 pp, $35. [ISBN: 0-387-12180-3] Thirty-three papers, first published from 1926 to 1971 and largely on the foundations of the field. Also a complete list of van der Waerden's publications through 1982. Should fascinate those interested in the history of algebraic geometry. JD-B

Algebraic Geometry, P. Lecture Notes in Mathematics-997: Algebraic Geometry--Open Problems. Ed: C. Ciliberto, F. Ghione, F. Orecchia. Springer-Verlag, 1983, viii + 411 pp, $19.50 (P). [ISBN: 0-387-12320-2] Proceedings of Ravello conference held May 31-June 5, 1982. Papers deal with current problems in algebraic geometry. Many of the papers include interesting and illuminating remarks on the historical origins of classical geometric problems currently being investigated, along with indications of specific problems yet to be resolved. DD

Algebraic Geometry, P. Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday. Ed: Michael Artin, John Tate. Birkhauser Boston, 1983. Volume I: Arithmetic, Progress in Math., V. 35, 359 pp, $29.95 [ISBN: 3-7643-3132-1]; Volume II: Geometry,

Progress in Math., V. 36, 481 pp, $29.95. [ISBN: 3-7643-3133-X] Contributed papers in Volume I concern arithmetic aspects of algebraic geometry (samples: results on Mordell-Weil groups, generators of Neron-Severi groups, height pairings, torsion points on abelian varieties). Volume II papers concern more geometric topics (results on blowing up, crystalline Torelli theorem, Kähler metrics, moduli spaces of curves, etc.). DD

Algebraic Geometry, P. Lectures on Moduli of Curves. D. Gieseker. Springer-Verlag, 1982, iii + 99 pp, $6.40 (P). [ISBN: 0-387-11953-1] Lectures given at the Tata Institute on the moduli problem for curves. Examines the stability (as in geometric invariant theory) of Hilbert points attached to space curves, analogous to Mumford's results on Chow points. Constructs the Deligne-Mumford moduli space of stable curves, proving irreducibility. DD

Differential Geometry, P. Symplectic Geometry. Ed: A. Crumeyrolle, J. Grifone. Research Notes in Math., No. 80. Pitman Pub, 1983, 250 pp, $22.95 (P). [ISBN: 0-273-08575-1] Nineteen papers, all with references, presented at the "Colloque Symplectique de Toulose," under the auspices of the Mathematical Society of France in March 1981. Papers reflect growth during the past fifteen years. JK

Differential Geometry, T(18: 1), S, P. Geometry of Riemannian Spaces. Elie Cartan. Transl: James Glazebrook. Lie Groups, Series A, V. XIII. Math Sci Pr, 1983, xiv + 506 pp, $50. [ISBN: 0-915692-34-1] The original French work dates back to 1926 with a revision in 1951 by Cartan. The present translation was completed in 1979. The preface by Robert Hermann says this is "the best place to begin, both in terms of the historical structure and the needs of a student today who wants an introduction to Cartan's methods." JS

Topology, T(17-18: 1), S, P, L. Topology for Analysis. Albert Wilansky. Kreiger Pub, 1983, xiii + 383 pp, $28.50. [ISBN: 0-89874-343-5] Intended to be used as handbook and reference as well as a text for the novice, the most unusual feature is an extended appendix containing 42 tables showing inter-relationships, examples, and counterexamples for various topological spaces. Text begins with elementary notions, proceeds through compactness, metrization, uniformity, groups, and function spaces. Very strong on examples, exercises, helpful commentary. Bibliography, index. (Original 1970 edition, TR, June-July 1971; Extended Review, December 1971.) JS

Operations Research, P. Operations Research in Progress. Ed: Gustav Feichtinger, Peter Kall. Theory & Decision Lib., V. 32. D Reidel Pub, 1982, ix + 520 pp, $56.50. [ISBN: 90-277-1464-9] Papers presented in a 1980 joint meeting of Swiss and Austrian Operations Research Societies. Divided into two major groupings: mathematical programming and optimal control theory; and stochastic models. Applications of operations research models to energy planning are to be found in both parts. AWR

Operations Research, T(17-18: 1), P. Optimality and Efficiency. D.J. White. Wiley, 1982, xvi + 227 pp, $49.95. [ISBN: 0-471-10223-7] A theoretical treatment of the notion of efficient solutions of multi-objective problems, set in the context of optimality. For students in operations research and management science with some advanced mathematical background. RM

Optimization, P. Optimization: Theory and Algorithms. Ed: Jean-Baptiste Hiriart-Urruty, Werner Oettli, Josef Stoer. Pure & Appl. Math., V. 86. Dekker, 1983, xi + 253 pp, $45 (P). [ISBN: 0-8247-7019-6] Fifteen papers given at a 1981 conference in France. There are three parts: Theory, Algorithms, Applications. Price seems high for a paperback from typed text. AWR

Probability, S*(13), L*. Lady Luck: The Theory of Probability. Warren Weaver. Dover, 1982, 392 pp, $6 (P). [ISBN: 0-486-24342-7] Reprint of the original 1963 edition, which has become a classic. A witty and interesting non-technical introduction to probability. RSK

Statistics, T(18), P, L. The Advanced Theory of Statistics: V. 2: Inference and Relationship, Fourth Edition. Maurice Kendall, Alan Stuart. Macmillan Pub, 1979, x + 748 pp, $59.95. [ISBN: 0-02-847820-7] Revision of the 1973 edition of the second volume of the authors' three-volume treatise (TR, February 1974). Changes are primarily based on recent research. RSK

Statistics, T*(13-14: 1, 2). Introduction to Probability and Statistics, Sixth Edition. William Mendenhall. Duxbury Pr, 1983, xii + 710 pp. [ISBN: 0-87150-365-4] Revision of the 1979 Fifth Edition of this well-known text. Major changes include the following: each chapter now begins with a case study, which is later analyzed; large sample inference has been split into two chapters and now includes p-values; a new chapter on multiple regression has been added; exercises have been revised and the "experiences with real data" have been dropped. (Second Edition, TR, November 1969; Third Edition, TR, April 1972, Extended Review, February 1973; Fourth Edition, TR, November 1975.) RSK

Computer Science, P. Parallel Algorithms on a Multiprocessor. Peter N. Oleinick. Comp. Sci.: Systems Prog., No. 4. UMI Research Pr, 1982, xv + 110 pp, $34.95. [ISBN: 0-8357-1327-X] This text is an advanced research monograph covering the topics of parallel processing methods on multiprocessing computers. The topics covered include performance evaluation, synchronization, and effects of these algorithms on operating system behavior. There is also an extended case study describing the implementation of a speech recognition system using the parallel techniques just described. The text assumes an extensive background in both mathematics and computer science, especially operating systems design and implementation. MS

Computer Programming, T(13). Introducing Pascal. Seymour V. Pollack. Holt, Rinehart & Winston, 1983, x + 371 pp, $14.95 (P). [ISBN: 0-03-060563-6] An introductory programming textbook that presupposes no background in computer science or mathematics. It pays a great deal of attention to the topics of structured programming and good programming habits, as well as to the syntax of the language. It covers all aspects of the language including such advanced features as records, sets, pointers, and recursion. MS

Computer Programming, T(14: 1). Assembler Language with ASSIST, Second Edition. Ross A. Overbeek, W.E. Singletary. Sci Res Assoc, 1983, x + 491 pp, $27.95. [ISBN: 0-574-21435-6] A text for an undergraduate course on assembly language programming and computer organization using the language called ASSIST, a student-oriented assembler developed specifically as a pedagogical tool which runs on the IBM 370 computer under either OS or DOS. The book assumes only a rather elementary background in programming some high level language. Otherwise, no specific hardware or software knowledge is required. MS

Computer Programming, S(14-16). Programming the 6809. Rodnay Zaks, William Labiak. Sybex, 1982, xii + 362 pp, $15.95 (P). [ISBN: 0-89588-078-4] Intended for anyone wishing to do assembly language programming for the Motorola 6809 microprocessor. Complete, including a detailed discussion of the 6809 machine organization, instruction set, and I/O techniques. Contains an introduction to programming including high level programming techniques, and useful applications. RM

Computer Programming, T?(13-14: 1), S. FORTRAN 77. Donald M. Monro. Edward Arnold Pub, 1982, vii + 360 pp, $19.95 (P). [ISBN: 0-7131-2794-5] Primarily language oriented, but gives some emphasis to interactive programming and structured techniques. Discusses common pitfalls encountered by students. Oriented towards scientists and engineers, treating algorithms on matrices (LU factorization, inversion, Gaussian reduction, etc.) in some detail. RM

Computer Programming, S(13). The Timex/Sinclair 1000 Basic Handbook. Douglas Hergert. Sybex, 1983, xvii + 164 pp, $7.95 (P). [ISBN: 0-89588-113-6] Designed especially for the implementation of BASIC on Timex/Sinclair 1000, this book summarizes BASIC commands and gives examples of use together with computer vocabulary with English descriptions. Most beneficial for new computer users. MT

Computer Programming, T(13: 1, 2). Introduction to Computer Programming: A Problem-Solving Approach. John M. Hartling, Larry E. Druffel, F. Jack Hilbing. Digital Pr, 1983, xvi + 468 pp, $24. [ISBN: 0-932376-21-5] A language-independent introduction to computer programming, this text covers the material usually treated in a first programming course such as assignment, expressions, iteration, arrays, etc., but it does it all in pseudo-code rather than in a language like BASIC, FORTRAN, or Pascal. The authors have chosen to concentrate on the broad concepts rather than the specific implementation details. The book is directed at beginning students at the high school or junior college level. MS

Computer Programming, T(14: 1), S*, P*, L. The Structured Alternative: Program Design, Style, and Debugging. Don Cassel. Reston Pub, 1983, xii + 242 pp, $24.95. [ISBN: 0-8359-7084-1] This text is directed at teaching the tools and techniques of problem solving and program development, rather than the syntax of a particular programming language. It addresses such issues as problem definition, top-down design, structured code, and programming style. It would be an ideal book to read concurrently with the study of a specific language. It would also be very helpful for an advanced programmer who wants to learn how to program elegantly as well as correctly. MS

Computer Programming, T(13: 1). Pascal Programming: A Beginner's Guide to Computers and Programming. Chris Hawksley. Cambridge U Pr, 1983, xiii + 188 pp, $24.95; $9.95 (P). [ISBN: 0-521-25302-0; 0-521-27292-0] This is an introductory computer programming text which teaches the language Pascal. It is, in the author's own words, a book for the "real beginner" who is most likely not a computer science student and who has been unable to handle the more rigorous approach of more advanced texts. It is a very "gentle" introduction that introduces the concepts slowly and with simple, non-mathematical examples. Even though it is written at a quite elementary level, it does adequately treat the topics of programming design, programming style, and algorithms. MS

Computer Programming, S(13-14). Writing Pascal Programs. J.S. Rohl. Comp. Sci Texts, No. 16. Cambridge U Pr, 1983, viii + 226 pp, $24.95; $11.95 (P). [ISBN: 0-521-25077-3; 0-521-27196-7] Intended to be used as a supplement to an introductory Pascal textbook, this book does not teach the language but, instead, consists of the development of complete and sometimes quite complex computer programs. These programs are intended to be used as examples to illustrate certain conceptual points about the language, or to illustrate some interesting feature. There are 15 chapters, each containing the complete, annotated development of one program. MS

Computer Programming, T(14: 1). Apple Assembly Language With LazerWare Software. W. Douglas Maurer. Computer Sci Pr, 1984, xiv + 402 pp, $17.95 (P). [ISBN: 0-914894-82-X] This text teaches the principles of assembly language programming and computer organization using the Apple II+, Apple IIe, or Apple III. It is intended for a second course in computer science and presupposes a background in high level language programming. The text is divided into 100 small "units" each of which addresses some small aspect of Apple assembly language. MS

Software Systems, T*(14: 1), L*. Software Design and Development. Philip Gilbert. SRA, 1983, xvi + 681 pp, $32.95. [ISBN: 0-574-21430-5] An introductory textbook on software engineering emphasizing the design and development of small- to medium-scale systems. Appendices include problems in

design, evaluation, and system formulation suitable for use as projects. AO

Software Systems, S*, P, L*. Inside CP/M: A Guide for Users and Programmers with CP/M-86 and MP/M2. David E. Cortesi. Holt, Rinehart & Winston, 1982, xix + 571 pp, $26.45 (P). [ISBN: 0-03-059558-4] A well-written and well-indexed treatment of the CP/M operating system and its standard utilities. It goes beyond the usual in its treatment of generalities concerning computers and hardware and in its provision of technical appendices on such things as the diskette organization. JAS

Software Systems, T(16-17), P. Computer Image Generation. Ed: Bruce J. Schachter. Wiley, 1983, xx + 236 pp, $29.95. [ISBN: 0-471-87287-3] This well written survey describes the algorithms and architectures used in major computer image generation devices, emphasizing visual flight training simulators as the most successful examples to date. Presupposes basic knowledge of computer graphics, but does not delve into coding details. GHM

Software Systems, S, P, L. UNIX Time-Sharing System: UNIX Programmer's Manual, Seventh Edition, Volume 2. Holt, Rinehart & Winston, 1983, vii + 616 pp, (P). [ISBN: 0-03-061743-X] The complete set of 38 reference documents for Version 7 UNIX in five parts: getting started, document preparation, programming, supporting tools and languages, and implementation and maintenance. Concludes with a short glossary and a very helpful index. LAS

Computer Science, T(16-17: 1), P, L. Measurement and Tuning of Computer Systems. Domenico Ferrari, Giuseppe Serazzi, Alessandro Zeigner. Prentice-Hall, 1983, xiii + 523 pp, $35. [ISBN: 0-13-568519-2] An introductory, practically oriented textbook on system performance evaluation. Emphasizes measurement techniques and tuning projects. Includes several case studies. AO

Computer Science, P. Proving Operating Systems Correct. Richard Alan Karp. Computer Science: Systems Prog., No. 16. UMI Research Pr, 1983, ix + 161 pp, $34.95. [ISBN: 0-8357-1365-2] Discusses techniques for the implementation and verification of concurrent systems (e.g., operating systems). A revised version of the author's Ph.D. thesis. AO

Computer Science, P. Lecture Notes in Computer Science-154: Automata, Languages and Programming. Ed: J. Diaz. Springer-Verlag, 1983, viii + 733 pp, $42.50 (P). [ISBN: 0-387-12317-2] Proceedings of the 10th international colloquium sponsored by the European Association for Theoretical Computer Science, covering all theoretical aspects of computer science. LAS

Computer Science, P*. Foundations of Computer Science IV: Distributed Systems. Ed: J.W. de Bakker, J. van Leeuwen. Math Centrum, 1983. Part 1: Algorithms and Complexity, iii + 122 pp, Dfl. 15,40 (P) [ISBN: 90-6196-254-4]; Part 2: Semantics and Logic, ii + 255 pp, Dfl. 34 (P). [ISBN: 90-6196-255-2] A collection of five papers presented at the Fourth Advanced Course on the Foundations of Computer Science in Amsterdam in June, 1982. The papers report on advanced research work in the field of theoretical computer science. Four of the five papers report on work on VLSI algorithms. The other paper is concerned with some theoretical results in distributed computation. MS

Computer Science, P. An Analysis of Pascal Programs. Lynn Robert Carter. Comp. Sci. Systems Prog., No. 6. UMI Research Pr, 1982, xi + 191 pp, $44.95. [ISBN: 0-8357-1331-8] Descriptive statistics are used to analyze the static structure of 89 Pascal programs in an attempt to characterize the structure of such programs. The purpose is to aid future compiler writers to make more enlightened decisions based on how the language is actually used, and to discover in which directions efforts for optimization should be concentrated. The results shed an interesting light on how features of the language are used, at least in the sample studied. RM

Control Theory, S(18), P. Contrôle des systèmes distribués singuliers. J.L. Lions. Gauthier-Villars, 1983, xxiii + 448 pp, 285FF. [ISBN: 2-04-015539-2] On the theory of control of systems whose equations of state are partial differential or integro-differential equations exhibiting singularities (e.g., instability, explosive phenomena or multiple solutions). JD-B

Control Theory, T(17), P. Dynamics of Feedback Systems. A.I. Mees. Wiley, 1981, x + 214 pp, $41.95. [ISBN: 0-471-27822-X] An attractively printed book which extends the view of an Adams prize winning essay, arguing that nonlinear feedback systems can be profitably studied using both the viewpoint of differential equations and of input-output systems. Intended for control engineers, applied mathematicians, and mathematical biologists; written at the beginning graduate student level. AWR

Applications (Artificial Intelligence), P. Pattern Recognition Principles, Fourth Printing. Julius T. Tou, Rafael C. Gonzalez. Appl. Math. & Computation, No. 7. Addison-Wesley, 1981, xxii + 377 pp, $36.50. [ISBN: 0-201-07586-5] Though no longer up-to-date, this is still a nice, systematic introduction to the basic philosophies and techniques of pattern recognition by computer up to 1974. (First Printing, TR March 1976.) GHM

Applications (Computer Engineering), T?(14-15), S, L. From Chips to Systems: An Introduction to Microprocessors. Rodnay Zaks. Sybex, 1981, xvi + 552 pp, $17.95 (P). [ISBN: 0-89588-063] Relatively complete introduction to microprocessors, from individual chip design to complex systems. Informative discussion of individual components, including evaluation of common architectures. Includes sections on interfacing, system development, programming, and applications. Updated version of 1977 edition. RM

Applications (Engineering), T(18: 1). Application and Implementation of Finite Element Methods. J.E. Akin. Comput. Math. and Applic. Academic Pr, 1982, xii + 372 pp, $55. [ISBN: 0-12-047650-9] A textbook on the implementation and design of computational procedures for finite element calculations, meant for a course following an introductory finite element course. The use of isoparametric elements and numerical integration techniques is emphasized. AO

Applications (Information Processing), T(16-17: 1), S, P, L. Digital Document Processing. H.S. Hou. Wiley, 1983, xii + 329 pp, $34.95. [ISBN: 0-471-86247-9] Provides an introduction to digital image processing, pattern recognition, digital typography, document retrieval, text processing, digital graphics, and computer communications as they relate to digital document processing. AO

Applications (Physics), S*, P*, L*. Quarks: The Stuff of Matter. Harald Fritzsch. Basic Books, 1983, xiii + 297 pp, $19. [ISBN: 0-465-06781-6] A sophisticated yet non-mathematical display of the fruits of recent high energy research into subnuclear structure: bosons, baryons, leptons, gluons, quarks, fermions. Deals in a matter-of-fact style with the potential of gauge groups such as SU(5) and SO(10) as the basis for a unified theory of matter. LAS

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Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

New Jersey Section

The fall meeting of the New Jersey Section was held on November 5, 1983 at Kean College of New Jersey, Union. There were 67 registrants.

Invited Addresses:

"Elliptic Functions Revisited," by Linda Keen, Lehman College, CUNY.
"Mathematical Understanding Through Problem-Solving," by Alan Schoenfeld, University of Rochester.
"Bringing the Black Chamber Into the Classroom: Cryptology as an Academic Discipline," by Cipher Deavours, Kean College of New Jersey.

North Central Section

The fall meeting of the North Central Section was held at South Dakota State University on October 28-29, 1983. The attendance was 78.

Invited Addresses:

"Boolean-Valued Set Theory," by Kay Smith, St. Olaf College.
"On the Utilization of Achievement and Avoidance Games for Teaching Mathematical Concepts," by Frank Harary, University of Michigan.

Contributed Papers:

* "Nonlinear Asymptotes to Graphs of Algebraic Functions," by Dan Kalman, Augustana College.
"Problem Solving in Computer Science," by Bill Marion, College of St. Catherine.
"School Teacher Retraining Program at NDSU," by Ronald Mathsen and Warren Shreve, North Dakota State University.
* "Undistinguished Algebra," by Murray Braden, Macalester College.
* "f(qx) = qf(x)-Monthly Problem 6419," by Michael B. Gregory, University of North Dakota.
"Mathematics Education in China," by Clayton Knoshaug, Bemidji State University.
* "Some Interesting Properties of the Matrix [[1,1], [1,1+x]]," by Gerald Bergum, South Dakota State

University.
* "1683-1983: Three Hundred Years of Determinants," by Walter Sizer, Moorhead State University.
"Graduate Mathematics Education at the University of Minnesota," by Robert Hardt, University of Minnesota.

Student Paper:

"Asymmetric Silverman Games on Discrete Sets," by R. Dow Rieder, Concordia College.

Ohio Section

The fall meeting of the Ohio Section was held on November 4-5, 1983 at Baldwin-Wallace College, Berea. The theme was differential equations. There were 103 registrants.

Invited Addresses:

"In Search of Stability in a Complex World," by Clark Robinson, Northwestern University.
"Some Models of Glacial Flows," by James Greenberg, Ohio State University.
"Mathematical Education in Mainland China," by James Leitzel, Ohio State University.
"Stability of Trojan Satellites," by Ken Meyer, University of Cincinnati.
"Intermediate Value Theorems and Differential Equations," by Alan Lazer, University of Miami and University of Cincinnati.

Short Presentations:

* "A Relationship Between the Modified Euler Method and e," by Thomas P. Dence, Bowling Green University.
* "Detecting Floating Overflow: Condition Handlers on a VAX," by Al Stickney, Wittenberg University.
* "A Model of the Recombination of ions of n Molecules," by William Wagner, Department of Public Welfare.
* "Knotted Elastica and Elastic Knots," by David Singer, Case Western Reserve University.
* "On Resonant Hamiltonian Systems," by Martin Kummer, University of Toledo.
* "Another Stability Property for Non-linear Ordinary Differential Equations," by Otomar Hajek, Case Western Reserve University.
* "Non-linear Neutron-Flux Equations for Fission Reaction-Diffusion: Steady States and Asymptotic Behavior," by Anthony Leung, University of Cincinnati.
* "Stability of a Barter Economy," by Boulem Bendjilali, University of Cincinnati.
"Traveling Waves for Forced Fisher's Equation," by Larry Turyn, Wright State University.

Panel Discussions:

"Discrete Mathematics, What Are You Doing?" by Charles Hampton (Moderator), College of Wooster.
"CUPM Recommendations on Teaching Differential Equations," by James Greenberg (Moderator), Ohio State University.
"FIPSE," by Stephen Slack (Moderator), Kenyon College.

Seaway Section

The fall meeting of the Seaway Section was held on November 4-5, 1983 at Niagara University. There were 69 registrants.

Invited Addresses:

"Some Work with Gifted High School Mathematics Students," by Gerald Rising, State University of New York at Buffalo.
"The Conway Criterion," by Doris Schattschneider, Moravian College, and Editor of Mathematics Magazine.

Short Presentations:

"The Problem of Problem Setting," by Peter J. O'Halloran, Executive Director of the Australian Mathematics Competition, and Visiting Professor at the University of Waterloo.
"Solving Inequalities," by Lawrence A. Trivieri, Mohawk Valley Community College.
"Minimal Sets and Ergodic Measures for βG," by Mostafa Nassar, State University of New York Center at Buffalo.
"On the Groups that Contain Subgroups Having Finite Indexes," by Ter-Jenq Huang, State University of New York College at Cortland.
"Do We Really Teach Problem Solving?" by Ruth Heintz, State University of New York College at Buffalo.
"A Program for Underprepared Mathematics Students," by Wendy L. Dunignan, Niagara University.
"A Counting Problem mod p," by Gerald Myerson, State University of New York Center at Buffalo.

AN UNSYMMETRIC FUBINI THEOREM

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Some of the most basic theorems of analysis involve interchanging the order of limiting processes. The Fubini Theorem is one of the most frequently used of these results. Fubini's Theorem asserts that under very general conditions we have the formulas

$$\int_Y \left[\int_Z \psi(y,z)\, d\sigma(z)\right] d\gamma(y) = \int_Z \left[\int_Y \psi(y,z)\, d\gamma(y)\right] d\sigma(z)$$

$$= \int_{Y\times Z} \psi(y,z)\, d(\sigma\times\gamma)(y,z).$$

Note that two equalities are involved and that essentially symmetric roles are played by the measures σ and γ. In the theorem below we give an unsymmetric Fubini Theorem involving one equality (equation (2) below) rather than two and a family of measures $\{\sigma_y : y \text{ in } Y\}$ rather than a single measure σ. In case $\sigma_y = \sigma$ for every y, the result reduces to a part of the familiar Fubini Theorem.

This result was worked out in the course of some work on the Feynman integral [8]. Several illustrations of how the result can be used can be found in that paper. Some of these applications occur in the middle of rather technical proofs, but others, notably in section 3 of [8], can easily be read independently. Special cases of the result below have been found by several authors. We are aware of a number of instances of this [1], [2], [3], [6], [9], [10] and suspect that there have been others.

The results and terminology which we use without explanation can be found in either of the standard references [4] or [11]. The measures involved are allowed to be complex valued (or C-valued). See [11, p. 117] for the definition of the total variation of such a measure. Given a subset E of $Y \times Z$ and y in Y, $E^{(y)}$ denotes the y-section of E; that is, $E^{(y)} = \{z \text{ in } Z : (y,z) \text{ is in } E\}$.

Proposition 1 begins to establish the measurability necessary for the theorem.

PROPOSITION 1. *Let $(Y, \mathcal{Y}, \gamma)$ be a σ-finite measure space and let $(Z, \mathcal{Z})$ be a measurable space. For γ-a.e. y, let σ_y be a C-valued, countably additive measure on $(Z, \mathcal{Z})$ of finite total variation $\|\sigma_y\|$. Suppose that, for any B in $\mathcal{Z}$, $\sigma_y(B)$ is a $\mathcal{Y}$-measurable function of y. Then for any E in the product σ-algebra $\mathcal{Y}\times\mathcal{Z}$, $\sigma_y(E^{(y)})$ is a $\mathcal{Y}$-measurable function of y.*

Proof. The Monotone Class Theorem [4, p. 27] is frequently useful and will be the key here. Let $\mathcal{C} = \{E \text{ in } \mathcal{Y}\times\mathcal{Z} : \sigma_y(E^{(y)}) \text{ is } \mathcal{Y}\text{-measurable in } y\}$. Let $E = A \times B$ be a measurable rectangle.

$$(A\times B)^{(y)} = \{z \text{ in } Z : (y,z) \text{ is in } A\times B\} = \begin{cases} B & \text{if } y \text{ is in } A,\\ \phi & \text{if } y \text{ is not in } A.\end{cases}$$

Hence $\sigma_y((A\times B)^{(y)}) = \sigma_y(B)\chi_A(y)$. But this is a measurable function of y, and so $A \times B$ is in $\mathcal{C}$.

Now let $E_1, \ldots, E_n$ be pairwise disjoint elements of $\mathcal{C}$. Then

$$\sigma_y\left((E_1 \cup \cdots \cup E_n)^{(y)}\right) = \sigma_y\left(E_1^{(y)} \cup \cdots \cup E_n^{(y)}\right) = \sigma_y\left(E_1^{(y)}\right) + \cdots + \sigma_y\left(E_n^{(y)}\right)$$

and so it is clear that $\mathcal{C}$ is closed under finite disjoint unions.

Now suppose that $\{E_n\}$ is a sequence from $\mathcal{C}$ such that $E_1 \supseteq E_2 \supseteq \cdots$. Let $E = \cap_{n=1}^{\infty} E_n$. For each y, $E_1^{(y)} \supseteq E_2^{(y)} \supseteq \cdots$ and $E^{(y)} = \cap_{n=1}^{\infty} E_n^{(y)}$. Hence $\sigma_y(E_n^{(y)}) \to \sigma_y(E^{(y)})$. Thus $\sigma_y(E^{(y)})$ is measurable being the pointwise limit of a sequence of measurable functions. The case of a monotone increasing sequence of sets can be handled similarly.

Thus $\mathcal{C}$ is a monotone class containing the algebra $\mathcal{A}$ of finite disjoint unions of measurable

rectangles and so, by the Monotone Class Theorem, $\mathcal{C} \supseteq \sigma(\mathcal{A}) = \mathcal{Y} \times \mathcal{Z}$. Hence $\mathcal{C} = \mathcal{Y} \times \mathcal{Z}$ as desired.

Proposition 2 is the key measurability result needed for the theorem. It will be obtained from Proposition 1 by a common procedure: first use a linearity argument and then a limiting argument.

PROPOSITION 2. *Let the assumptions of Proposition* 1 *be satisfied. Then for any bounded, C-valued,* $\mathcal{Y} \times \mathcal{Z}$ *measurable function* $\psi(y, z)$ *on* $Y \times Z$, $\int_Z \psi(y, z)\, d\sigma_y(z)$ *is a* $\mathcal{Y}$*-measurable function of* y.

Proof. First suppose $\psi(y, z) = \chi_E(y, z)$ where E is in $\mathcal{Y} \times \mathcal{Z}$. Then

$$\int_Z \psi(y, z)\, d\sigma_y(z) = \int_Z \chi_{E^{(y)}}(z)\, d\sigma_y(z) = \sigma_y(E^{(y)}),$$

and this is a measurable function of y by Proposition 1. The result follows easily for ψ a simple function by a linearity argument. For general ψ, let $\{\psi_n\}$ be a sequence of $\mathcal{Y} \times \mathcal{Z}$-measurable simple functions such that $\|\psi_n\|_\infty \leqslant \|\psi\|_\infty$ and $\psi_n \to \psi$ uniformly. By the Dominated Convergence Theorem, for γ-a.e. y,

$$\int_Z \psi(y, z)\, d\sigma_y(z) = \lim_{n \to \infty} \int_Z \psi_n(y, z)\, d\sigma_y(z).$$

Since $\int_Z \psi(y, z)\, d\sigma_y(z)$ is the pointwise limit of a sequence of $\mathcal{Y}$-measurable functions, it is itself $\mathcal{Y}$-measurable.

We now come to the unsymmetric Fubini Theorem.

THEOREM. *Let the assumptions of Proposition* 1 *be satisfied and suppose, in addition, that* $\|\sigma_y\| \leqslant h(y)$, *where* h *is in* $L_1(Y, \mathcal{Y}, \gamma)$. *Then if* μ *is defined on* $\mathcal{Y} \times \mathcal{Z}$ *by*

$$\mu(E) := \int_Y \sigma_y(E^{(y)})\, d\gamma(y), \tag{1}$$

μ *is a C-valued, countably additive measure on* $\mathcal{Y} \times \mathcal{Z}$ *with* $\|\mu\| \leqslant \|h\|_1$.

Further, if $\psi(y, z)$ *is bounded and* $\mathcal{Y} \times \mathcal{Z}$ *measurable, then* $\int_Z \psi(y, z)\, d\sigma_y(z)$ *is in* $L_1(Y, \mathcal{Y}, \gamma)$, *and we have*

$$\int_Y \left[\int_Z \psi(y, z)\, d\sigma_y(z)\right] d\gamma(y) = \int_{Y \times Z} \psi(y, z)\, d\mu(y, z). \tag{2}$$

Proof. For every E in $\mathcal{Y} \times \mathcal{Z}$, $\sigma_y(E^{(y)})$ is $\mathcal{Y}$-measurable by Proposition 1. Since $|\sigma_y(E^{(y)})| \leqslant \|\sigma_y\| \leqslant h(y)$, $\sigma_y(E^{(y)})$ is in $L_1(Y, \mathcal{Y}, \gamma)$. Thus $\mu(E)$ exists for every E in $\mathcal{Y} \times \mathcal{Z}$. To show that μ is countably additive, let $\{E_n\}$ be a pairwise disjoint sequence from $\mathcal{Y} \times \mathcal{Z}$. Then, for each y, $\{E_n^{(y)}\}$ is a pairwise disjoint sequence from $\mathcal{Z}$. Further, for each integer N, $\sum_{n=1}^N |\sigma_y(E_n^{(y)})| \leqslant \|\sigma_y\| \leqslant h(y)$. Hence, by the Dominated Convergence Theorem,

$$\begin{aligned}
\mu\left(\bigcup_{n=1}^\infty E_n\right) &= \int_Y \sigma_y\left(\left(\bigcup_{n=1}^\infty E_n\right)^{(y)}\right) d\gamma(y) = \int_Y \sigma_y\left(\bigcup_{n=1}^\infty E_n^{(y)}\right) d\gamma(y) \\
&= \int_Y \left(\sum_{n=1}^\infty \sigma_y(E_n^{(y)})\right) d\gamma(y) = \sum_{n=1}^\infty \int_Y \sigma_y(E_n^{(y)})\, d\gamma(y) \\
&= \sum_{n=1}^\infty \mu(E_n).
\end{aligned}$$

To show that $\|\mu\| \leqslant \|h\|_1$, let $E_1, \ldots, E_n$ be any finite, pairwise disjoint sequence from $\mathcal{Y} \times \mathcal{Z}$. By definition of the total variation of a measure, it suffices to show that $\sum_{j=1}^n |\mu(E_j)| \leqslant \|h\|_1$. Since

$\Sigma_{j=1}^{n}|\sigma_y(E_j^{(y)})| \leqslant \|\sigma_y\| \leqslant h(y)$, we can write

$$\sum_{j=1}^{n}|\mu(E_j)| = \sum_{j=1}^{n}\left|\int_Y \sigma_y\left(E_j^{(y)}\right)d\gamma(y)\right| \leqslant \sum_{j=1}^{n}\int_Y\left|\sigma_y\left(E_j^{(y)}\right)\right|d\gamma(y)$$

$$= \int_Y \sum_{j=1}^{n}\left|\sigma_y\left(E_j^{(y)}\right)\right|d\gamma(y) \leqslant \|h\|_1.$$

We know from Proposition 2 that $\int_Z\psi(y,z)\,d\sigma_y(z)$ is $\mathcal{Y}$-measurable. Also $|\int_Z\psi(y,z)\,d\sigma_y(z)| \leqslant \|\psi\|_\infty\|\sigma_y\| \leqslant \|\psi\|_\infty h(y)$, and so $\int_Z\psi(y,z)\,d\sigma_y(z)$ is in $L_1(Y,\mathcal{Y},\gamma)$.

For $\psi(y,z) = \chi_E(y,z)$, E in $\mathcal{Y}\times\mathcal{Z}$, formula (2) reduces to (1). (2) follows by linearity for simple functions. Given a bounded $\mathcal{Y}\times\mathcal{Z}$ measurable function ψ, let $\{\psi_n\}$ be a sequence of simple functions bounded by $\|\psi\|_\infty$ such that $\psi_n \to \psi$ uniformly. Note that

$$\left|\int_Z\psi_n(y,z)\,d\sigma_y(z)\right| \leqslant \|\psi\|_\infty\|\sigma_y\| \leqslant \|\psi\|_\infty h(y) \in L_1(Y,\mathcal{Y},\gamma).$$

This justifies the second of the three uses of the Dominated Convergence Theorem in the following string of equalities which will complete the proof:

$$\int_{Y\times Z}\psi(y,z)\,d\mu(y,z) = \lim_{n\to\infty}\int_{Y\times Z}\psi_n(y,z)\,d\mu(y,z)$$

$$= \lim_{n\to\infty}\int_Y\left[\int_Z\psi_n(y,z)\,d\sigma_y(z)\right]d\gamma(y)$$

$$= \int_Y\left[\lim_{n\to\infty}\int_Z\psi_n(y,z)\,d\sigma_y(z)\right]d\gamma(y)$$

$$= \int_Y\left[\int_Z\psi(y,z)\,d\sigma_y(z)\right]d\gamma(y).$$

It seems likely that the result can be extended in various ways, but our theorem seems general enough to cover most situations of interest and simple enough to avoid measure theoretic technicalities which might obscure the basic simplicity of the result.

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ANOTHER PROOF THAT A_5 IS SIMPLE

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Finite group theory is fundamentally sophisticated counting. One of the high points of an undergraduate abstract algebra course or beginning group theory course is the utilization of Sylow's Theorems and various counting arguments to show the nonexistence of simple groups of specific orders. For example, Fraleigh [2] shows that there is no simple group of order 20, 30, 36, or 48 and gives 96 and 160 as exercises; Saracino [6] does 24, 28, and 30 and gives 56 and 200 as exercises. Rose [5] does all integers $\leqslant 100$, except 60, and gives 132, 144, 300, and 1000 as exercises; Scott [7] does 264, 420, 1008, and 1080 and gives all integers $\leqslant 200$, except 60 and 168, and a dozen more above 200 as exercises. Herstein [4] gives all integers < 60 as exercises. In contrast, when it comes to proving A_5 (the group of 60 even permutations on 5 symbols) is simple, nearly every author argues as follows (see [3, p. 92] for details): Suppose there is a nontrivial normal subgroup N of A_5. There are three possible cycle structures for the elements of N, and in each case, one may conjugate by a suitably chosen element of A_5 to show that N contains all three cycles. Since the three cycles generate A_5, we have $N = A_5$.

In this note, we offer a counting argument that A_5 is simple. Two elementary observations are needed first.

LEMMA. *Let N be a normal subgroup of a finite group G. If $x \in G$ and $(o(x), |G/N|) = 1$, then $x \in N$.*

Proof. Since $o(xN)$ divides both $o(x)$ and $|G/N|$, we have $o(xN) = 1$. Thus, $xN = N$ and $x \in N$.

LEMMA. *A_5 has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.*

Proof. Observe that there are $(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/5 = 24$ elements of the form (abcde); observe that there are $(5 \cdot 4 \cdot 3)/3 = 20$ elements of the form (abc); and observe that there are

$$\frac{1}{2}\left[\left(\frac{5 \cdot 4}{2}\right)\left(\frac{3 \cdot 2}{2}\right)\right] = 15$$

elements of the form $(ab)(cd)$ where (ab) and (cd) are disjoint cycles.

THEOREM. *A_5 is simple.*

Proof. Suppose N is a nontrivial proper normal subgroup of A_5. Then, by Lagrange's Theorem, $|N| = 2, 3, 4, 5, 6, 10, 12, 15, 20$, or 30. If $|N| = 3, 6, 12$, or 15, then $(3, |A_5/N|) = 1$ and N must contain the 20 elements of order 3. If $|N| = 5, 10$, or 20, then $(5, |A_5/N|) = 1$ and N must contain the 24 elements of order 5. If $|N| = 30$, then N must contain the 20 elements of order 3 and the 24 elements of order 5. If $|N| = 4$, then $(2, |A_5/N|) = 1$ and N must contain the 15 elements of order 2. Finally, if $|N| = 2$, then $|A_5/N| = 30$. But, Sylow's Theorem and a counting argument show that any group of order 30 must have a normal subgroup of order 3 or a normal subgroup of order 5 (otherwise, the group would have 20 elements of order 3 and 24 elements of order 5). Then, the pullback of this normal subgroup of A_5/N is a normal subgroup of A_5 of order 6 or 10. However, we have already shown that A_5 has no such subgroups. Thus, in each case for $|N|$, we have reached a contradiction. This proves A_5 is simple.

Since the above proof of the simplicity of A_5 relies on arithmetical arguments, rather than properties of permutations, it applies equally well to the group of rotations of a dodecahedron and the group of rotations of an icosahedron. One need only precede the proof given above with the following observation which is readily seen from models (see also [1, p. 170]).

LEMMA. *The group of rotations of a dodecahedron and the group of rotations of an icosahedron have 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2.*

Recall that SL(2, 7) is the special linear group of 2×2 matrices of determinant 1 over the integers modulo 7. Also, PSL(2, 7) is the factor group of SL(2, 7) by $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right\}$, and PSL(2, 7) has order 168.

EXERCISE. Use a counting argument to prove PSL(2, 7) is simple.

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A GENERALIZED PROBLEM OF LEAST SQUARES

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In regression analysis [1], the process of determining an interpolation function for a given set of data under the least squares criterion, one assumes the independent variables to be exact and not subject to error or variation. For instance, in elementary statistics one usually uses vertical deviations. In this context x is called the independent variable which is under control and assumed to be exact (occasionally, some textbooks [2] mention using horizontal deviations; in this situation x and y interchange roles). If one allows all the variables to reflect error, the problem of determining the "best-fitting" function for a given set of data becomes more complicated. In this paper I will consider this generalized least squares problem for the determination of an interpolation function in two variables in detail.

Consider a set of n points $(x_1, y_1), \ldots, (x_n, y_n)$. Without loss of generality, one can assume that the n points are noncollinear. The generalized linear interpolation function for the data is the line

$$y - b - mx = 0, \tag{1}$$

such that the sum of squares of the perpendicular distances Σd_j^2 (j runs from 1 to n) from the given n points to the line (1) is a minimum, where

$$d_j = |y_j - b - mx_j| \big/ \sqrt{1 + m^2}\,.$$

The object is to determine the parameters b and m in (1) which minimize Σd_j^2.

First note that x_j and y_j are the fixed numbers which one has observed. Denote

$$f(b, m) = \Sigma d_j^2 = \frac{1}{1 + m^2}\Sigma\left(y_j - b - mx_j\right)^2.$$

$$\partial f/\partial b = 0 \text{ and } \partial f/\partial m = 0 \text{ give} \tag{2}$$

$$b = \bar{y} - m\bar{x},$$

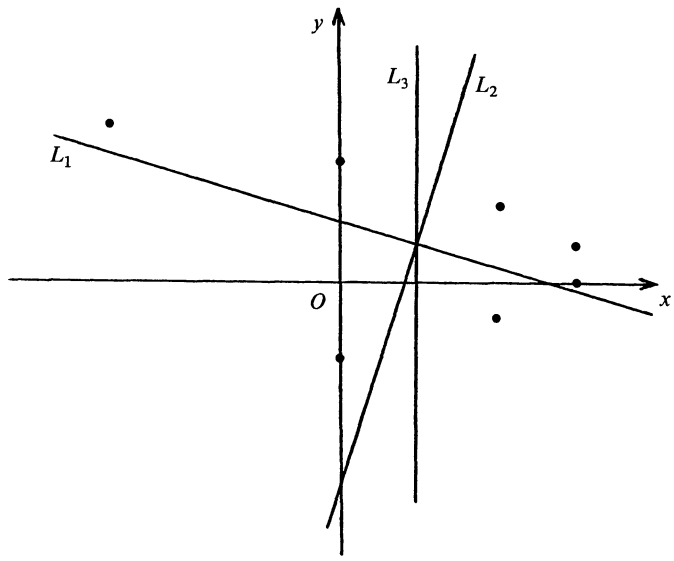
and

$$Am^2 - Bm - A = 0, \tag{3}$$

where $\bar{x}$ and $\bar{y}$ are the arithmetic means of $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ respectively, and

$$A = \Sigma x_j y_j - n\bar{x}\bar{y} \text{ and } B = \Sigma y_j^2 - n\bar{y}^2 - \Sigma x_j^2 + n\bar{x}^2.$$

Let $\hat{m}_1$ and $\hat{m}_2$ be the roots of (3). If $A \neq 0$, then



From Eq. (2) one obtains $\hat{b}_1 = -2 + \sqrt{13}$ and $\hat{b}_2 = -2 - \sqrt{13}$,
The three lines

$$L_1: 2y - (3 - \sqrt{13})x + 2(2 - \sqrt{13}) = 0$$

$$L_2: 2y - (3 + \sqrt{13})x + 2(2 + \sqrt{13}) = 0$$

and

$$L_3: x - 2 = 0. \text{ (See figure.)}$$

The lines L_1, L_2, and L_3 give the values of Σd_j^2, 19.52, 120.48 and 112 respectively.

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THE TEACHING OF MATHEMATICS

EDITED BY MARY R. WARDROP AND ROBERT F. WARDROP

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WHY THE PRODUCT TOPOLOGY?

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A student is often puzzled when he or she first encounters the product topology. The usual topology on an infinite product $\Pi_\alpha X_\alpha$ of topological spaces X_α has as a basis the sets of the form $\Pi_\alpha U_\alpha$, where each U_α is open *and* $U_\alpha = X_\alpha$ for all but finitely many values of α. Why this last

restriction? Why not take as a basis all the sets of the form $\Pi_\alpha U_\alpha$ where each U_α is open in X_α (as is done in the case of finite product)? This latter is a basis for a topology often called the box topology; see [1, p. 113]. While the box topology agrees with the product topology when the number of spaces is finite, it is quite different for infinite products.

In the product topology, the product of connected spaces is connected. This is not true in the box topology. Consider, for example, $\mathbb{R}^\omega$—the countable product of real lines. The set

$$A = \{(x_1, x_2, \ldots) | \{x_i\} \text{ is a bounded sequence}\}$$

is both open and closed in the box topology, and thus $\mathbb{R}^\omega$ is not connected. (See [1, p. 152].)

In the product topology the product of compact spaces is compact. This, too, fails in the box topology. Consider I^ω—the countable product of copies of the unit interval, I. If $A_0 = [0, 1)$ and $A_1 = (0, 1]$, then the collection of all open sets of the form $A_{\varepsilon_1} \times A_{\varepsilon_2} \times \cdots$, where $\varepsilon_i = 0$ or 1, is an uncountable open cover of I^ω with no proper subcover. For if $A_{\varepsilon_1} \times A_{\varepsilon_2} \times \cdots$ is excluded from the cover, the point $(\varepsilon_1, \varepsilon_2, \ldots)$ is not covered.

Continuous functions $f\colon X \to \Pi_\alpha X_\alpha$ in the product topology are characterized as being those whose coordinate functions $f_\alpha (= \pi_\alpha f)\colon X \to X_\alpha$ are each continuous. To see that this characterization does not hold when the box topology is used, consider the function $f\colon \mathbb{R} \to \mathbb{R}^\omega$ defined by $f(x) = (x, x, x, \ldots)$. An easy argument shows that f is not continuous (see [1, p. 115]).

The following provides a characterization of continuous functions into product spaces with the box topology.

THEOREM. *Let X and X_α ($\alpha \in \Lambda$) be metric spaces. A function $f\colon X \to \Pi_{\alpha\in\Lambda} X_\alpha$ is continuous in the box topology if and only if each coordinate function $f_\alpha (= \pi_\alpha f)\colon X \to X_\alpha$ is continuous and each $x \in X$ has a neighborhood on which all but a finite number of f_α's are constant.*

Proof. Assume that f is continuous but does not satisfy the stated conditions. Choose $x \in X$ for which these conditions fail. Then we can construct an infinite sequence $\alpha_1, \alpha_2, \ldots$ in Λ and a sequence of positive reals $\varepsilon_1, \varepsilon_2, \ldots$ such that for each i one can find $x_i \in N_{1/i}(x)$ so that

$$d\big(f_{\alpha_i}(x_i), f_{\alpha_i}(x)\big) > \varepsilon_i.$$

It follows that for *no* open set U about x is

$$f(U) \subset \prod_{i=1}^{\infty} N_{\varepsilon_i}\big(f_{\alpha_i}(x)\big) \times \Pi_{\alpha \neq \alpha_i} X_\alpha.$$

Conversely, suppose U_x is an open set about x such that f is constant on U_x for all $\alpha \notin \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$. If $\Pi_\alpha V_\alpha = V$ is a basic open set in the box topology about $f(x)$, then

$$U = f_{\alpha_1}^{-1}(V_{\alpha_1}) \cap f_{\alpha_2}^{-1}(V_{\alpha_2}) \cap \cdots \cap f_{\alpha_n}^{-1}(V_{\alpha_n}) \cap U_x$$

is an open set about x such that $f(U) \subset V$.

As an immediate corollary we obtain the following:

COROLLARY. *Let X and X_α ($\alpha \in \Lambda$) be metric spaces, X compact. If $f\colon X \to \Pi_\alpha X_\alpha$, let $f_\alpha = \pi_\alpha f$ be the coordinate function of f. Then f is continuous in the box topology if and only if all f_α's are continuous and only a finite number of them are not constant.*

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ANSWERS TO PHOTOS ON PAGE 124

L. E. J. Brouwer (1881–1966) and Errett Bishop (1928–1983), the founding fathers of intuitionism and constructive analysis, respectively.

THE DISK AND SHELL METHOD

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In most calculus books there is little effort given to showing that the cylindrical shell method and disk method give the same value when computing the volume of a solid of revolution. Indeed it is not obvious that these two distinct methods should give the same result. In some texts this is demonstrated when the trapezoid bounded by the x-axis, $y = mx + b$, $x = a$ and $x = b$ is revolved about the y-axis.

In this paper we shall show that the cylindrical shell and disk methods give the same value if the region revolved about the y-axis is bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis, provided $f(x)$ is a differentiable function on $[a, b]$ and $f(x)$ is one-to-one. The proof is simple and uses two theorems which the students have recently learned (substitution formula and integration by parts). This proof can easily be included in a calculus course.

Consider the solid of revolution K produced by revolving the region bounded by $y = f(x)$, $x = a$, $x = b$ and the x-axis, about the y-axis. We use the shell method, which involves summing the volumes of cylindrical shells, to define the volume of K to be $\lim_{\|P\|\to 0}\sum_{i=1}^{n} 2\pi x_i f(x_i)\,\Delta x_i$. If $f(x)$ is differentiable on $[a, b]$ and hence continuous there, this limit exists and is equal to $\int_a^b 2\pi x f(x)\,dx$.

Suppose the region is bounded by the function $x = g(y)$, $y = c$, $y = d$ and the y-axis. In the disk method, which involves summing the volumes of disks, we consider

$$\lim_{\|P\|\to 0}\sum_{i=1}^{n}\pi\left[g(y_i)\right]^2\Delta y_i.$$

If $g(y)$ is continuous on $[c, d]$, this limit exists and is equal to $\int_c^d \pi[g(y)]^2\,dy$.

THEOREM. *Let $0 \leqslant a < b$, and let $y = f(x)$ be differentiable, nonnegative and $1-1$ on $[a, b]$, with $f(a) = c$ and $f(b) = d$ where $c < d$.* Also let $f'(x)$ be continuous on $[a, b]$ and let $x = g(y)$ iff $y = f(x)$. If R is the region bounded by $y = f(x)$, the x-axis, $x = a$ and $x = b$ and R is revolved about the y-axis, then the value obtained by using the disk method is equal to the value obtained by using the cylindrical shell method. Equivalently $\pi[b^2d - a^2c] - \int_c^d \pi[g(y)]^2\,dy = \int_a^b 2\pi x f(x)\,dx$.*

Proof. The region R can also be described as the region bounded by $x = m(y)$, $x = b$, $y = 0$ and $y = d$, where

$$m(y) = \begin{cases} a & \text{if } 0 \leqslant y < c, \\ g(y) & \text{if } c \leqslant y \leqslant d. \end{cases}$$

We observe from the way that $m(y)$ is defined that it is continuous on $[0, d]$. If we evaluate the volume obtained by revolving the region R about the y-axis by using the disk method, we find this to be $\int_0^d \pi b^2\,dy - \int_0^d \pi m(y)\,dy$. This is equal to

$$\int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi[b^2d - a^2c] - \int_c^d \pi[g(y)]^2\,dy.$$

By the substitution formula, the latter expression is equal to $\pi[b^2d - a^2c] - \int_a^b \pi x^2 f'(x)\,dx$. A straightforward application of the integration by parts formula and algebraic simplification shows that

$$\int_a^b 2\pi x f(x)\,dx = \pi[b^2d - a^2c] - \int \pi x^2 f'(x)\,dx,$$

and the argument is complete.

*The result is true in case $d < c$, but a slight alteration is needed in the argument.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS), H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the address given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by June 30, 1984. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3035. *Proposed by L. E. Mattics, University of South Alabama.*

Let p be an odd prime. For $1 \leqslant m \leqslant p-1$ and $0 \leqslant i \leqslant p-1$ set

$$s(m,i) = \binom{mp}{i} - \binom{mp}{i+p} + \binom{mp}{i+2p} - \cdots.$$

Prove

$$s(m,0) \equiv 0 \pmod{p^{m+1}} \qquad \text{for} \qquad 1 \leqslant m \leqslant p-2,$$

and

$$s(m,i) \equiv p^m(-1)^{i+1} m! i^{-m} \pmod{p^{m+1}} \qquad \text{for} \qquad i > 0.$$

($s(p-1,0) \equiv p^{p-1} \pmod{p^p}$ is established in E 2685 [1977, 820; 1979, 131–132].)

E 3036. *Proposed by F. Lazebnik, University of Pennsylvania and Y. Pilipenko, Kiev University, USSR.*

Define a sequence $\{a_n\}$ by $a_1 = a$ and $a_{n+1} = a_n^2 - 2$. For which values of a does this sequence converge?

E 3037. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus.*

Find all positive integers n such that $\phi(n) | n$.

E 3038. *Proposed by Tetsundo Sekiguchi, University of Arkansas.*

Let ABC be a triangle in the plane. Show that

$$\sin A + \sin B + \sin C \leqslant \frac{3\sqrt{3}}{2} \quad \text{and} \quad \sin A \sin B \sin C \leqslant \frac{3\sqrt{3}}{8}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

The Diophantine Equation $1 + 3^a = 5^b + 3^c$

E 2929 [1982, 131]. *Proposed by Leo J. Alex, State University College, Oneonta, NY.*

Find all solutions to the equation $1 + 3^a = 5^b + 3^c$ in integers a, b, c.

Solution by J. Suck, Essen, Germany. The only solutions in integers are $(a, b, c) = (a, 0, a)$, $(3, 2, 1)$.

Proof. Suppose (a, b, c) is a third solution. Obviously $a, b, c > 0$. Clearly $b \neq 0$, so that $3^a \equiv 4 + 3^c \bmod 20$, and $(a, c) \equiv (3, 1) \bmod 4$. Hence $5^b \equiv 9 \bmod 16$, $b \equiv 2 \bmod 4$. Using mod 13, we conclude $(a, c) \equiv (0, 1) \bmod 3$. Thus $a \equiv 3$, $c \equiv 1 \bmod 6$, and $5^b \equiv 4 \bmod 7$, so that $b \equiv 2 \bmod 6$. If $c > 1$, this is a contradiction mod 9. If $c = 1$, $a > 3$, then $5^b \equiv -2 \bmod 81$, so that $b \equiv 20 \bmod 5^4$. This shows that $3^a \equiv 5^{20} + 2 \equiv 37 \bmod 109$, which is impossible. (Note that 3, 5 have period 27 modulo 109.)

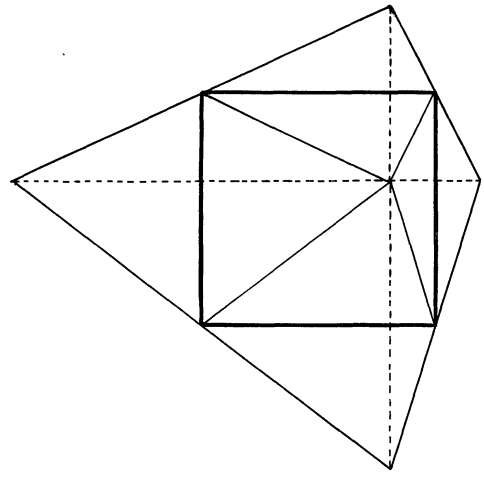
Also solved by V. Hernandez (Spain), O. P. Lossers (Netherlands), V. D. Mascioni (student, Switzerland), M. R. Modak (India), University of South Alabama Problem Group, C.-T. Yen (Taiwan), and the proposer.

Square Inscribed in a Triangle

E 2930 [1982, 131]. *Proposed by the editors.*

Find the largest square that can be inscribed in some triangle of area 1.

Solution by O. P. Lossers, Eindhoven I. T., Netherlands, I. J. Schoenberg, University of Wisconsin, and F. B. Strauss, U.T.E.P., El Paso, TX. If a point of a square is reflected in each side, the images are the vertices of a quadrilateral having the vertices of the square at the midpoints of its sides and of area twice as large as the square. (A proof without words, requiring only the recognition of congruent triangles, is appended below.)



Any half-plane containing the square contains at least three of the reflective images of the point. Hence any intersection of three such half-planes contains at least one of the images. In particular, any triangle in which the square is inscribed contains at least one reflective image of every point in the square. Hence, if the regions of the triangle outside the square are reflected about the sides of the square, each point of the square is covered at least once. Thus the area of the part of the triangle outside the square is at least as large as that of the square, i.e., the area of the square is at most half the area of the triangle.

If the point to be reflected is chosen on the boundary of the square, then three of its images are collinear and the quadrilateral is in fact a triangle circumscribed to the square and of twice the square's area.

In summary, the area of a square inscribed into a triangle of area one is at most one half and there exist triangles of area one admitting an inscribed square of area one half: The largest square we set out to find has area one-half.

Also solved by B. Andersen (Denmark), P. H. Anderson, R. Bagby, D. M. Bloom, D. Bode (Germany), S. Bulman-Fleming (Canada), P. L. Chabot, C. W. Dodge, J. Dou (Spain), M. P. Eisner, M. Gikas, Maxim Goldberg, Michael Goldberg, D. Halprin (Australia), V. Hernandez (Spain), L. Hoehn, L. Kuipers (Switzerland), L. H. Lange, H. J. Ludwig, C. Lyons, D. Mick & Olk, W. A. Newcomb, C. E. Nolan, I. Paasche (Germany), C. F. Pinzka, D. A. Rawsthorne, J. W. Reed (Canada), J. Schaer (Canada), T. Sekiguchi, M. Sholander, R. A. Shore, M. Skalsky, R. S. Stacey (Germany), K. L. Stellmacher, B. A. Troesch, University of South Alabama Problem Group, D. M. Wells, J. Wiener, M. Woltermann, C.-T. Yen (Taiwan), and P. Y. Wu (Taiwan).

Lange referred to G. D. Chakerian and L. H. Lange, Geometric extremum problems, Math. Mag., 44 (1971) 57–69, and M. T. Bird, Maximum rectangle inscribed in a triangle, Math. Teacher, 64 (1971) 759.

Schoenberg gave several solutions and complements.

Polynomial in Two Variables

E 2940 [1982, 273]. *Proposed by Andrew Leonard, Indiana University.*

If $f(x, y)$ is defined for all real x, y, and if f is a polynomial in x for every y, and a polynomial in y for every x, prove that f is a polynomial in two variables.

Editorial note: This problem appeared previously in this MONTHLY as 4897 [1960, 295]. The results appeared in F. W. Carroll's Mathematical Note entitled "A polynomial in each variable separately is a polynomial," this MONTHLY, 68 (1961) 42. For further references see 4897 [1961, 187].

Representations of n^2 as Distinct Power Sums

E 2951 [1982, 424]. *Proposed by Barbara Turner, California State University, Long Beach.*

Let $d_k(n) = \sum_{j=1}^{n}(2j-1)^k$, the sum of the kth powers of the first n odd positive integers, for $k \geqslant 0$, $n \geqslant 1$. Show that if p and q are to be relatively prime, $p > q > 0$, the only nontrivial solution which satisfies $[d_k(n)]^p = [d_m(n)]^q$ for all n is the identity

$$\left[\sum_{j=1}^{n}(2j-1)^0\right]^2 = \left[\sum_{j=1}^{n}(2j-1)^1\right]^1.$$

Solution by Nicholas Tzanakis, University of Crete, Greece. The hypothesis $[d_k(n)]^p = [d_m(n)]^q$ for every $n \geqslant 1$, is stronger than necessary. In fact, we shall prove the following result: If

$$[d_k(2)]^p = [d_m(2)]^q, \quad \text{i.e., } (1+3^k)^p = (1+3^m)^q \tag{1}$$

where $0 < q < p$ and $(p, q) = 1$, then $k = 0$, $p = 2$; $m = 1$, $q = 1$.

First consider the case $q > 1$. Put $p = qs + r, 0 < r < q$. From (1), it is clear that $1 + 3^k = c^q$ and $1 + 3^m = c^p$, with c an integer > 1 and, on the other hand, $m > k$. Therefore $c^q - 1 | c^p - 1$. But now

$$c^p - 1 = c^r(c^{qs} - 1) + c^r - 1,$$

from which we conclude that $c^q - 1 | c^r - 1$ and this is, clearly, impossible.

Next, consider the case $q = 1$.

If $k = 0$, then $1 + 3^m =$ power of 2, and since $m \geqslant 1$, we see mod 3 that, in fact, this must be an even power of 2. Put $1 + 3^m = 2^{2t}$ so that $2^t + 1 = 3^m$ and $2^t - 1 = 1$. Then $t = 1$, $m = 1$ and, in view of (1), $p = 2$. Thus we found the solution $k = 0$, $p = 2$; $m = 1$, $q = 1$.

If $k > 0$, we first exclude the value $p = 2$. Indeed, the relation $(1 + 3^k)^2 = 1 + 3^m$ implies $3^k(2 + 3^k) = 3^m$, which is impossible, since $2 + 3^k$ is not divisible by 3. Thus $p > 2$ and on expanding the binomial in the left-hand side of (1) we get

$$3^m = p \cdot 3^k + \sum_{j=2}^{p-1} \binom{p}{j} 3^{kj} + 3^{kp}. \tag{2}$$

Suppose that $3^r \| p (r \geqslant 0)$ (in general, $a^x \| b$ means that x is the greatest power of a dividing b). Take any $j = 2, \ldots, p-1$ and let $3^s \| j (s \geqslant 0)$. Then, since $\binom{p}{j} = \binom{p-1}{j-1} p/j$, it follows that the exponent of 3 in the canonical decomposition of $\binom{p}{j} 3^{kj}$ is at least $r + kj - s$. However,

$$r + kj - s > k + r \tag{3}$$

because

$$kj \geqslant \begin{cases} k3^s & \text{if } s > 0, \\ 2k & \text{if } s = 0. \end{cases}$$

Thus, in view of (3),

$$\sum_{j=2}^{p-1} \binom{p}{j} 3^{kj} \equiv 0 \pmod{3^{k+r+1}}. \tag{4}$$

Note also that

$$kp \geqslant \begin{cases} k3^r & \text{if } r \geq 0, \\ 3k & \text{if } r = 0 \text{ (since } p > q > 2), \end{cases}$$

so that $kp > k + r$, and we have

$$3^{kp} \equiv 0 \pmod{3^{k+r+1}}.$$

This relation, together with (4), proves that the greatest power of 3 dividing the right-hand side of (2) is 3^{k+r}. Therefore, in view of (2), $m = k + r$. But this is a contradiction since, by (2), $3^{k+r} = 3^m > p3^k \geqslant 3^{k+r}$. The proof of the stated result is now complete.

Also solved by M. D. Ašić, K. L. Bernstein, R. Breusch, L. L. Foster, F. Gerrish (England), V. Grinberg, W. Janous (Austria), L. Jones, L. C. Larson, O. P. Lossers (The Netherlands), A. Markovich (student, Yugoslavia), V. D. Mascioni (student, Switzerland), J. B. M. Melissen (The Netherlands), W. A. Newcomb, I. J. Schoenberg, J. Schwaiger (Germany), R. E. Shafer, J. Suck (Germany), J. Weiner, and the proposer.

Additional Acknowledgments

E 2719. Also solved by A. M Cohen (Wales).
E 2799. Also solved by R. Stong.
E 2865. Also solved by B. W. Brunson, S. K. Venkatesan.
E 2871. Also solved by L. H. de Figueiredo (Brazil).
E 2876. Also solved by D. V. Widder.
E 2879. Also solved by D. Singmaster (England).
E 2887. Also solved by K. L. Bernstein, N. Glick.
E 2896. Also solved by L. L. Foster, R. Morelli, S. Singh.
E 2897. Also solved by R. Morelli.
E 2907. Also solved by S. Singh.
E 2920. Also solved by M. Bencze (Rumania).
E 2921. Also solved by M. Bencze (Rumania).
E 2925. Also solved by M. Bencze (Rumania).
E 2929. Also solved by M. R. Modak (India).
E 2934. Also solved by J. W. Fickett.
E 2940. Also solved by A. Boghossian (Saudi Arabia), N. Martin.

ADVANCED PROBLEMS

Solutions of these Advanced problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by June 30, 1984. The solver's full post-office address should be on each sheet.

6451. *Proposed by C. L. Mallows, Bell Laboratories, Murray Hill, NJ.*

Construct a function f on $(0,\infty)$ that satisfies $f(0) = 0$, $f(1) = 1$, $f(x) = f(2 - x)$ for $0 < x < 2$, $f'(x) = 2f(2x)$ for $x > 0$, and $f(x) \not\to 0$ as $x \to \infty$.

6452. *Proposed by Giovanni Emmanuele and Alfonso Villani, Catania, Italy.*

Construct a sequence of nonnegative random variables $\{\xi_n\}_{n\geq 1}$ such that $\sup_{n\geq 1}\mathbb{E}\xi_n^p < \infty$ for all $p > 0$, and $\mathbb{P}(\sup_{j\geq 1}\xi_{n_j} < \infty) = 0$ whenever $1 \leqslant n_1 < n_2 < \cdots < n_j < \cdots$.

6453. *Proposed by Richard Askey, University of Wisconsin, Madison.*

Show that, for any real x,

$$\lim_{\lambda\to\infty}\prod_{n=0}^{\infty}\left(1 + \frac{\lambda x^2}{(n+\lambda)^2}\right) = e^{x^2}.$$

SOLUTIONS OF ADVANCED PROBLEMS

A Commutative Ring

6395 [1982, 502]. *Proposed by P. M. Cohn, Bedford College, London, England.*

Let R be a commutative ring and $A = (a_{ij})$ an $n \times n$ matrix over R. Let S be the ring generated by R and n^2 indeterminates x_{ij} such that, on writing $X = (x_{ij})$, the relations $AX = XA = I$ (in matrix form) hold.

Is S necessarily commutative?

Solution by Anon, Erewhon-upon-Spanish River. Yes. Let C denote the center of S. If $r \in R$, then $rA = Ar$, so $rX - Xr = (XA)(rX) - Xr = Xr(AX) - Xr = Xr - Xr = 0$, hence $r \in C$, so $R \subseteq C$. If $s \in S$, then $sA = As$ since $a_{ij} \in R \subseteq C$. Therefore $sX - Xs = XA(sX) - Xs = Xs(AX) - Xs = Xs - Xs = 0$, hence s also commutes with all generators of S, so $s \in S$, $S \subseteq C$, $S = C$.

Also solved by A. G. Heinicke (Canada) and the proposer.

A Sum of Probability Distributions

6396 [1982, 502–503]. *Proposed by N. A. Marlow, American Telephone and Telegraph Company, and M. Tortorella, Bell Laboratories.*

Suppose F is a right-continuous probability distribution function on $\mathbb{R}$ with $\mu = \int_{\mathbb{R}} x\, dF(x)$, $-\infty < \mu < \infty$. Put

$$F_1 = F, F_2(x) = \int_{\mathbb{R}} F(x-u)\, dF(u) = F * F, \quad \text{and } F_{n+1} = F_n * F, n = 2,3,\ldots.$$

(a) If $\mu \neq 0$, show that

$$\sum_{n=1}^{\infty}\left[F_n(n\mu) - F_{n+1}(n\mu)\right] \tag{1}$$

diverges.

(b) If $\mu = 0$ and $\sigma^2 = \int_{\mathbf{R}} x^2\, dF(x) < \infty$, show that (1) converges.

Solution by Ellen Hertz, United States Army Operational Test & Evaluation Agency, Falls Church, Virginia. Let $X_1, X_2, \ldots$ be independent random variables each having F as distribution function; let $S_n = X_1 + X_2 + \cdots + X_n$, and let I be the interval $(-\mu, 0]$ or $(0, -\mu]$ according as μ is positive or negative.

(a) The stated result is only partially true. We have

$$\sum_{k=1}^{n} \left[F_k(k\mu) - F_{k+1}(k\mu)\right] - F_1(\mu) + F_{n+1}(n\mu) = \operatorname{sgn}\mu \sum_{k=2}^{n} P(S_k - k\mu \in I).$$

According to W. Feller, *An Introduction to Probability Theory and Its Applications*, vol. 2, pp. 200–203, the distribution of $X_1 - \mu$ is persistent, and this implies that

$$\sum_{k=2}^{\infty} P(S_k - k\mu \in I) = \infty$$

when the distribution of $X_1 - \mu$ is either not arithmetic or it is arithmetic with an atom in I. Consequently, in either of these cases, the series (1) is divergent. When the distribution by $X_1 - \mu$ is arithmetic with no atoms in I, the series (1) may converge; for example, the sum of series (1) is 0 when $\mu < 0$ and $F(x) = 0$ for $x < \mu$, $F(x) = 1$ for $x \geqslant \mu$. In this case the distribution of $X_1 - \mu$ has 0 as its only atom and $0 \notin (0, -\mu]$.

(b) We have

$$\sum_{k=1}^{n} \left[F_k(0) - F_{k+1}(0)\right] = F_1(0) - F_{n+1}(0)$$

$$= F_1(0) - P(S_{n+1} \leqslant 0) \to F_1(0) - \frac{1}{2}$$

by the central limit theorem.

Partially solved by Gaston Giroux (Canada), Victor Hernandez (Spain), O. P. Lossers (The Netherlands), K. Aswath Rao, and the proposer. These did not deal with the possible convergence of (1) when $\mu \neq 0$.

Continuous Functions on the Unit Disk

6397 [1982, 503]. *Proposed by Edgar Feldman and Michael Vulis, CUNY Graduate Center, New York.*

(i) Let $D = \{x \in \mathbb{R}^n \mid \|x\| < 1\}$ be the unit disk, and let $f\colon D \to D$ be a diffeomorphism of D such that

(a) $f(f(x)) = x$ for every $x \in D$, i.e., $f \circ f = \mathrm{id}$;

(b) there exists a neighborhood U of 0 in D such that $f|_U = \mathrm{id}$.

Show that $f = \mathrm{id}$.

(ii)* Is the theorem true when f is not a diffeomorphism, but a homeomorphism?

Solution by B. Schaaf, Kerkstraat 65, Mierlo, Holland. The answer to (ii) is yes as follows from the lemma proved below. Let Σ be the set of Lebesgue measurable sets in D, and let m denote Lebesgue measure on D normalized so that $m(D) = 1$.

LEMMA. *Let $T\colon D \to D$ be a continuous function such that*

(α) $T^r = \mathrm{id}$ *for some integer* $r \geqslant 2$;

(β) *there exists a set* $A \in \Sigma$ *such that* $m(A) > 0$ *and* $T|_A = \mathrm{id}$.

Then $T = \mathrm{id}$.

Proof. It follows easily from (α) that T is bijective and hence bicontinuous. Further, T satisfies the conditions of a theorem of de Guzmán [2] which shows that given $V \in \Sigma$ there is a function J integrable and nonnegative on V such that $m(TV) = \int_V J(y)\,dy$. Hence if $m(TV) > 0$, then $m(V) > 0$, and it follows by (α) that $m(V) > 0$ if and only if $m(T^{-1}V) > 0$. Define an operator P on $L_1 = L_1(D, \Sigma, m)$ by setting $Pg = g \circ T$ for $g \in L_1$. Then (D, Σ, m, P) is a Markov process (see Foguel [1]). Furthermore, the process is conservative; for if not, then by result 2.6 in [1], there is a set $B \in \Sigma$ such that $T^{-1}B \subset B$ and $m(B - T^{-1}B) > 0$, so that

$$B = T^{-r}B \subset T^{-r+1}B \subset \cdots \subset T^{-1}B \subset B,$$

contrary to the condition $m(B - T^{-1}B) > 0$. Now let

$$g^*(x) = \sum_{j=0}^{\infty} (P^j g)x \quad \text{for } x \in D.$$

Then, by result 2.4 in [1], if $0 \leqslant g \in L_\infty(D, \Sigma, m)$, then either $g^*(x) = 0$ a.e. on D or $g^*(x) = \infty$ a.e. on D. Taking

$$g(x) = \sum_{j=0}^{r-1} |T^{j+1}(x) - T^j(x)| \quad \text{for } x \in D,$$

we get that $(Pg)x = g(x)$ and $g(x) = 0$ on A, and hence that $g^*(x) = 0$ on A, a set of positive measure. Therefore, $g^*(x) = 0$ a.e. on D, and so $g(x) = 0$ a.e. on D. On account of the continuity of T, it follows that $|T(x) - x| = 0$ on D, i.e., $T = \mathrm{id}$.

References

1. S. R. Foguel, The Ergodic Theory of Markov Processes, Van Nostrand Mathematical Studies, 21.
2. M. de Guzmán, A change-of-variables formula without continuity, this MONTHLY, 87 (1980) 736–739.

Also solved by A. V. Nabutovskii (U.S.S.R.), who observed that a result of M. H. A. Newmann (*A theorem on periodic transformations of spaces*, Quart. J. Math., 2 (1931) 1–9) leads easily to a version of the above lemma in which D is a connected topological n-manifold, T is a homeomorphism on D, and A is an open subspace of D.

Part (i) was also solved by the Chico Problem Group and the proposers.

A Metric Space Having a Given Group of Isometries

6398 [1982, 503]. *Proposed by Edgar Feldman and Michael Vulis, CUNY Graduate Center, New York.*

Is it true that for any group G of cardinality less than or equal to that of the continuum, one can construct a metric space M such that G is its full group of isometries? *What if $|G|$ has greater power?

Solution by the proposers. Assume that the cardinality of G is at most that of the continuum. Choose $G \times \{0,1\}$ as the underlying set for the metric space M and define a distance function on this set. Define all nonzero distances on $G \times \{0\}$ to be equal to r_1 and on $G \times \{1\}$ to be equal to r_2, where $r_1 \neq r_2$. Define the distances between $(a,0)$ and $(a,1)$ to be equal to a third constant r_3. All other distances will be different from r_1, r_2, r_3. Define distances between pairs (a, i) and (b, j) where $i \neq j$ so that $d((a,0),(b,1)) = d((x,0),(y,1))$ if and only if there exists $g \in G$ such that $x = ga$ and $y = gb$. This can be done since the cardinality of G is no larger than that of the continuum. Moreover, all nonzero distances can be chosen to be between 1 and 2 in order to have the triangle inequality.

Now we are guaranteed that any isometry of M sends $G \times \{0\}$ and $G \times \{1\}$ onto themselves and that if $(a,0)$ is sent to $(b,0)$, then $(a,1)$ must go to $(b,1)$, for every $a, b \in G$.

It is easy to check that M has G as its full group of isometries.

The question of what happens when G has cardinality greater than that of the continuum remains open.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Introduction to Higher Algebra. By Maxime Bocher. Dover, New York, 1964 (reprint of a 1907 MacMillan publication). xi + 321 pp.

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Reviewing a book written about 75 years ago surely is a challenging task: what new can be said about old mathematics? Well, there is one thing: in spite of its age, Bocher's book is alive, interesting, and a delight to read! Of course, the reader must fit the book into the temper of its time, but even so, the "classical" (by necessity) theorems are still important, and the neatness of their proofs compares well with the best of modern exposition. Perhaps an objective criterion of the book's continuing significance is the response to this question: Did I gain anything that I didn't already know from a reading of Bocher? This is to certify that I did indeed learn facts not previously in my possession, quite a few [mostly involving polynomials in several variables], and ones that probably I would not otherwise have met. I suspect that at least some other mathematicians would have a similar experience.

Having read Bocher's book through to its last pages, reaching the end in a growing aura of elegance, I turned to the writing of this review. Glowing with praise it was to be! As the writing progressed, however, I gradually became aware of certain criticisms that could be voiced against Bocher's choice of material. Most of my objections, though, were addressed by Bocher at some point in the book, and always in the same way: author's choice! (Examples will be given below.) The book is surely meritorious, even by 1983 criteria, but it *can* be criticized, and by either 1907 or 1983 standards.

Of course, one should not expect ideas and styles developed since 1907 to be in Bocher. There are, for example, no exact sequences; everything is in characteristic zero and usually over the complex numbers, maps between spaces are usually secondary to coordinate manipulations (but maps *are* present), determinants are very visibly to the forefront. Surprisingly, Bocher does have the word "category," as in "category of pairs of bilinear forms" and "category of collineations." Two of the ways in which the temper of its time show in Bocher's text are: (i) complex coefficients are assumed when Bocher makes *no* statement about the intended coefficients; and (ii) although determinants are freely used, with only a brief discussion of their properties [the definition is *not* given], there is a careful introduction to matrices and matrix algebra.

For whom did Bocher write his book? According to the preface, for the "American student approaching the higher parts of mathematics," with the book "intended... really... as an introduction to higher algebra" and "not as a compendium." Bocher goes on to say that while no technical preparation is needed beyond quadratic equations, a knowlege of determinants, and induction, the book is not for the wholly immature reader; two or three years training in the elements of higher mathematics are desirable. His clean and concise style fits this statement. In today's scenario, a junior level mathematics major at any large American university would presumably meet Bocher's prerequisites. So it would be appropriate to compare Bocher's book against the best known of today's algebra texts for juniors and seniors, for example, against Herstein's well known *Topics in Algebra.* Jacobson's recent *Basic Algebra I* might also be a fair comparison text, although perhaps placed above Bocher's intended level. These choices are suggested upon noting that Bocher's book probably was as widely used in its heyday as Herstein's and Jacobson's books are now.

A quick inspection of the Bocher and Herstein books reveals some common topics and some others in one, but not both. Common topics include polynomial theory, linear algebra, and groups, but the two books usually do not emphasize the same aspects. Example: Symmetric

polynomials are prominent in Bocher as objects interesting in their own right, but occur in Herstein only through the connection with Galois theory. Second example: Invariants (e.g., under the action of a set of linear transformations) are repeatedly emphasized in Bocher, less so in Herstein, and in Bocher are classified as "geometrical," "arithmetical," or "algebraic." Geometrical invariants are points, lines, or metric properties such as perpendicularity, all of which are presumably coordinate free concepts; invariants that are integers are arithmetical; and algebraic invariants are usually polynomials in coordinates, so fall under what is now called classical invariant theory. The invariants in Herstein are those of linear algebra (elementary divisors, inertia) and of Galois theory (fixed fields). Bocher has more linear algebra invariants than Herstein, but doesn't have the connection with finite abelian groups (which Herstein does.)

A major difference between the books is that Bocher omits Galois theory, Herstein includes it. Another difference is the degree of emphasis on rings and modules—these are central concepts for Herstein, they go unmentioned in Bocher.

Some of the differences are authors' choices, of course, but others reflect the passage of time. An example of author's choice is the statement in Bocher's preface that "no apology seems necessary for the omission of... Galois's theory" [and no apology is given]. The emphasis on rings and modules in Herstein probably reflects the passage of time. (The terms "ring" and "module" were already part of mathematical language by 1900, but principally in the context of algebraic number theory. Time has extended their use to wider areas of mathematics, including linear algebra.) Also, Herstein has the Wedderburn theorem on the commutativity of finite division rings, but this was proved only in 1905, certainly too late for mention in Bocher's 1907 book.

Stylistic differences include the more chatty format of Herstein, Bocher's frequent use of projective coordinates, and the polish of Herstein's abstract reasoning. But note: coordinates *are* interesting, at least they are in Bocher's skillful hands. And while both books include geometrical ideas to support the algebra, they do so in rather different ways. (More on this below.) Finally, Bocher's treatment of groups is much lighter than Herstein's, to be expected perhaps, since Bocher elects to omit Galois theory.

With respect to the geometry, the two books differ considerably. First, there isn't much geometry in Herstein; there is a lot in Bocher. Herstein's geometry is vector space geometry; that is, the introductory tools of geometric algebra and analysis, and little more. Bocher has some vector space geometry, but more often turns to the geometry of curves and surfaces—frequently, but not always, quadric surfaces. Thus we find much mention of tangent lines and planes, conjugate points, polar points, etc., with more emphasis on dimension three than on any other. Only in a short section towards the end are characteristic values and vectors to be found. But Bocher prefers the projective view, so instead of a characteristic vector he discusses a fixed point under a collineation. Perfectly satisfactory, and certainly more oriented towards algebraic geometry.

Somewhat remarkably, Bocher mentions just twice [very briefly both times] what is now called the inner product space, or put more primitively, perpendicularity. The concept is not in the index. That perpendicularity almost isn't mentioned in a mathematics text, even a 1907 one, is unusual enough to attract notice, and all the more remarkable in that Bocher studies bilinear and quadratic forms, including pairs of forms, in some detail, with various diagonalization theorems carefully laid down. Towards the end of the book, Bocher explains away the absence of this idea in these terms: "... properties may be either projective or metrical. Thus it would be a metrical property of a collineation if it carried over some pair of perpendicular lines into a pair of perpendicular lines.... We shall be concerned only with projective properties...." Fair enough, an author's choice. But, is it fair treatment of a beginning student?

Orthogonality is in Herstein's book, though. Thus, we have the curious anomaly that the fundamentally geometric idea of perpendicularity is quite visible in Herstein's "abstract" algebra, but is hidden in Bocher's algebraic "geometry." The explanation behind this difference, as mentioned above, of course, is this: the small amount of geometry in Herstein is *linear* geometry,

the geometry of vector spaces, whereas geometry to Bocher usually means curves and surfaces; that is, *algebraic* geometry. As an example of a nonlinear theorem, Bocher has this fact: A polynomial equation $\phi(a_0,\ldots, a_n) = 0$ in homogeneous coordinates $a_0,\ldots, a_n$ specifies a geometric property only if the polynomial ϕ is homogeneous. This assertion would be out of place in Herstein.

Now, it is true that a geometrization of algebra began in the 1930's with Witt, reaching maturity with Artin's 1957 book *Geometric Algebra*. But this was, in large measure, an axiomatic (i.e., coordinate free) formulation of linear algebra. The algebraic geometry of surfaces was flourishing decades earlier. Unfortunately, linear geometry is needed everywhere, and especially in the study of bilinear and quadratic forms. Thus Bocher's treatment of the forms, while surely interesting, and to some extent geometric, definitely shows its age when compared to a well done modern discussion using geometric algebra. The key idea that's missing is that a diagonalization theorem is also an orthogonalization theorem. [In passing, it might be mentioned that this approach to the forms is beautifully presented in Mal'cev's *Linear Algebra*.]

On the other hand, if Bocher can be criticized for failing to emphasize the metrical aspects of linear algebra, Herstein can be criticized for paying so little attention to geometry, and especially for no mention of algebraic geometry. Each to his own taste! To say that Herstein's book is an undergraduate text, with algebraic geometry too advanced a concept to be mentioned, is not a defense: the essential unity of major components of mathematics should already begin to be visible at the undergraduate level.

As far as expository style is concerned, a comparison of Bocher and Herstein shows how dramatically expository standards have changed, and in the reviewer's eyes, improved. The point at issue here is *motivation* and *direction*. Herstein repeatedly informs his reader where he is heading, what the objectives are, and how it all fits together. Bocher never does this, instead he just places a panoply of theorems before his reader. Now, the professional mathematician uses his broad perspective to tie such a panoply into a lovely coherent whole—that's why Bocher's book reads well to the trained eye—but the untrained student usually has difficulty doing the same. It's really not good enough to say "here's a fact," "here's another," and imply that you will see how they fit together when you are more experienced. This comment is worth making because some modern texts are still written with weak motivation. (Usually their sales are equally weak!) But Bocher writes in 1907, not the 1980's, so we must overlook his shortcomings. And after all, his shortcomings are slight since he surely has a unity of spirit and outlook: always, a search for invariants and a constant recourse to a geometric outlook. But he doesn't have the quintessence of motivation: a difficult problem posed near the start of the book and solved near the end. Nor does he have the smaller quintessences of a sequence of interesting problems, each posed, then solved after some theory.

How does Bocher's text compare with Jacobson's *Basic Algebra I*? It really isn't quite fair to make this comparison, since Jacobson's intended audience is much more advanced than Bocher's. But let's make it anyway! A first remark is this: Bocher has the needs of invariant theory and algebraic geometry more in the forefront of his mind than does Jacobson, but the gap is less than between Bocher and Herstein. And Jacobson has Galois theory, beautifully done. Apart from this, we see something quite dramatic: Jacobson is superbly efficient in taking advantage of every advance in mathematical thinking that the passing decades have brought, and that could be fitted into his book. Comparing the two texts really makes visible how striking some of these advances are. Example: Both books have the resultant, but Jacobson goes on to use it to study decision methods for determining whether an algebraic curve has a real point. And while Jacobson has less motivational content than Herstein (and needs less, since he writes to a more advanced audience), he has more than Bocher. A final point of comparison is that Jacobson's writing style is every bit as concise (but nonetheless clear) as is Bocher's. Neither book is for children!

Should Bocher's book be used as a text in today's modern algebra classes? Certainly not. The advances of the last 75 years in both mathematical style and mathematical fact are simply too

great to be ignored. But Bocher can surely be recommended as supplementary reading for the serious student, both for enrichment and for historical perspective. After all, the nineteenth century mathematicians said very much that is worthwhile, some of which Bocher's Introduction captures well. Today's student who wishes some exposure to the classical style (that's a synonym for the style of the 1800's), and this should be every student, probably cannot do better than to read Bocher. It is for this audience that Bocher's book is highly recommended.

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Curriculum Development in Mathematics. By Geoffrey Howson, Christine Keitel, and Jeremy Kilpatrick. Cambridge University Press, Great Britain at the Pitman Press, Bath, 1981. viii + 288 pp.

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The failings—one could say the failure—of mathematics education on all levels are so well known that substantiation of this fact is hardly necessary. Intelligent people, many of whom have done superb work in nonmathematical areas, fail to acquire even a modicum of mathematical knowledge and indeed are repelled by mathematics of any sort. Surely these people would not deny the importance of mathematics not only for science and engineering but also for its impact on all branches of our culture including the very branches in which these intelligent people have made their mark. Such an anomalous situation calls for an explanation and remedy. The role of mathematics in our civilization is too fundamental and vital to allow such a situation to be tolerated.

The explanation involves what is certainly the major feature of mathematics education, the curriculum. The curriculum is fashioned by mathematicians. But mathematicians, especially in our time, make assumptions which vitiate what they seek to achieve. One assumption is that mathematics proper is intrinsically attractive and should interest all students. Indeed in the past twenty or so years we have had curriculum developments that not only concentrate on mathematics proper but stress many fine points—rigor—that eluded the best mathematicians from Greek times to about 1900. This approach adds insult to injury. One such development, popularly known as the new math, fortunately fell flat on its face, and now there is no need to whip a dead horse. But a return to the older, somewhat more intuitive curriculum is also not the answer. This type of curriculum was a failure for many generations before the new math was fashioned.

Should we then abandon the teaching of all mathematics beyond the practical elementary arithmetic except perhaps to future scientists and engineers and those few "perverted" young minds who for some queer reason take a liking to mathematics? Of course not. Mathematicians generally know the importance and relevance of the subject even though they fail to convey it in the curricula. This is the major failure that must be remedied.

This failing has been emphasized by no less a man than Hermann Weyl. In his MONTHLY article of October 1951, Weyl said:

> One may say that mathematics talks about things which are of no concern at all to man. Mathematics has the inhuman quality of starlight, brilliant and sharp but cold. But it seems an irony of creation that man's mind knows how to handle things the better the farther removed they are from the center of his existence. Thus we are cleverest where knowledge matters least: in mathematics, especially in number theory.

The proper approach to mathematics education was urged by Alfred North Whitehead as far back as 1912. In his essay "The Aims of Education," Whitehead said:

> In scientific training the first thing to do with an idea is to prove it. But allow me for one moment to extend the meaning of prove; I mean—to prove its worth. ... The solution which I am urging is to eradicate the fatal disconnection of subjects which kills the vitality of our modern curriculum. There is only one subject matter for education and that is Life in all its manifestations. Instead of this single unity, we offer children Algebra, from which nothing follows; Geometry, from which nothing follows.... Our course of instruction should be planned to illustrate simply a succession of ideas of obvious importance.

In some of his essays in his book *Essays in Science and Philosophy*, Whitehead elaborates on the above theme.

In substantiation of Whitehead's suggestion I might note that I have asked any number of high school teachers why anyone should learn how to solve quadratic equations or trigonometric identities. Did they have any occasion to use this knowlege outside the classroom? The answer was an abashed silence.

The answer to curriculum deficiencies is in broad terms relevance or application. These may vary from games and puzzles to physical and social science applications depending upon the ages of the students. Such motivations are available. But they must not be pieces of candy thrown out occasionally to cheer up the students. They must be an integral part of the curriculum and they must be inserted in the texts in the proper places where they will be most effective.

To many professional mathematicians this motivation may be judged as desecration or a waste of time that could be better devoted to more mathematics. But even God's words must be interpreted for the layman.

If the contention that mathematics courses must contain motivations drawn from outside of mathematics is correct, then it is a corollary that teachers must be trained to present such material as carefully and as assiduously as they now present mathematics proper.

If any additional evidence is needed to support the contention that mathematics courses must teach not only what mathematics is but also what mathematics does, the book under review supplies it at least indirectly. The essence of the book is a review of the many new curricula fashioned during the last twenty or so years not only in the United States but also in several foreign countries. It is a valuable record even though it is a record of failures to improve mathematics education.

The authors state that the boom in curricula has about ended. Has work on new curricula ended? No! Now many teachers are on new binges: problem-solving, back to basics, developmental mathematics, and the New York State Department of Education three-year high school curriculum.

Anyone planning a new curriculum should certainly study this history of the various factors that motivated and in the end defeated the objectives of the many movements. The authors are experienced and competent.

MISCELLANEA

120.

There is no branch of mathematics, however abstract, that will not eventually be applied to the phenomena of the real world.

—N. I. Lobachevsky, quoted in *Mathematicians on Mathematics*, compiled by N. Ya. Vilenkin, Znanie, Moscow, 1982.

Infinity and the Mind. The Science and Philosophy of the Infinite. By Rudy Rucker. Birkhäuser, Boston, 1982. x + 342 pp., $15.95.

PHILIP J. DAVIS
Applied Mathematics, Brown University, Providence, RI 02912

If you are looking for a white water trip down the streams of Infinityland for yourself, your students, or as a graduation present for young relatives, rush out and buy this book written by logician and sci-fi author Rudy Rucker. I guarantee you will get your money's worth in thrills and spills.

What kind of thing will you see on this guided tour? Well, there are the usual objects: the transfinite ordinals and cardinals; the processes of recursion, self-referral, diagonalization; the paradoxes of Berry, Richards, and the Liar; Hilbert's Hotel; infinitesimals. All of these are viewed in an up-to-date manner, often from a computer angle. In addition, there are subjects and objects less often seen, or unknown to older trippers: the process of tetration, Conway numbers, and the real fat cardinals such as the inaccessibles, the hyperinaccessibles, the indescribables and the ineffables. One has a view also of universal libraries, fractals, of mind-machine tensions, and of the dialectic split between the "one" and the "many."

At the end of the trip, one comes off having met infinities of the absolute, actual, potential, mathematical, physical, geometrical, numerical, logical, psychological, parapsychological, metaphysical, mystical, metamystical, theological, telepathic, and thanatotic types, all residing in one neo-Renaissance mindscape. In a word, infinity here has been related to just about everything with the possible exception of politics. I didn't find (perhaps I didn't look hard enough) anything about petit bourgeois or workers' infinities.

Anyway, the price of admission buys you all this, salted and peppered with comic strips and clips from sci-fi classics, plus a dash of Gödelolotry, plus intimations of the dark, hoaxing occultism of Jorge Luis Borges, plus—for the intrepid—detailed, but hopefully "popular" accounts of big cardinal theory and the incompleteness theorems of logic, plus, lest we forget, a metempsychotic tear shed for the memory of John Lennon. All this and euphoria too.

Rucker's book fits nicely into the current tendency of restoring a balance between the rational and the irrational. Whether as good business, as deeply felt belief, as a reaction to dessicated logicism, as a response to social alienation, it is simply a fact of today's world, that cult and occult, theosophy and mysticism have been flourishing as they have not for many centuries. The interaction between magic, religion, and mathematics, moving both ways, can be fertile; but in the past two centuries, centuries of positivistic tendency among scholars, the topic has either been avoided or put down. Rucker plays footsie with it—which distresses me somewhat—but it remains a serious task for scholarship in the years ahead.

So, as we rush down (or slowly ascend) the sunset-illuminated, transcendental waterfalls of the infinite, Peace. Let us think, as Gregory Chaitin would have us do, that all true theorems and much more are to be found in the bowels of Ω.

Join Geometries. A Theory of Convex Sets and Linear Geometry. By Walter Prenowitz and James Jantosciak. Springer-Verlag, New York, 1979. xxii + 534 pp., $20.80.

ROBERT E. JAMISON-WALDNER
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Any attempt to enliven the contemporary interest in geometry is certainly worthy of appreciation. Aside from its venerable place in mathematical history, geometry is valuable as a medium for the exercise of visual *intuition*. Such intuition is not only helpful in "picturing" the broad outlines

of complex processes but is also, I believe, a necessary right-brain counterbalance to the analytic, algorithmic, and often mechanical aspects of the computer age. There is a paradox here because geometry is also the acknowledged birthplace of the axiomatic method which emphasizes rigorous deduction rather than intuitive insight. It is precisely around this paradox that my feelings about the book under review became jumbled up. *Join Geometries* is not just a fresh treatment of elementary geometry; it is a wholly new approach based on a new axiom system. To evaluate it effectively requires some consideration of the purpose of axiomatics, and for me this is a key issue.

Everyone realizes that the discovery of non-Euclidean geometry was a turning point for mathematics. The issue goes far beyond deciding whether physical space is Euclidean or not, an oversimplification often conveyed in popular accounts. Ultimately, the discovery led to the realization that axiom systems do not *describe* reality but merely *model* it. Ironically, this realization did not decrease but rather extended the applicability of the axiomatic method. (Incidentally, a nice survey of the current mathematical approach to the geometry of space is given by Penrose in *Mathematics Today* (L. A. Steen, Editor).)

In some ways Gödel's theorems had an analogous effect on the role of axiomatics within mathematics. Since we can never completely pin down even our most fundamental mathematical systems, we have generally come to think of axioms not as a description of absolute mathematical truth but as a means to abstract for systematic study structures that seem useful or interesting. This mature view was nicely summarized by R. L. Wilder [MONTHLY, 74 (1967) 115–127] when he called the axiomatic method a "tool for the codification of what our intuitions tell us."

Indeed, axiomatics (definition-theorem-proof) provides the accepted means for testing, organizing, and ultimately communicating this intuition. As such axiomatics forms what might be called the "visible side" of mathematics, which tends to stick out all the more because of the considerable difficulty most people have in understanding how it functions (in contrast to understanding, say, a spoken—or even a computer—language). It is not surprising, then, that axiomatic (or even symbolic) manipulation is often confused with the essence of mathematics. The danger in this view is clearly pointed out by Courant and Robbins in the introduction to their classic book *What is Mathematics*? where they warn that "a serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from definitions and postulates. . . ." One needs to be aware that the formal structure is only a representation of a basic intuitive, conceptual understanding. This is important in good research and essential for effective teaching. By the way, it is sometimes felt that the conceptual basis should be grounded in physical "reality." But I also think that such concepts as "number," "infinity," "arrangement," "symmetry," *etc.* are basic mathematical notions. Thus while I would concur with Morris Kline's opinion in a recent *Omni* interview that large portions of current research are "useless" (because the intuitive underpinning is lost), I do not agree that the twin prime problem is an example.

Ultimately, what is desired is a balance between abstraction and intuition. This was described beautifully by no less than David Hilbert in the preface to *Geometry and the Imagination*:

> In mathematics, as in any scientific research, we find two tendencies present. On the one hand, the tendency toward *abstraction* seeks to crystallize the *logical* relations inherent in the maze of materials that is being studied, and to correlate the material in a systematic and orderly manner. On the other hand, the tendency toward *intuitive understanding* fosters a more immediate grasp of the objects one studies, a live *rapport* with them, so to speak, which stresses the concrete meaning of their relations.

That this balance can be achieved beautifully in geometry is illustrated not only by *Geometry and the Imagination* but by Young's Carus monograph on *Projective Geometry* and more recently by Grünbaum and Shephard's work on *Tilings and Patterns*.

With this background in mind, I would now like to propose two criteria for judging an axiom system:

1. Does it unify existing concepts or examples?
2. Does it allow for the efficient exposition and proof of the major results in a theory?

(The first of these, annunciated in 1910 by E. H. Moore in his *Form of General Analysis*, might not find as much acceptance today because of increasing emphasis on new results and applications rather than conceptual understanding.) Unfortunately, *Join Geometries* falls short on both counts. It is admitted (p. 53) that the theory did not arise to "unify common properties of known... systems" but rather "has its genesis in one particular situation"—namely, Euclidean n-space. Moreover, the only two examples given that do not arise directly from Euclidean space are presented more as sources for counterexamples than for their intrinsic interest. In regard to the second criterion, it is not until the end of the book that enough machinery is available to state and prove such basic results as Helly's theorem (p. 484), Carathéodory's theorem (p. 485), and the representation of a polytope as the intersection of finitely many half-spaces (p. 522). The root of these problems is summed up in a warning to the student on page 108 that he "may tend to lean too heavily on Euclidean intuition," which may pose an obstacle in "trying to grasp and assimilate the theory." In short, there is a confusion (which is all too common) of axiomatics for mathematics.

There is a related paradox in the book and that concerns the relationship of algebra to geometry. The authors are rightly disturbed by the contemporary tendency to view classical geometry as a subdiscipline of linear algebra. The linear algebra model has so successfully met the above two criteria for axiom schemes, that the peril is ever-present of confusing geometry with the algebraic manipulation of coordinates. Moreover, the fact that n-tuples provide a handy way of representing data has thrust on Euclidean n-space many interpretations that are not a priori geometrical. The antidote is not to discard the highly effective axiom scheme of vector spaces, but to keep clearly in mind the geometric significance of the algebraic manipulations. It is paradoxical that in *Join Geometries* the proposed antidote is another axiom scheme that is also highly *algebraic* in flavor.

The basic axioms for *Join Geometries* come from earlier papers of Prenowitz [cf. MONTHLY 68 (Nr. 1, Part II) (1961)] (which, in my opinion, still give the best presentation of the material) and are based on convexity as a fundamental notion. I think this is a good idea although its development in *Join Geometries* has not been successful. It is probably worthwhile to mention some approaches that promise more success. The basic insight is that linear span and convex hull are very similar in certain ways: they are both *closure operators*—not in the topological sense, but in the algebraic sense. (More discussion may be found in my article on alignments in *Convexity and Related Combinatorial Geometry*, Dekker Lect. Notes, vol. 76.)

There are basically four kinds of closure based on four types of combinations $\lambda_1 v_1 + \cdots + \lambda_n v_n$ of vectors:

1) linear: no restriction on the scalars λ_i
2) affine: $\lambda_1 + \cdots + \lambda_n = 1$
3) positive: $\lambda_i \geqslant 0$ for all i
4) convex: $\lambda_1 + \cdots + \lambda_n = 1$ and $\lambda_i \geqslant 0$ for all i.

The affine case is subsumed in the linear case by embedding the space under consideration into a hyperplane away from the origin. The study of linear closure has been very successfully abstracted in theory of *matroids* or *combinatorial geometries*. This was founded by Whitney and others in the 1930's and an excellent introduction may be found in the article by Brylawski and Kelly in the MAA *Studies in Combinatorics*. Positive closure seems to be modelled well by the theory of *oriented matroids* which was started in the mid-1970's by papers of Bland, Las Vergnas, and Lawrence in the *Journal of Combinatorial Theory*. There seems to be the promise that the case of convex closure will be usefully modelled by *antimatroids* (which satisfy an anti-exchange law) or *convex geometries*. These structures are only in the germinal stage of development, appearing in different form in the work of Edelman, Korte, Markowsky, myself and others. It is worth noting that oriented matroids are really the umbrella structure, as each oriented matroid has both an underlying matroid and underlying antimatroid.

Why do these efforts appear effective? Matroids have successfully unified such topics as

algebraic independence, graph theory, and transversal (matching) theory. Through the greedy algorithm, they have found applications as far removed from pure mathematics as the scheduling of milk pick-up from dairies. Oriented matroids bring together arrangements of hyperplanes and orientations of graphs, and antimatroids relate the structure of chordal graphs, ordered sets, and certain formal languages. In all cases, the *basic* results are easily derivable in the general theory.

In conclusion, I would like to make a final plea for the intuitive in mathematics. The heyday of axiomatics for its own sake seems over. The confusion of mathematics with its tool axiomatics may be on the decline. But will the pendulum swing too far? Are we now facing an era of confusing mathematics with its applications? Is mathematics only a tool of the sciences, science only a tool of technology, and technology only a cog in the great mechanism of economics? Or are there human values that somehow underlie all of these? I believe there are.

As R. W. Hammer pointed out in his article on the "Unreasonable Effectiveness of Mathematics" [MONTHLY, 87 (1980) 81–90], mathematics may be effective because the only problems we tackle are those that admit mathematical treatment. When we encourage an appreciation for the intuitive in our research and teaching, we are encouraging a frame of thought that goes beyond this mere methodology. In our age of ever-increasing quantification, it is especially important that we keep some grasp on the basic goals that underlie the whole scientific and technological superstructure. Otherwise, we risk the confusion of means for ends, of insignificant for important. This is precisely what the Parsi mystic Meher Baba [*Discourses*, vol. I] warned of nearly fifty years ago: "All this confusion is due to the fact that man's mind is often dominated by mathematical ideas, even when it is concerned with estimates of a spiritual nature."

LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

Editor:

This letter is in the nature of an author's inquiry. I believe that students' interest in and enthusiasm for an elementary course may often be determined by what the instructor says during the first class period. With this in mind, I would like to ask readers of the MONTHLY to let me know how they begin their introductory calculus course. From the replies a report will be prepared for possible publication.

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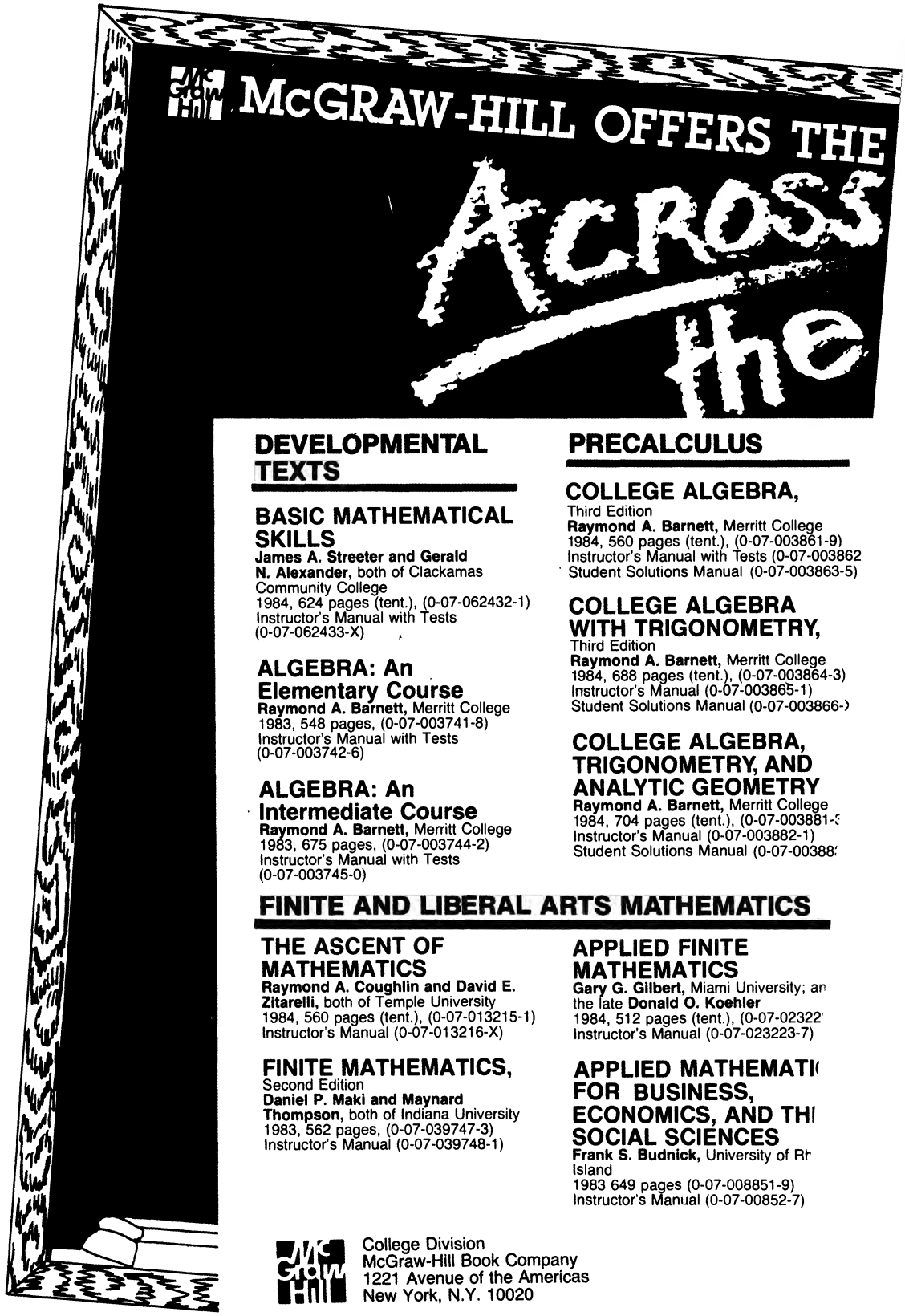
MISCELLANEA

121.

Kronecker Revised

Man gave us the integers;
the rest is Dieudonné.

—Richard K. Guy



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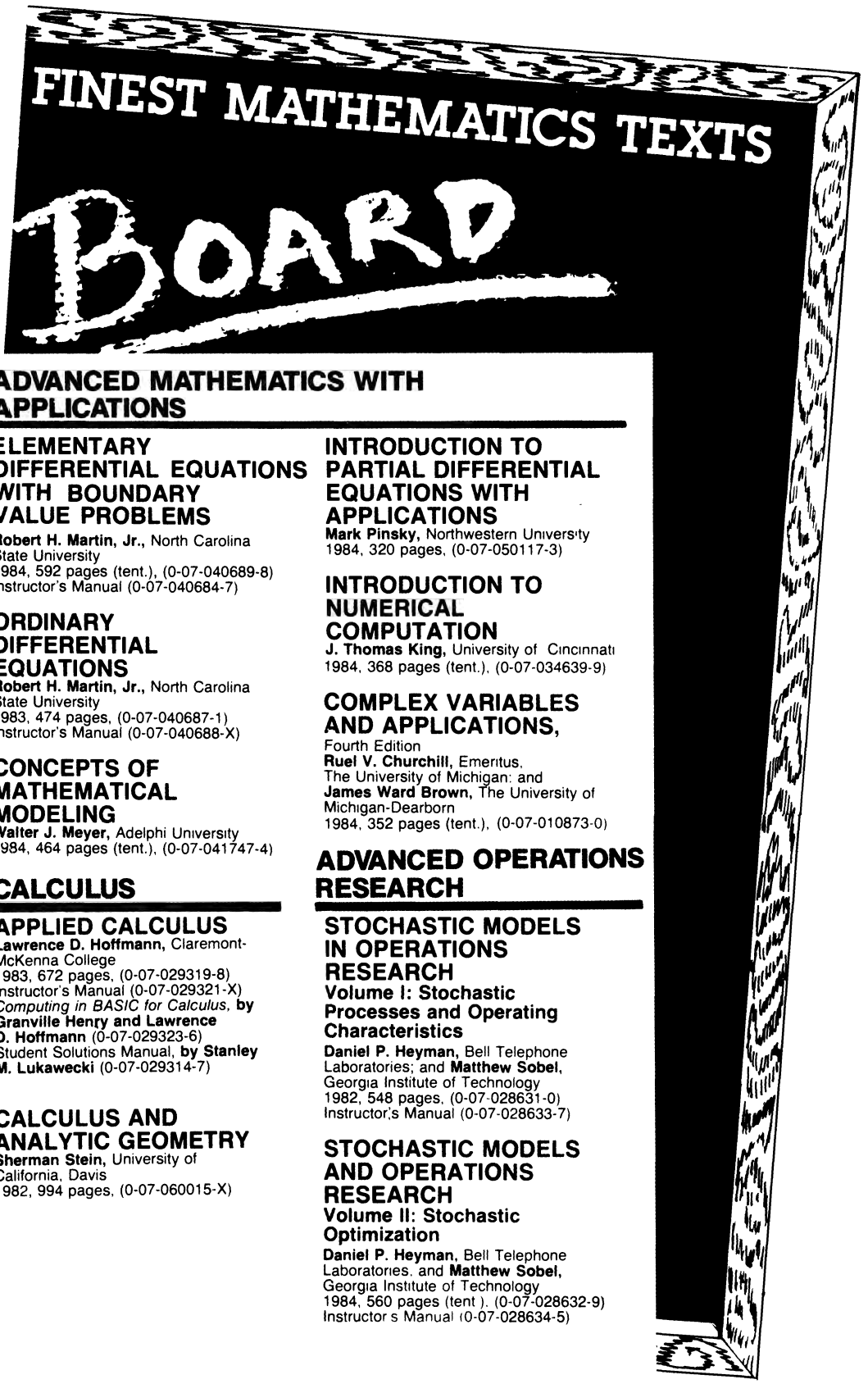
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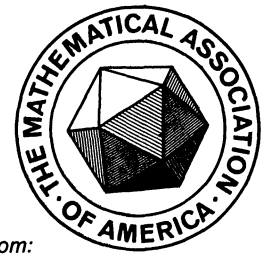
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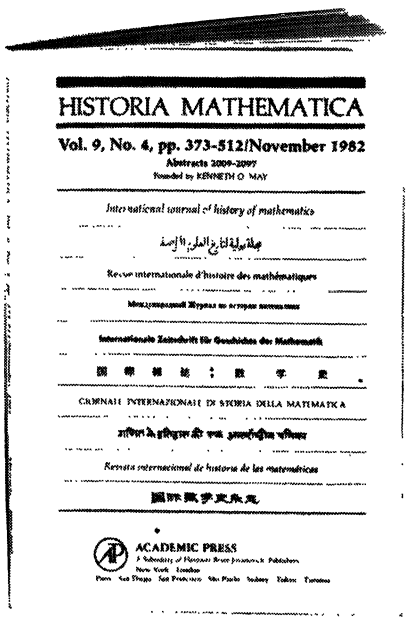


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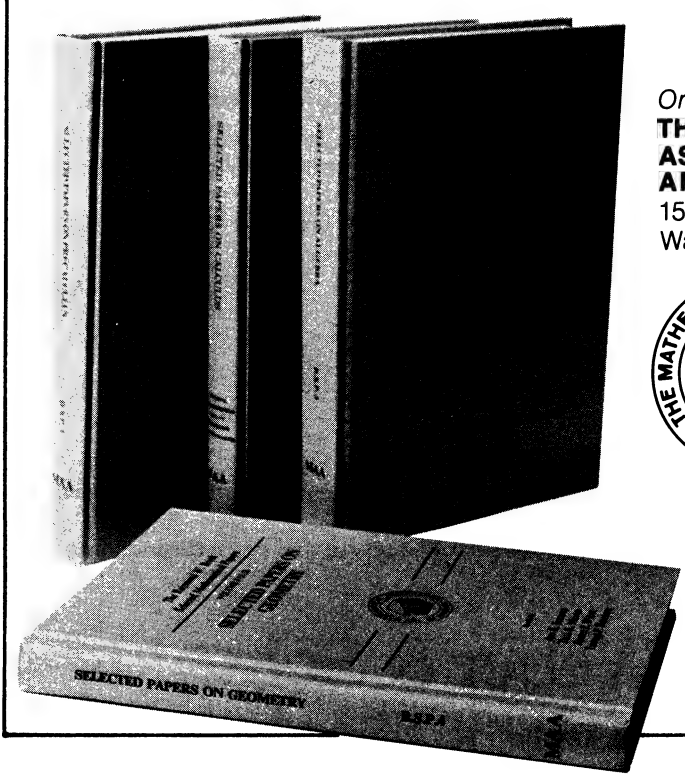
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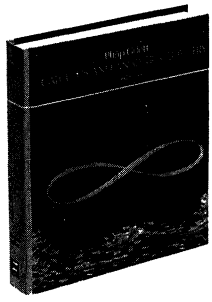
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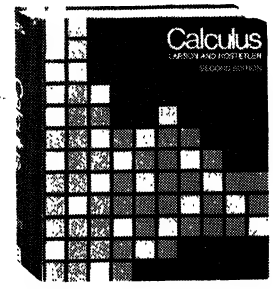


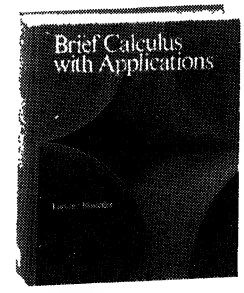
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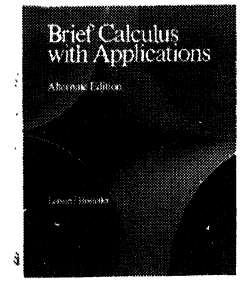
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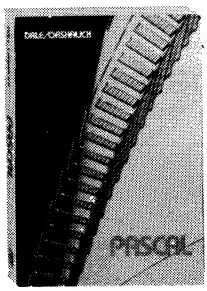
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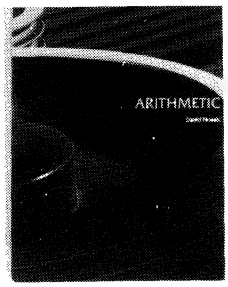
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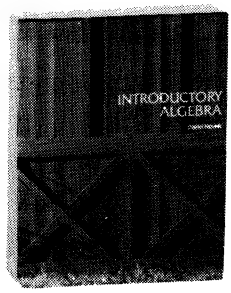
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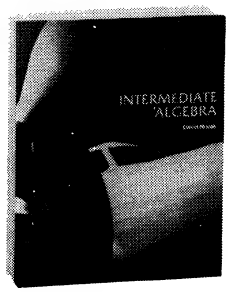
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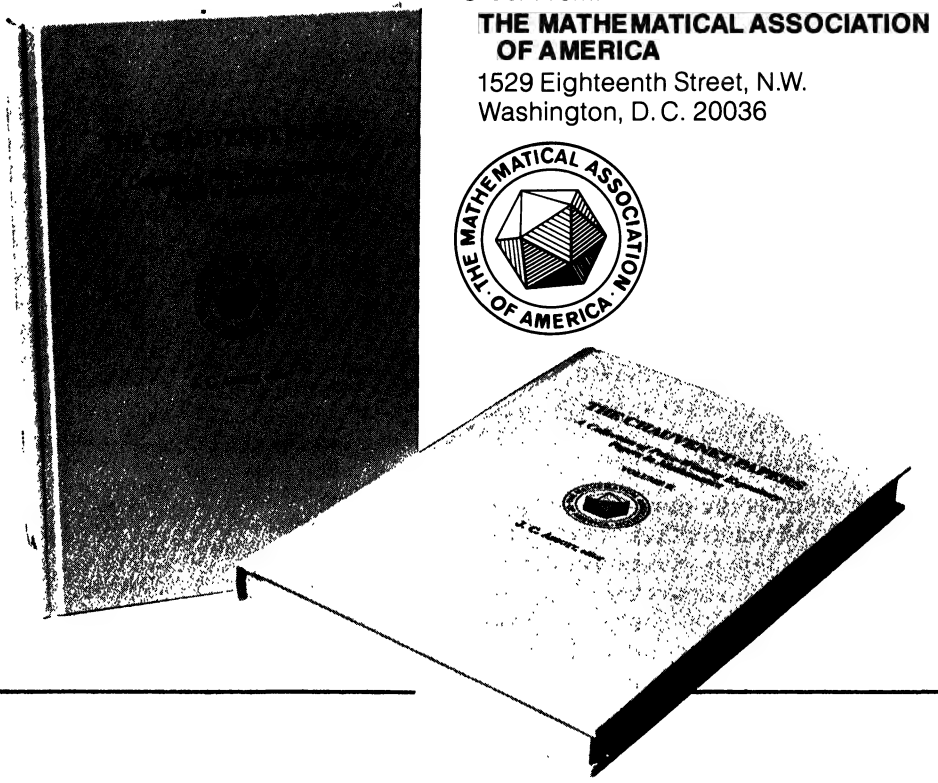
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DETECTION OF A DEFECTIVE COIN WITH PARTIAL WEIGHT INFORMATION

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1. Introduction. The problem of detecting the single defective coin with a beam scale in the minimum number of weighings is a well-known puzzle, on which numerous references [**1**], [**2**], [**6**], [**8**], [**9**], [**11**], [**12**] can be found in the literature. However, the similar problem using a spring scale with weights g of a good coin and d of a defective coin known has an easy solution, and hence is often bypassed in the spring scale literature [**3**], [**4**], [**5**], [**7**], [**10**]. In fact, even if only g is known, then we can determine whether a set of coins contains the defective one by checking whether its average weight is different from g. When that information is available, the halving method, i.e., each weighing involves half or nearly half of the coins still not known to be good, is obviously optimal. But what happens when g is unknown? The difficulty is that the average weight of a set no longer tells us whether that set of coins contains the defective coin or not. Presumably this difficulty can be overcome by weighing separately three different coins (from this the value of g can be deduced); but this can hardly be expected to be optimal! It turns out indeed that there are surprisingly efficient ways of decoding g from the results of combined weighings.

Of course, "coins" and "spring scale" is just one way to formulate the problem. Any instrument measuring an additive function of the elements (for instance an ammeter giving the total amount of current flowing through a circuit built up from identical elements wired up parallelly) defines a problem isomorphic to the defective coin problem. In this context, it is less far-fetched to suppose unknown the values of the good and defective elements. It is indeed quite possible that the measurements are affected by slow external variations so that the precise current standard value has to be determined at the start of each batch of experiments. Another reason is that there may be no "good" elements, because "standard" elements are determined by majority only; why should we then know their value a priori?

Once it is admitted that only partial weight information may be known, it may be useful to consider the extreme case where we don't have any weight information. By proving a lower bound for the extreme case, one obtains in a single stroke a bound for each of those subproblems with various partial information.

2. Some General Remarks. We will refer to a subset of coins involved in a single weighing as a *sample*. A *scheme* is a collection of samples. It is called *detecting* if it stops only when the results of all samples weighed are sufficient to determine the defective coin. Two schemes are considered *equivalent* if one can be obtained from the other by relabeling the coins. In the rest of the paper, we consider equivalent schemes as the same.

In general, several samples can be weighed simultaneously. For example, in the electrical case noted above, we can measure the currents in several sections of the circuits (using fan-out) in one set-up. For some practical applications, like the testing of VLSI components, minimizing the number of set-ups can be just as important as minimizing the number of weighings, as each set-up may be complicated and lengthy. Thus we introduce the concept of *stage*, which refers to those

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A minibiography of F. K. Hwang appeared in this MONTHLY in vol. 89, p. 235.

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samples which are weighed simultaneously. Note that the samples in the same stage must also be selected simultaneously without knowing the results of each other's weighing. However, the results of weighing in one stage are available for the selection of samples (or the decision of no more samples) in all subsequent stages. A scheme is called *nonadaptive* if it consists of a single stage and is called *adaptive* otherwise. The most extreme adaptive scheme is the *sequential* scheme in which each stage consists of a single sample. A scheme can also be called an *s-stage* scheme if the number of stages is known in advance. Since an s-stage scheme can always be artificially treated as an $(s+1)$-stage scheme, an optimal s-stage scheme cannot require less weighings than an optimal $(s+1)$-stage scheme. However, we will not be interested in a scheme if there exists another scheme with less stages but same number of weighings. We present our results on optimal adaptive and nonadaptive schemes in that sense.

Let $W_N(n)$ and $W_A(n)$ denote the numbers of weighings under an optimal nonadaptive scheme and an optimal sequential scheme, respectively, for n coins. Let g denote the weight of a good coin and $d \neq g$ the weight of a defective coin (we call $d-g$ the *deviation*). A state of weight information I is defined as the set of all possible pairs of good and defective weights (g, d). We will always talk about $W_N(n)$ and $W_A(n)$ with respect to a given I, though often we use the characterization of I instead of I itself. For example, "g known" means all possible pairs (g, d) such that g equals a given constant.

We first list a few folklore results for the "g known" case.

LEMMA 1. *If g is known, any detecting scheme for n coins is also detecting for $n-1$ coins; hence $W_N(n)$ and $W_A(n)$ are both nondecreasing in n.*

Proof. Obvious by observing that we can always add an imaginary coin of weight g to the $n-1$ coins.

LEMMA 2. $W_N(n) \geqslant W_A(n) \geqslant \lceil \log_2 n \rceil$ *if g is known, where $\lceil x \rceil$ denotes the smallest integer not less than x.*

Proof. The first inequality follows from the fact that a nonadaptive scheme can also be used sequentially. The second inequality follows from the fact that in the beginning there are n possibilities for the defective coin and that each weighing has only two outcomes: either the average weight is g or it is $g + (d-g)/m$, where m is the sample size.

COROLLARY. $W_N(n) \geqslant W_A(n) \geqslant \lceil \log_2 n \rceil$ *with respect to any I.*

THEOREM 1. $W_N(n) = W_A(n) = \lceil \log_2 n \rceil$ *if g is known.*

Proof. It suffices to prove $W_N(2^k) \leqslant k$ if g is known. We do this by constructing a scheme, called the *binary scheme* and denoted by B_k, which meets the requirement. Let the 2^k coins be labeled by the binary representation of the 2^k numbers from 0 to $2^k - 1$ and let the ith sample of B_k consist of all coins whose ith bit is 1.

Let r be the $(0,1)$ vector of k elements whose ith element is 1 if and only if the ith sample of B_k contains a defective (this information is available since g is known). Then clearly, the defective coin is the one whose label is r. ■

No obvious proof exists for Lemma 1 if g is not known. The trick of adding an imaginary coin does not work since we don't know how to assign its weight. In particular, for $n = 2$ and g unknown, no number of weighings can tell us which of the two coins is defective. However, for $n = 3$, the scheme of weighing each coin separately is certainly detecting. Therefore we have to construct $W_N(n)$ and $W_A(n)$ for every n when g is not known.

We now give a lemma which will play an important role in proving our main results.

LEMMA 3. *The average weights of three samples of different sizes determine g uniquely unless the size of one of the samples is equal to the sum of the sizes of the other two.*

Proof. Since the sizes m_1, m_2 and m_3 of the samples are different, equality of two of the average weights implies that every coin in the two corresponding samples is good (thus determining g).

When the three average weights $\overline{w}_1$, $\overline{w}_2$ and $\overline{w}_3$ are different, at most one of the samples can be all good. Four cases arise, in which the following conditions have (respectively) to be satisfied:

$$C_1\begin{cases}\overline{w}_1 = g_1\\ \overline{w}_2 = g_1 + (d_1 - g_1)/m_2\\ \overline{w}_3 = g_1 + (d_1 - g_1)/m_3\end{cases} \qquad C_2\begin{cases}\overline{w}_1 = g_2 + (d_2 - g_2)/m_1\\ \overline{w}_2 = g_2\\ \overline{w}_3 = g_2 + (d_2 - g_2)/m_3\end{cases}$$

$$C_3\begin{cases}\overline{w}_1 = g_3 + (d_3 - g_3)/m_1\\ \overline{w}_2 = g_3 + (d_3 - g_3)/m_2\\ \overline{w}_3 = g_3\end{cases} \qquad C_4\begin{cases}\overline{w}_1 = g_4 + (d_4 - g_4)/m_1\\ \overline{w}_2 = g_4 + (d_4 - g_4)/m_2\\ \overline{w}_3 = g_4 + (d_4 - g_4)/m_3\end{cases}$$

It is easily verified that simultaneous satisfiability of C_4 and C_i, $i = 1,2,3$, implies $g_4 = g_i$, $d_4 = d_i$, and $g_4 = d_4$; but this is impossible by hypothesis. Furthermore, satisfiability for C_i implies for $\{i, j, k\} = \{1,2,3\}$

$$\overline{w}_i = (m_j\overline{w}_j - m_k\overline{w}_k)/(m_j - m_k).$$

Therefore, simultaneous satisfiability of C_i and C_j implies

$$m_k\overline{w}_k = m_j\overline{w}_j + (m_k - m_j)\overline{w}_i = m_i\overline{w}_i + (m_k - m_i)\overline{w}_j,$$

which can be reduced to

$$(m_i + m_j - m_k)\overline{w}_i = (m_i + m_j - m_k)\overline{w}_j,$$

an absurdity since $m_i + m_j \neq m_k$ and $\overline{w}_i \neq \overline{w}_j$.

Thus, under the stated condition, at most one of the four systems above can be solved; but this of course determines g uniquely, since each system has rank two. ■

3. Nonadaptive Detection Schemes. In the context of nonadaptive schemes, the basic notion is that of a type. The *type* of a coin with respect to a weighing scheme is the incidence vector of the coin to the samples of the scheme. Thus, a coin included in each weighing has type $\underline{1} = (1,1,\ldots,1)$, and a coin included in no weighing has type $\underline{0} = (0,0,\ldots,0)$. These two types are called *homogeneous*, and the others *nonhomogeneous*. Two types with 0 and 1 interchanged are called *complementary*. We will denote the complement of type $\underline{t}$ by $\underline{t}^c$. Clearly a scheme is not detecting when it has two coins of the same type, since all results of the weighings would be the same whichever of these coins is defective. For a detecting scheme we will identify a coin by its type.

The following theorem shows that in the case of no weight information we can do as well as if g is known, except when n is of the form 2^k or $2^k - 1$, where possibly one more weighing is needed.

THEOREM 2. *For $k \geqslant 3$ and $2^{k-1} - 1 \leqslant n \leqslant 2^k - 2$, $W_N(n) \leqslant k$ with no weight information.*

Proof. From Lemma 3, it suffices to show that, for $k \geqslant 3$ and any n with $2^{k-1} - 1 \leqslant n \leqslant 2^k - 2$, it is possible to omit $2^k - n$ types from B_k in such a way that three different sample sizes are used, and that no two of them sum to the third. By Lemma 3, g is uniquely determined, and hence such a scheme is detecting.

If $n = 2^k - 2$, omit coins $(1,0,\ldots,0)$ and $(1,1,0,\ldots,0)$ from B_k. Then the samples have three different sizes $2^{k-1} - 2$, $2^{k-1} - 1$ and 2^{k-1}; clearly, no two sizes sum to the third size.

From $n = 2^k - 3$ down to $n = 3 \cdot 2^{k-2}$, further omit successively coins of the types $(1,1,\varepsilon_3,\ldots,\varepsilon_k)$ but not type $\underline{1}$. At each step, the sizes of the first two samples are decreased by one, whereas the sizes of the other (larger) samples either decrease by one or remain the same; thus there are at least three different sizes at each stage. Moreover, no two sizes sum to a third,

since even for $n = 3 \cdot 2^{k-2}$, the sum of the two smallest sizes (2^{k-2} and $2^{k-2} + 1$) is bigger than the largest size ($3 \cdot 2^{k-3} + 1$).

From $n = 3 \cdot 2^{k-2} - 1$ down to $n = 2^{k-1} + 1$, further omit successively coins of the types $(0, 0, \varepsilon_3, \ldots, \varepsilon_k)$ but not $(0, 0, 1, \ldots, 1)$. The two smallest sizes stay fixed at 2^{k-2} and $2^{k-2} + 1$, whereas the largest size drops from $3 \cdot 2^{k-3} + 1$ down to $2^{k-2} + 2$. Thus there are again always three different sizes, and no two of them sum to a third.

When $n = 2^{k-1}$, one may further omit $\underline{1}$ if $k > 3$, since then $2^{k-2} - 1 + 2^{k-2} > 2^{k-2} + 1$. If $k = 3$, it suffices to weigh separately three of the four coins to find out the defective.

Similarly, for $n = 3$, it suffices to weigh separately all three coins. For $n = 2^{k-1} - 1$ and $k > 3$, omit all types $(t_1, \ldots, t_{k-1}, 1)$ but not $(0, \ldots, 0, 1)$, as well as $(1, 0, \ldots, 0)$ and $(1, 1, 0, \ldots, 0)$. The samples' sizes are then respectively $2^{k-2} - 2$, $2^{k-2} - 1$, 2^{k-2}, and 1. The first three are different, and no two of them sum to the other. Thus the proof is complete. ■

The following result gives a necessary and sufficient condition on the weight information I for $W_N(2^k) = k$; hence, it is a generalization of Theorem 1.

THEOREM 3. *$W_N(2^k) = k$ if and only if I does not contain pairs (g', d') and (g'', d'') such that*

$$g'' = g' + (d' - g')/2^{k-1}.$$

Proof. First we note that B_k is the only scheme for k weighings and 2^k coins such that each coin is of a different type. Therefore, $W_N(2^k) = k$ if and only if B_k is detecting.

If I contains (g', d') and (g'', d'') satisfying the condition of Theorem 3, then the two homogeneous complementary types $\underline{0}$ and $\underline{1}$ are indistinguishable when the weight of all samples is

$$w = 2^{k-1}g'' = (2^{k-1} - 1)g' + d',$$

since the defective coin is $\underline{0}$ if the weight pair is (g'', d''), but is $\underline{1}$ if the weight pair is (g', d').

If I does not contain such pairs (g', d') and (g'', d''), we show that B_k is detecting by showing that the value of g can be deduced. Note that the k samples of B_k yield two average weights $\bar{w}'$ and $\bar{w}''$, since all samples containing the defective coin have one common average weight and all samples not containing the defective coin have another common average weight. Clearly, either $\bar{w}' = g$ or $\bar{w}'' = g$. Note that if $\bar{w}' = g$, then

$$\bar{w}'' = g + (d - g)/2^{k-1}.$$

The condition that I does not contain (g', d') and (g'', d'') such that

$$g'' = g' + (d' - g')/2^{k-1}$$

implies that we cannot have both $\bar{w}'$ and $\bar{w}''$ equal to some g values in I. Therefore, g is uniquely determined. ■

COROLLARY 3.1. *When only a linear function $d + \lambda g$ is known, $W_N(2^k) > k$, except when $\lambda = (2^{k-1} - 1)$.*

Proof. By the Corollary of Lemma 2, $W_N(2^k) \geqslant k$. Hence, by Theorem 3, $W_N(2^k) > k$ if and only if the system

$$\begin{cases} d' + \lambda g' = d'' + \lambda g'' \\ g'' - g' = (d' - g')/2^{k-1} \end{cases}$$

has solutions for (d', g') and (d'', g'').

It is easily shown that the system has solutions if and only if the following system has:

$$\begin{cases} 2^{k-1}g'' = d' + (2^{k-1} - 1)g' \\ 2^{k-1}d'' = (2^{k-1} - \lambda)d' + \lambda g'. \end{cases}$$

Clearly, if $\lambda = 2^{k-1} - 1$, then $g'' = d''$ and there is no solution for (g', d') and (g'', d''). Otherwise, the input of any pair (g', d') with $g' \neq d'$ into the equation generates a pair (g'', d'') with $g'' \neq d''$. ■

COROLLARY 3.2. *$W_N(2^k) = k$ if g and d are known to be among a set of 2^{k-1} consecutive integers.*

Proof. From the assumptions, $d' \neq g'$ and

$$-1 < (d' - g')/2^{k-1} < 1$$

for any pair (g', d') in I. Hence, there do not exist different g' and g'' in I satisfying

$$g'' - g' = (d' - g')/2^{k-1}. \blacksquare$$

Next we study the $n = 2^k - 1$ case. Let $B_k(t_1,\ldots,t_q)$ denote the scheme obtained from B_k by omitting types $t_1,\ldots,t_q$. We note that all detecting schemes with k samples and n coins, $n < 2^k$, can be written as $B_k(t_1,\ldots,t_q)$ for some $q = 2^k - n$ choices of t_i.

THEOREM 4. *For $k \geq 2$, $W_N(2^k - 1) = k$ if and only if at least one of the following three systems of equations has no solutions such that (g', d'), (g'', d'') are both in I:*

$$E_1\begin{cases} d' - g' = g'' - d'' \\ g'' - g' = (d' - g')/2^{k-1}, \end{cases}$$

$$E_2\begin{cases} d' - g' = g'' - d'' \\ g'' - g' = (d' - g')/(2^{k-1} - 1), \end{cases}$$

$$E_3\begin{cases} d'' = g' \\ g'' - d'' = (d' - g')/(2^{k-1} - 1). \end{cases}$$

Proof. As we commented before, the only detecting schemes for $2^k - 1$ coins in k weighings are $B_k(t)$ for some t, so we need only be concerned with the conditions under which $B_k(t)$ is detecting.

When a homogeneous type is omitted, the coin of the remaining homogeneous type is defective if and only if the average weight of all samples are equal. As different complementary pairs of types induce different partitions of the samples (into two sets with distinct average weights), such a scheme is detecting if and only if it distinguishes between nonhomogeneous complementary types. It is straightforward to verify that $B_k(\underline{0})(B_k(\underline{1}))$ can distinguish t and t^c if and only if $E_1(E_2)$ has no solution.

When a nonhomogeneous type t is omitted, the scheme $B_k(t)$ contains samples of size $2^{k-1} - 1$, as well as samples of size 2^{k-1}. Suppose first that there exist two samples of the same size s_1 with different average weights $\bar{w}_1$ and $\bar{w}_1'$. Note that the fact that we have at least three samples implies that $k > 2$. Now if a sample of the other size s_2 has one of these average weights, it must be all good. On the other hand, if all samples of size s_2 contain the defective coin, thus having the average weight $\bar{w}_2 \neq \bar{w}_1$ or $\bar{w}_1'$, then the average weight $\bar{w}_1$ or $\bar{w}_1'$ furthest apart from $\bar{w}_2$ must be g. This is because

$$\{\bar{w}_1, \bar{w}_1'\} = \left\{g, g + \frac{d-g}{s_1}\right\} \quad \text{and} \quad \bar{w}_2 = g + \frac{d-g}{s_2}.$$

Consequently,

$$|\bar{w}_2 - g| > \left|\bar{w}_2 - \left(g + \frac{d-g}{s_1}\right)\right| \quad \text{for } k > 2.$$

Thus g is uniquely determined in either case and $B_k(\underline{t})$ is detecting by Theorem 1.

So suppose now that all samples of same size have the same average weight. Clearly, if the two averages are equal, then the defective coin must be $\underline{0}$. If not, then $\underline{t}^c$ or $\underline{1}$ must be defective. It is then straightforward to verify that $B_k(\underline{t})$ can distinguish $\underline{t}^c$ and $\underline{1}$ if and only if E_3 has no solution. ■

COROLLARY 4.1. *For $k \geqslant 2$, $W_N(2^k - 1) = k$ if the value of a linear function $d + \lambda g$ is known.*

Proof. If $\lambda = (2^{k-1} - 1)$, then Corollary 4.1 follows from Corollary 3.1 since omitting $\underline{0}$ does not affect anything. Therefore we assume $\lambda \neq (2^{k-1} - 1)$. We show that there do not exist (g', d') and (g'', d'') satisfying E_3 and the equation $d' + \lambda g' = d'' + \lambda g''$ simultaneously.

Suppose to the contrary that there exist two such pairs. Then

$$d' - d'' = \lambda(g'' - g')$$

and

$$d' - g' = d' - d'' = \lambda(g'' - g') = \lambda(d' - g')/(2^{k-1} - 1).$$

But this is impossible since $d' \neq g'$ and $\lambda \neq (2^{k-1} - 1)$ by our assumptions. ■

COROLLARY 4.2. *Suppose that g and d are integers and that d is known to be in an interval of length less than $2^k - 1$. Then $W_N(2^k - 1) = k$.*

Proof. E_3 implies that

$$d' - d'' = (2^k - 1)(g'' - g').$$

So $(d', g') \neq (d'', g'')$ forces that $|g'' - g'| \geqslant 1$ and $|d' - d''| \geqslant 2^k - 1$, which contradict the assumption. Hence, E_3 has no solution. ■

COROLLARY 4.3. *For $k \geqslant 2$, $W_N(2^k - 1) = k$ if the sign of the deviation is known.*

Proof. $d' - g' = g'' - d''$ has no solution. ■

COROLLARY 4.4. *For $k \geqslant 3$, $W_N(2^k - 1) = k$ if the absolute value of the deviation is known.*

Proof. $g'' - d'' = (d' - g')/(2^{k-1} - 1)$ has no solution. ■

COROLLARY 4.5. *For $k \geqslant 2$, $W_N(2^k - 1) = k$ if the set of possible values of g and the set of possible values of d are disjoint.*

Proof. $g' = d''$ has no solution. ■

By noting that $W_N(2) = 1$ if and only if g or d is known, $W_N(2) = 2$ if and only if the sign of the deviation is known and $W_N(2) = \infty$ otherwise, we complete our treatment of the nonadaptive schemes under partial weight information.

4. Adaptive Detection Schemes. For an adaptive scheme to be detecting, it is no longer necessary that any two coins must have different types. For instance, the halving method (a sequential scheme) mentioned in Section 1 is detecting when the standard weight is known. But either all the coins of the first sample, or all those of its complement are of the same type; what happens is that all these coins are eliminated as candidates for the defective.

By Theorem 4 and the Corollary of Lemma 2 we conclude immediately

THEOREM 5. *For $k \geqslant 3$ and $2^{k-1} + 1 \leqslant n \leqslant 2^k - 2$, $W_A(n) = k$ with no weight information.*

Therefore, we need only be concerned with the two cases $n = 2^k, 2^k - 1$.

The following results show that even a two-stage scheme is, in general, more powerful than a nonadaptive scheme.

THEOREM 6. *For $k \geqslant 4$, $W_A(2^k) = k$ with no weight information.*

Proof. In phase 1, weigh two samples s_1 and s_2 of 2^{k-1} coins each with 2^{k-2} coins in common. If $\overline{w}_1 \neq \overline{w}_2$, then the defective coin is in the symmetrical difference $s = (s_1 - s_2) \cup (s_2 - s_1)$ of 2^{k-1} coins. If $\overline{w}_1 = \overline{w}_2$, then the defective coin is in $\bar{s} = (s_1 \cap s_2) \cup (\bar{s}_1 \cap \bar{s}_2)$. In phase 2, apply B_{k-1} to s or $\bar{s}$ according to the outcomes of phase 1, except that:

(i) The first sample in phase 1 is treated as the first sample of B_{k-1} by ignoring the 2^{k-2} coins in that sample known to be good.

(ii) The second sample of B_{k-1} is enlarged by adding a coin known to be good from the outcome of phase 1.

To show that this scheme is detecting, we first note that there are three samples of different sizes, i.e., sizes 2^{k-1}, $2^{k-2} + 1$ and 2^{k-2}, with no two of them summing to the third. From Lemma 3, g can be uniquely determined. By ignoring coins known to be good, the first sample in phase 1 together with all samples of phase 2 constitute the scheme B_{k-1} for s (or $\bar{s}$). By Theorem 1, B_{k-1} is detecting. ■

COROLLARY. *For $k \geqslant 4$, $W_A(2^k - 1) = k$ with no weight information.*

Proof. Since there is always one coin never tested in the above scheme for 2^k coins, omitting that coin does not affect anything. ■

The only cases not covered by Theorem 5, Theorem 6 and its Corollary are for $n = 2, 3, 4, 7, 8$. It is straightforward to verify that $W_A(n) = W_N(n)$ with respect to any I (no adaptive scheme can do better than the optimal nonadaptive scheme in each of these cases), except for $n = 7$. We show that $W_A(7) = 3$ by giving the following 2-stage algorithm. Let the seven coins be labeled by $0, 1, \ldots, 6$. Phase 1 consists of two samples $s_1 = \{0, 1, 2, 3\}$ and $s_2 = \{2, 3, 4\}$. If $\overline{w}_1 \neq \overline{w}_2$, $s_3 = \{0, 2\}$; if $\overline{w}_1 = \overline{w}_2$, $s_3 = \{5\}$. It is easy to verify that this scheme is detecting.

5. A Summary for the Case of No Weight Information.

$$W_N(n) = W_A(n) = \lceil \log_2 n \rceil$$

except

$$W_N(2) = W_A(2) = \infty,$$

$$W_A(3) = W_A(4) = 3, \quad W_A(8) = 4,$$

$$W_N(2^k - 1) = W_N(2^k) = k + 1 \quad \text{for } k \geqslant 2.$$

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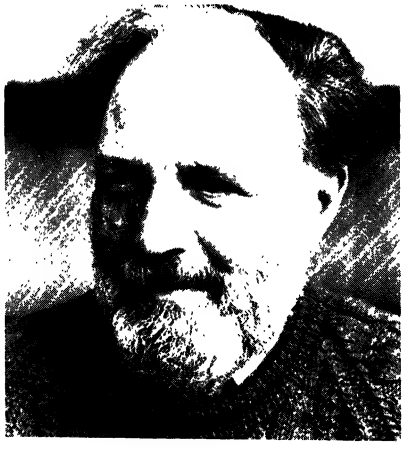
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They didn't prove the Riemann-Roch Theorem, but then Riemann and Roch didn't prove theirs either. (See p. 216.)

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MISCELLANEA

124.

Nothing can be more delightful than to listen to a person laboring through a string of absurdities, in order to prove an axiom.

—Oliver Wendell Holmes, "The Autocrat of the Breakfast Table," in *The New England Magazine*, 1 (1831) 428–431.

ANSWERS TO PHOTOS ON PAGE 180

M. F. Atiyah and I. M. Singer, famous for, among other things, their joint work on the index theorem.

PROOF CHECKING THE RSA PUBLIC KEY ENCRYPTION ALGORITHM[1]

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The development of mathematics toward greater precision has led, as is well known, to the formalization of large tracts of it, so that one can prove any theorem using nothing but a few mechanical rules.—Gödel [**11**]

But formalized mathematics cannot in practice be written down in full, and therefore we must have confidence in what might be called the common sense of the mathematician... We shall therefore very quickly abandon formalized mathematics...—Bourbaki [**1**].

1. Introduction. A formal mathematical proof is a finite sequence of formulas, each element of which is either an axiom or the result of applying one of a fixed set of mechanical rules to previous formulas in the sequence. It is thus possible to write a computer program to check mechanically whether a given sequence is a formal proof. However, formal proofs are rarely used. Instead, typical proofs in journal articles, textbooks, and day-to-day mathematical communication use informal notation and leave many of the steps to the reader's imagination. Nevertheless, by transcribing the sentences of the proof into a formal notation, it is sometimes possible to use today's automatic theorem-provers to fill in the gaps between published steps and thus mechanically check some published, informal proofs.

In this paper we illustrate this idea by mechanically checking the recently published proof of the invertibility of the public key encryption algorithm described by Rivest, Shamir, and Adleman [**17**]. We will briefly explain the idea of public key encryption to motivate the theorem proved.

In [**17**] a mathematical function, here called CRYPT, is defined. CRYPT(M, e, n) is the encryption of message M with key (e, n). The function has the following important properties:

1. It is easy to compute CRYPT(M, e, n).
2. CRYPT is "invertible," i.e., if M is encrypted with key (e, n) and then decrypted with key (d, n), the result is M. That is, CRYPT(CRYPT(M, e, n), d, n) = M, under suitable conditions on M, n, e and d.
3. Publicly revealing CRYPT and (e, n) does not reveal an easy way to compute (d, n). Public key encryption thus avoids the problem of distributing keys via secure means. Each user (e.g., a computer on a network) generates an encryption key and a corresponding decryption key, publicizes the encryption key to enable others to send private messages, and never distributes the decryption key.

The function defined in [**17**] is CRYPT(M, e, n) = $M^e \bmod n$; in addition, algorithms are given for constructing e, d, and n so that CRYPT has the three properties above. The first two properties are proved in [**17**]. The third property is not proved; instead the authors of [**17**] argue that "all the obvious approaches to breaking our system are at least as difficult as factoring n."

Robert S. Boyer received his Ph.D. in mathematics under W. W. Bledsoe at the University of Texas at Austin in 1971. J Strother Moore received his Ph.D. in computational logic under B. Meltzer at the University of Edinburgh in 1973. The automatic theorem-prover described in this paper spent its formative years proving such theorems as the commutativity of Peano multiplication in the Metamathematics Unit of the University of Edinburgh, where Boyer and Moore began collaborating in 1971. After four years of development at Stanford Research Institute, the theorem-prover could prove the uniqueness of prime factorizations and the many other theorems described in *A Computational Logic*. In 1981 Boyer and Moore joined the Computer Sciences faculty of the University of Texas at Austin as associate professors. They also hold research positions in the Institute for Computing Science and Computer Applications, The University of Texas at Austin, Austin, Texas 78712. For the past two years the theorem-prover's talents have been applied to the theorems arising in Boyer and Moore's graduate courses in programming languages and theory of computation, where it maintains a solid A average.

[1]The research reported here was supported by National Science Foundation Grant MCS-8202943 and Office of Naval Research Contract N00014-81-K-0634.

Since there is no known algorithm for efficiently factoring large composites, the security property of CRYPT is obtained by constructing n as the product of two very large (200 digit) primes.

In this paper we focus on mechanically checking the proofs of the first two properties. A precise statement of the "invertibility" property is: CRYPT(CRYPT(M, e, n), d, n) = M, if n is the product of two distinct primes p and q, $M < n$, and e and d are multiplicative inverses in the ring of integers modulo $(p - 1)*(q - 1)$. Our mechanical proof of this theorem requires that we first prove many familiar theorems of number theory, including Fermat's theorem: $M^{p-1} \bmod p = 1$, when p is a prime and $p \not| M$.

2. A Sketch of the Theorem-Prover. The theorem-prover we use is the current version of the system described in [2]. The theorem-prover deals with a quantifier free first order logic providing equality, recursively defined functions, mathematical induction, and inductively constructed objects such as the natural numbers and finite sequences.

The theorem-prover is a large interactive computer program. The main inputs provided by the user are new recursive definitions and conjectures to prove.

Before a proposed definition is admitted as a new axiom, certain conditions are mechanically checked to assure that there exists one and only one function satisfying the definition. The most important condition is that there exist a measure of the arguments of the function that is decreasing in a well-founded sense in each recursive call in the definition. The mechanized definitional principle can guess simple measures and well-founded relations; more complicated ones can be supplied by the user. Once a candidate measure and relation are found, the mechanical theorem-prover is invoked to prove theorems sufficient to admit the proposed definition.

Given a conjecture to prove, the theorem-prover orchestrates the application of many proof techniques under heuristic control. The main proof techniques used are:

Simplification–The system applies axioms, definitions, and previously proved theorems as rewrite rules to simplify expressions. For example, if f is a defined function, it is sometimes useful to replace an instance of $f(x)$ by the corresponding instance of the definition of f. To avoid looping the simplifier contains elaborate heuristics to control the use of recursive definitions. One of the main heuristics is to expand $f(x)$ to introduce a recursive call provided the arguments to the call already occur in the conjecture. Axioms and previously proved theorems are also used as rewrite rules. For example, the theorem

$$\text{prime}(p) \rightarrow [p|a*b \leftrightarrow (p|a \vee p|b)]$$

is used to replace instances of $p|a*b$ by $(p|a \vee p|b)$ whenever the hypothesis prime(p) can be established by simplification. The simplifier also contains decision procedures for propositional calculus, equality, and those formulas of rational arithmetic that can be built up from variables, integers, $+$, $-$, $=$, $<$, and $\sim$.

Elimination of Undesirable Function Symbols–The system uses axioms and previously proved lemmas to eliminate certain function symbols from the conjecture being proved. For example, it is a theorem that for each natural number i and each positive integer j there exist natural numbers $r < j$ and q such that $i = r + qj$. By replacing i with $r + qj$, the system can transform the expression $i \bmod j$ to simply r and i/j to simply q.

Strengthening the Conjecture to be Proved–It is frequently the case that to prove some theorem by induction, it is necessary to prove a stronger theorem than that initially posed. Our system contains several heuristics for guessing stronger

conjectures to try to prove. One heuristic involves "using" equality hypotheses by substituting one side for the other elsewhere in the conjecture and then strengthening the conjecture by throwing away the equality hypothesis. Another heuristic replaces certain nonvariable expressions in the conjecture by new variables.

For example, consider proving $(n^i)^j = n^{i*j}$, by induction on j. The induction step is

$$(n^i)^j = n^{i*j} \rightarrow (n^i)^{j+1} = n^{i*(j+1)}.$$

The conclusion simplifies to $n^i*(n^i)^j = n^{i+i*j}$. The system then applies the first heuristic above, using and throwing away the equality hypothesis, to obtain the goal

$$n^i*n^{i*j} = n^{i+i*j}.$$

The second heuristic then produces the goal $n^i*n^k = n^{i+k}$ by replacing $i*j$ with the new variable k. This final goal, a natural lemma about exponentiation, is then proved by a second appeal to induction.

Induction–When all else fails, it is useful to try mathematical induction. The selection of an "appropriate" induction is based on an analysis of the recursive functions mentioned in the conjecture. For example, since $n!$ is recursively defined in terms of $(n-1)!$, when n is not 0, the presence of $n!$ in a conjecture, $p(n)$, suggests a simple induction on n. The base case is $p(0)$. In the induction step, n is non-0, the inductive hypothesis is $p(n-1)$, and the induction conclusion is $p(n)$. Observe that the $n!$ in the conclusion can now be expanded by the simplifier and will produce a term involving $(n-1)!$, about which we have a hypothesis. Similarly, since we define $i \bmod j$ recursively in terms of $(i-j) \bmod j$, when $0 < j \leq i$, the occurrence of $i \bmod j$ in a conjecture suggests an inductive argument in which we suppose $0 < j \leq i$ and take as an inductive hypothesis the conjecture with i replaced by $i-j$.

Typically, a conjecture to be proved contains many different recursive functions and they each suggest different induction schemas. Our induction mechanism contains many heuristics for combining and choosing between the suggested inductions. That the inductions invented by the system are valid may be proved by considering the well-foundedness theorems proved when recursive functions are admitted.

Readers interested in more details of the theorem-prover should see [2] in which the system, as of May, 1978, is described at a level of detail sufficient to permit reproduction of our results. Several chapters of [2] are devoted to detailed annotated proofs by the system, including its proof of the uniqueness of prime factorizations. Improvements made to the system since the publication of [2] include the addition of the above mentioned decision procedures for equalities and simple arithmetic inequalities, the extension of the definitional principle to include reflexive functions as described in [16], and a metafunction facility permitting the incorporation of new simplifiers after they have been mechanically proved correct [3].

Finally, we have added a primitive "hint" facility so that the user can tell the theorem-prover how to prove a theorem when its heuristics lead it down blind alleys. There are two types of hints used in this paper. The first permits the user to say "use Lemma x with instantiation y." The interpretation of this hint is to obtain the lemma named, instantiate its variables as directed by y, and add the resulting formula as a hypothesis to the conjecture being proved. The system then

applies its usual heuristics. The second type of hint is "induct as suggested by the recursion in f" where f is a previously admitted recursive function.

The theorem-prover is automatic in the sense that once it begins a proof attempt, no user guidance is permitted. However, every time it accepts a definition or proves a theorem, it stores the definition or theorem for future use. By presenting the theorem-prover with an appropriate sequence of lemmas to prove, the user can "lead" it to proofs it would not otherwise discover. Thus the distinction between a proof checker and an automatic theorem-prover blurs once the system remembers and uses previously proved facts. An automatic theorem-prover merely enables the user to leave out some of the routine proof steps. A sufficiently good automatic theorem-prover might enable the user to check an "informal" proof by presenting to the machine no more material than one would present to a human colleague.

When we began the encryption proofs, we initialized the theorem-prover to the current version of the lemma library listed in Appendix A of [2]. The library contains several hundred previously proved theorems. Most of the theorems in this library were irrelevant to the encryption proofs (e.g., there are many theorems about list processing functions such as REVERSE, FLATTEN, and SORT). However, among the theorems in the library are many elementary facts about addition, multiplication, and integer division with remainder. The deepest number theory result in the library is the uniqueness of prime factorizations.

3. Correctness of CRYPT. To show that $M^e \bmod n$ is easy to compute—even when the numbers involved contain hundreds of digits—Rivest, Shamir and Adleman exhibit an algorithm for computing it in order $\log_2(e)$ steps. Below we define CRYPT as a recursive version of their algorithm and prove that it computes the desired function. The notation $e/2$ below denotes integer division, i.e., the floor of the rational quotient.

The material contained in boxes in this paper represents material typed by the user and checked by the theorem-prover. The boxed material very closely resembles traditional "informal" proofs. To make this more obvious to readers unfamiliar with our formal notation, we have taken the liberty of transcribing the user type-in into conventional mathematical English. Use of the phrase "Hint:" in boxed material notes those occasions on which we gave the system explicit hints.

It may not be immediately obvious to the casual reader that each line follows from the previous ones. We have been careful to give the reader no more or less information than was given the machine and challenge the reader to do the machine's job: verify each line of the boxed material.

Box 1

We define the encryption algorithm as the recursive function CRYPT:

DEFINITION.
CRYPT(M, e, n)
 =
if e *is not a natural number or is* 0,
then 1;

else if e *is even*,
then
 (CRYPT(M, $e/2$, n))$^2 \bmod n$;
else
 (M*(CRYPT(M, $e/2$, n)$^2 \bmod n$))$\bmod n$.

LEMMA. (x*($y \bmod n$))$\bmod n = (x*y) \bmod n$.

COROLLARY. (a*(b*($y \bmod n$)))$\bmod n = (a*(b*y)) \bmod n$.
(Hint: let x be $a*b$ in the preceding lemma.)

THEOREM. CRYPT(M, e, n) *is equal to* $M^e \bmod n$, *provided* n *is not* 1.

In Section 6 we give the actual user type-in for the material in Box 1. In order to reinforce in the reader's mind the fact that the theorem-prover assents to these claims only after proving them, we offer the following comments.

Before accepting the definition of CRYPT the theorem-prover guesses that e decreases in each recursive call and then proves it by showing that when e is a non-0 natural number, $e/2$ is strictly smaller than e.

CRYPT uses the "binary method" of computing M^e (see [15]), which is based on the observation:

$$M^e = \begin{cases} (M^{e/2})^2, & \text{if } e \text{ is even,} \\ M*(M^{e/2})^2, & \text{if } e \text{ is odd.} \end{cases}$$

However, by doing multiplications modulo n, CRYPT keeps the intermediate results manageably small and computes $M^e \bmod n$.

The first lemma above—i.e., that $(x*(y \bmod n)) \bmod n$ is $(x*y) \bmod n$—establishes that the intermediate mods can be dropped. This lemma is obviously important in establishing that CRYPT computes $M^e \bmod n$. We brought this fact to the system's attention before even attempting to have the system prove properties of CRYPT.

How did the theorem-prover prove $(x*(y \bmod n)) \bmod n = (x*y) \bmod n$? It first tried simplification, but no known rewrites could be applied under our heuristics. The system then decided to eliminate $(y \bmod n)$ by replacing y with $r + nq$, where $r < n$. To permit this, the system case split on whether y is a natural number and n is a positive integer. The "pathological" cases, where y was not numeric or n was nonpositive, yielded immediately to simplification. In the case where y was a natural number and n was positive, the system replaced y with $r + nq$, where $r < n$. Thus $(y \bmod n)$ became r and the left hand side of the conjecture became $x*r \bmod n$. On the right, $x*y$ became $x*(r + nq)$. The simplifier then distributed the multiplication over the addition (using a previously proved lemma in the library) and obtained $(x*r + n*q*x) \bmod n$, which was further rewritten to $x*r \bmod n$ by the lemma that $i + nj \bmod n$ is $i \bmod n$. The left and right hand sides were then identical. The machine spent about 23 seconds of cpu time on the proof in Interlisp-10 on a DEC 2060.

The corollary above, that $(a*(b*(y \bmod n))) \bmod n$ is $(a*(b*y)) \bmod n$, follows trivially from the previous line, by letting x be $(a*b)$ and applying the associativity of multiplication. Since this observation is uninteresting to a human proof checker, the need for it in our mechanical proof exposes a deficiency in our mechanical theorem-prover. Why is this line needed by the machine? The reason has to do with the order in which rewrite rules are applied. Consider the term $((a*b)*(y \bmod n)) \bmod n$. The lemma just proved can be applied as a rewrite rule from left to right, to eliminate the intermediate mod and produce $(a*b)*y \bmod n$, to which we can then apply associativity to get $a*(b*y) \bmod n$. However, if we apply associativity first, we obtain $a*(b*(y \bmod n)) \bmod n$ and we can no longer use the first lemma from left to right.[2] The second observation solves this problem.

We did not anticipate the machine's need for the corollary. Instead, immediately after proving the lemma, we thought the machine could prove that CRYPT computes $M^e \bmod n$. We commanded it to do so and watched its proof attempt on the screen. (Imagine watching a colleague proving the theorem on the blackboard.) We saw the term $(a*(b*(y \bmod n))) \bmod n$ arise and remain "unsimplified" even though we knew it was $(a*(b*y)) \bmod n$. At that point we interjected with the corollary.

We now consider the machine's acceptance of the final theorem in Box 1, the claim that CRYPT computes the desired function. The first time we submitted the claim we did not include the hypothesis that $n \neq 1$, because the hypothesis is not noted by Rivest, Shamir and Adleman,

[2] This is the Knuth-Bendix problem in rewrite driven simplification. See [13] for an elegant solution to the problem in certain cases.

who imply that the algorithm always computes $M^e \bmod n$. However, the theorem-prover failed to prove the simpler conjecture and exhibited a formula showing that the encryption algorithm does not compute $M^e \bmod n$ when e is 0 and n is 1. In practice, n is always larger than 1, so the additional hypothesis is no burden.

The theorem-prover proved the final claim by induction on e. The base case is that e is not a natural number or is 0. In the induction step, it supposes e is positive and assumes the conjecture for $e/2$. Observe that this induction is precisely the one suggested by the recursion in CRYPT. The proof required about 6 minutes of cpu time.

4. Fermat's Theorem. The proof of the invertibility of CRYPT in [17] assumes the reader is familiar with elementary number theory up through Fermat's theorem. While a production model proof checker for informal proofs would come factory equipped with a good number theory library, we had no such library when we began the encryption proofs. We therefore had the system prove the following theorems:

- Many elementary facts about remainder and exponentiation.
- Suppose p and q are distinct primes, $a \bmod p = b \bmod p$, and $a \bmod q = b \bmod q$. Then $a \bmod p*q = b \bmod p*q$.[3] Hence, under the additional hypothesis $b < p*q$, $a \bmod p*q = b$.
- Suppose p is a prime and p does not divide M. Then $M*x \bmod p = M*y \bmod p$ iff $x \bmod p = y \bmod p$.[4] Hence, by letting y be 1, if p is a prime, $M*x \bmod p = M \bmod p$ iff either $p|M$ or $x \bmod p = 1$.
- The Pigeon Hole Principle: If L is a sequence of length n, every element of L is a positive integer, no element occurs twice in L, and every element of L is less than or equal to n, then L is a permutation of the sequence $[n\ n-1 \ldots 2\ 1]$.
- The following straightforward observations about permutations and the concept of the product of the elements in a sequence:

 *If $L1$ is a permutation of $L2$, then the product of the elements in $L1$ is equal to that of the elements in $L2$.

 *The product of the elements in $[n\ n-1 \ldots 2\ 1]$ is $n!$.

 *Hence, if L is a permutation of $[n\ n-1 \ldots 2\ 1]$, then the product of the elements in L is $n!$.

 *If p is a prime and $n < p$, then p does not divide $n!$.

We then had the theorem-prover check the proof of Fermat's theorem in [14]. (See Box 2.)

To prove Lemma 1 the system inducts on n and uses the previously proved lemma that intermediate mods can be dropped.

To get the system to prove the next lemma, that $j*M \bmod p$ is not a member $S(i, M, p)$, we had to instruct it to induct on i. (If left to its own devices, it chooses here to induct on i and j simultaneously.) The proof is as follows. The base case, when $i = 0$, is easy because $S(0, M, p)$ is empty. In the induction step we assume that $j*M \bmod p$ is not a member of $S(i-1, M, p)$ and must prove that it is not a member of $S(i, M, p)$. But by definition $S(i, M, p)$ is the sequence consisting of $i*M \bmod p$ followed by the sequence $S(i-1, M, p)$. The induction hypothesis establishes that $j*M \bmod p$ is not an element of the latter. It suffices to prove that $j*M \bmod p \neq i*M \bmod p$. Suppose the contrary. Then $j \bmod p = i \bmod p$, since $p \not| M$. Thus $j = i$, contradicting $i < j$.

[3] Cf. Theorem 53 of Hardy and Wright's *An Introduction to the Theory of Numbers*.

[4] Cf. Theorem 55 of Hardy and Wright.

Box 2

DEFINITION. We define $S(n, M, p)$ to be the sequence:

$$[n*M \bmod p, (n-1)*M \bmod p, \ldots, 1*M \bmod p].$$

LEMMA 1. *The product of the elements in* $S(n, M, p) \bmod p$ *is equal to* $n!*M^n \bmod p$.

LEMMA. *If* p *is a prime that does not divide* M *and* $i < j < p$, *then* $j*M \bmod p$ *is not a member of* $S(i, M, p)$. (Hint: induct on i.)

LEMMA. *If* p *is a prime that does not divide* M *and* $n < p$, *then no element of* $S(n, M, p)$ *occurs twice.*

LEMMA. *If* p *is a prime that does not divide* M *and* $n < p$, *then each element of* $S(n, M, p)$ *is a positive integer.*

LEMMA. *If* $p > 0$, *then each element of* $S(n, M, p)$ *is less than or equal to* $p - 1$.

LEMMA. $S(n, M, p)$ *has* n *elements.*

FERMAT'S THEOREM. *If* p *is a prime that does not divide* M, *then* $M^{p-1} \bmod p = 1$.

(Hint: From Lemma 1 we have that the product of the elements in $S(p-1, M, p) \bmod p$ is $(p-1)!*M^{p-1} \bmod p$. But from the Pigeon Hole Principle we have that $S(p-1, M, p)$ is a permutation of $[p-1, \ldots 2, 1]$.)

The remaining lemmas above are proved by similar inductions of the system's own invention.

The system's proof of the main theorem is then as follows. The hints lead it to conclude that $(p-1)!*M^{p-1} \bmod p = (p-1)! \bmod p$. Since p does not divide $(p-1)!$, we can cancel $(p-1)!$ from both sides and get $M^{p-1} \bmod p = 1$.

5. Invertibility of CRYPT. We now prove that CRYPT(CRYPT(M, e, n), d, n) = M, if n is the product of two distinct primes p and q, $M < n$, and e and d are multiplicative inverses in the ring of integers modulo $(p-1)*(q-1)$.

Box 3

LEMMA 2. *For all primes* p, $(M*M^{k*(p-1)}) \bmod p = M \bmod p$.

COROLLARY. *If* p *and* q *are prime, then*

$$(M*M^{k*(p-1)*(q-1)}) \bmod p = M \bmod p$$

and

$$(M*M^{k*(p-1)*(q-1)}) \bmod q = M \bmod q.$$

(Hint: take two instantiations of (2).)

LEMMA 3. *If* p *and* q *are distinct primes,* M *is a natural number less than* $p*q$, *and* $x \bmod (p-1)*(q-1)$ *is* 1, *then* $M^x \bmod p*q = M$.

RSA THEOREM. *If* p *and* q *are distinct primes,* n *is* $p*q$, M *is a natural number less than* n *and* $e*d \bmod (p-1)*(q-1)$ *is* 1, CRYPT(CRYPT(M, e, n), d, n) $= M$.

The proof of Lemma 2 can be seen by rearranging the exponents and mods so that $(M*M^{k*(p-1)}) \bmod p$ becomes $(M*(M^{p-1} \bmod p)^k \bmod p) \bmod p$. Fermat's Theorem can then be used to replace $M^{p-1} \bmod p$ by 1. The corollary is obvious.

To prove Lemma 3 the system first observes that, for some k, x is $k*(p-1)*(q-1)+1$.

Thus, $M^x \bmod p*q$ is $(M*M^{k*(p-1)*(q-1)}) \bmod p*q$. Now recall the previously mentioned result that if p and q are distinct primes and $a \bmod p = b \bmod p$ and $a \bmod q = b \bmod q$, then $a \bmod p*q = b \bmod p*q$. Letting a be $M*M^{k*(p-1)*(q-1)}$ and b be M and appealing to the corollary above, the system concludes that $M^x \bmod p*q$ is $M \bmod p*q$, which, in turn, is M since $M < p*q$.

Finally, to prove the RSA Theorem itself, the system appeals to the correctness of CRYPT and the hypothesis that $n = p*q$ to reduce the conclusion to $(M^e \bmod p*q)^d \bmod p*q = M$. It then eliminates the intermediate mod, collects the exponents e and d and appeals to Lemma 3.

6. Sample Input to the Theorem-Prover. To illustrate the sense in which the boxed material is an English transcription of the user supplied type-in to our theorem-prover, we give below the type-in for the material in Box 1. We use the prefix syntax of Church's lambda calculus and McCarthy's LISP. It would be straightforward to arrange for the system to read and print according to a more elaborate grammar, but we prefer the simplicity of prefix notation.

```
DEFINITION.
(CRYPT M E N)
   =
(IF (ZEROP E)
    1
    (IF (EVEN E)
     (REMAINDER (SQUARE (CRYPT M (QUOTIENT E 2) N))
                N)
     (REMAINDER
       (TIMES M
         (REMAINDER (SQUARE (CRYPT M (QUOTIENT E 2) N))
                    N))
       N)))

THEOREM. TIMES.MOD.1 (rewrite):
(EQUAL (REMAINDER (TIMES X (REMAINDER Y N)) N)
       (REMAINDER (TIMES X Y) N))
THEOREM. TIMES.MOD.2 (rewrite):
(EQUAL (REMAINDER (TIMES A (TIMES B (REMAINDER Y N)))
                  N)
       (REMAINDER (TIMES A B Y) N))
Hint: Use TIMES.MOD.1 with X replaced by (TIMES A B).
THEOREM. CRYPT.CORRECT (rewrite):
(IMPLIES (NOT (EQUAL N 1))
         (EQUAL (CRYPT M E N) (REMAINDER (EXP M E) N)))
```

Readers interested in the complete list of definitions and theorems typed by the user should see section 8 of [5].

7. Conclusion. We have shown how an existing mechanical theorem-prover was used to check a recently published proof. Among the other mathematically interesting proofs performed by our theorem-prover are:

- WILSON'S THEOREM. *If p is a prime, then* $(p-1)! \bmod p = p - 1$ [18];
- the law of quadratic reciprocity [19];
- the termination over the integers of the Takeuchi function [16]:

$$\begin{aligned}\mathbf{tak}(x, y, z) = \text{if } & x \le y \\ & \text{then } y \\ & \text{else } \mathbf{tak}(\mathbf{tak}(x-1, y, z), \\ & \qquad \mathbf{tak}(y-1, z, x), \\ & \qquad \mathbf{tak}(z-1, x, y));\end{aligned}$$

- the soundness and completeness of a decision procedure for propositional calculus [2];
- the existence of nonprimitive recursive functions;
- the Turing completeness of the Pure LISP programming language [8]; and
- the recursive unsolvability of the halting problem for Pure LISP [6].

We take these examples as evidence that proof checking mathematics is not only a theoretical but also a practical possibility. We doubt that the mechanical theorem-provers of today could be easily used to check theorems at the frontiers of mathematics. The less ambitious motivation behind much automatic theorem-proving research—certainly ours—is to mechanize the often mundane and tedious proofs arising in connection with computer programs. For example, our theorem-prover has been used to prove thousands of theorems related to the correctness of various programs [4], [7], communications protocols [9], and computer security [10]. Because of the high cost of bugs in software, the increasing impact of software due to cheap microprocessors, and the relatively shallow nature of most program correctness proofs, we expect to see, within the decade, commercial use of mechanical theorem-provers and formal logic in software development. The construction of an automatic theorem-prover that can advance the frontiers of mathematics, however, must still await another Gödel or Herbrand.

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NOTES

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For instructions about submitting Notes for publication in this department see the inside front cover.

ON TWO HALVES BEING TWO WHOLES

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The sum of the parts is not always equal to the whole, at least in the sense of Lebesgue outer measure. In particular, if $m(A)$ denotes the Lebesgue outer measure of a subset A of the set of real numbers R and A' is the complement of A, then $m(A \cap I) + m(A' \cap I) \geqslant m(I)$ for any set I. Strict inequality can occur when A is a nonmeasurable set. In this note we extend an argument in [4], formalize an Archimedean property, and then give some examples of sets A such that

$$m(A \cap I) = m(I) = m(A' \cap I)$$

for all intervals I. Furthermore we demonstrate a decomposition result, showing that the sum $m(A \cap I) + m(A' \cap I)$ counts the "nonmeasurable part" of A in I twice.

Let I be a nonempty open interval of R. For any subset A of R let

$$\beta(A|I) = \sup\{m(A \cap J)/m(J) : J \text{ is a nonempty open bounded subinterval of } I\},$$

$$\alpha(A|I) = \inf\{m(A \cap J)/m(J) : J \text{ is a nonempty open bounded subinterval of } I\}.$$

Intuitively, β and α measure the extremal densities of the set A in I. We shall show that the only possible values for β and α are 0 and 1. Before doing so we state some clear relationships in the following two lemmas.

LEMMA 1. *Let $I_1 \subset I_2$ and $A_1 \subset A_2$. Then*

$$\alpha(A_2|I_1) \geqslant \alpha(A_1|I_2) \quad \text{and} \quad \beta(A_1|I_1) \leqslant \beta(A_2|I_2).$$

LEMMA 2. *Let A be any subset of R, and let I be a nonempty open interval. If there exists an open bounded subinterval J of I such that $\beta(A|J) = 1$ or $\alpha(A|J) = 0$, then $\beta(A|I) = 1$ or $\alpha(A|I) = 0$, respectively. Furthermore, if $\alpha(A|J) = 1$ for every nonempty open bounded subinterval J of I, then $\alpha(A|I) = 1$.*

THEOREM 1. *Let I be a nonempty open interval. For any set A with $m(A \cap I) > 0$, we have*
(i) $\beta(A|I) = 1$ *and*
(ii) $\alpha(A|I) = 0$ or $\alpha(A|I) = 1$.

Proof. In view of Lemma 2, it is sufficient to consider I as bounded. To show (i) suppose that b is a positive number less than 1 such that $m(A \cap J) \leqslant bm(J)$ for all open intervals J of I. Let ε be a positive number less than $(1/b - 1)m(A \cap I)$. Choose U to be an open subset of I covering $A \cap I$ such that $m(U) < m(A \cap I) + \varepsilon$. Write $U = \bigcup_{n=1}^{p} J_n$ where the J_n are pairwise disjoint open subintervals of I and p is either a positive integer or ∞. Let $U'' = \bigcup_{k=1}^{p'} J_k''$ where J_k'' are precisely those J_n such that $m(A \cap J_n) > 0$. As such the J_k'' are pairwise disjoint and p' is either a positive integer or ∞. Then

$$m(A \cap I) + \varepsilon > \sum_{n=1}^{p} m(J_n) \geqslant \sum_{k=1}^{p'} m(J_k'') = \sum_{k=1}^{p'} \left(m(J_k'')/m(A \cap J_k'')\right) m(A \cap J_k'')$$

$$\geqslant \sum_{k=1}^{p'} (1/b)\, m(A \cap J_k'') = (1/b) \sum_{n=1}^{p} m(A \cap J_n) \geqslant (1/b)\, m(A \cap I).$$

This implies the contradiction that $\varepsilon > (1/b - 1)m(A \cap I)$. Hence $\beta(A|I) = 1$.

To show (ii) we first consider the measurable case; that is, let A be a measurable set and suppose $\alpha(A|I) < 1$. Let J be an open subinterval of I such that $m(A \cap J) < m(J)$. Thus $m(A' \cap I) > 0$ and so $\beta(A'|I) = 1$ by (i). Then given an ε with $0 < \varepsilon < 1$ there is an open interval $J_0 \subset I$ such that

$$m(A' \cap J_0)/m(J_0) > 1 - \varepsilon = \big(m(A' \cap J_0) + m(A \cap J_0)\big)/m(J_0) - \varepsilon,$$

leaving $m(A \cap J_0)/m(J_0) < \varepsilon$, and hence $\alpha(A|I) = 0$. Finally we establish (ii) for sets in general. Suppose A is a set with $\alpha(A|I) < 1$. Let ε be a positive number such that $\alpha(A|I) + 2\varepsilon < 1$. Then there is an open interval $J_0 \subset I$ with $m(A \cap J_0) < (\alpha(A|I) + \varepsilon)m(J_0)$. Let V be an open subset of J_0 covering $A \cap J_0$ with measure less than $(\alpha(A|I) + 2\varepsilon)m(J_0)$. By the above, since V is measurable we have $\alpha(V|J_0) = 0$. By Lemma 1 we have $\alpha(J_0 \cap A|J_0) = 0$; that is, $\alpha(A|J_0) = 0$. Hence by Lemma 1 again, we have $\alpha(A|I) = 0$.

A subset A of R is an *Archimedean* set if the set of all real numbers r such that $A + r = A$ is dense in R. We show now that any Archimedean set of real numbers which occupies some measure of R must occupy the entire measure of R.

THEOREM 2. *Let A be an Archimedean set with positive measure. Then for any interval I, $m(A \cap I) = m(I)$.*

Proof. We may assume that I has nonempty interior; otherwise we are done. Let J be any nonempty open bounded subinterval of I, denoted by $J = (a, b)$. Let n be a positive integer. By Theorem 1, there exists a nonempty open interval J_0 and a positive integer n_0 with

$$m(J_0)/2 < m(A \cap J_0) \quad \text{and} \quad 1/(2n_0) \leqslant m(J_0) < 1/n_0 \leqslant m(J)/n.$$

Let k be the maximum number of adjacent open subintervals J_i of length $1/n_0$ contained in J, where

$$J_i = \big(a + (i-1)/n_0, a + i/n_0\big), \quad \text{with } 1 \leqslant i \leqslant k.$$

Note that $k/n_0 \to m(J)$ as $n \to \infty$. Since A is an Archimedean set, for each i there is an r_i in R with $J_0 + r_i \subset J_i$ and $A + r_i = A$. Since Lebesgue measure is translation invariant, we have

$$\begin{aligned} m(J_i)/4 \leqslant m(J_0)/2 &< m(A \cap J_0) = m\big((A \cap J_0) + r_i\big) \\ &= m\big(A \cap (J_0 + r_i)\big) \leqslant m(A \cap J_i). \end{aligned}$$

Since the J_i are pairwise disjoint open intervals of J, we have

$$m(A \cap J) \geqslant \sum_{i=1}^{k} m(A \cap J_i) \geqslant km(J_i)/4 \to m(J)/4 \quad \text{as } n \to \infty.$$

Hence $m(A \cap J) \geqslant m(J)/4$ for all J. By Theorem 1, $m(A \cap I) = m(I)$.

We now give three examples in which both the set and its complement occupy all of R, in the sense of Lebesgue outer measure.

EXAMPLE 1. Let H be a Hamel base for R, where we think of R as a vector space over the rational numbers Q. Let h_0 be an element in H. Let

$$A = \left\{ \sum_{i=1}^{n} q_i h_i : q_i \text{ is in } Q, h_i \text{ is in } H, h_i \neq h_0, n \text{ is a positive integer} \right\}.$$

Note that R is the union of the disjoint sets $A + qh_0$, where q ranges over Q. Since Lebesgue measure is translation invariant and subadditive, $m(A) > 0$ and $m(A') > 0$. Since both A and A' are Archimedean sets, Theorem 2 shows that $m(A \cap I) = m(I) = m(A' \cap I)$ for all intervals I.

EXAMPLE 2. Define an equivalence relation on R wherein $x \sim y$ iff $x - y$ is a rational number

with odd denominator when expressed as a fraction in lowest terms. From each equivalence class pick an α and let B be the set of all numbers $\alpha + p/q$ for all odd integers p and q and all α. Then B' is the set of all numbers $\alpha + 2p/q$ for all integers p and odd integers q and all α. It follows easily that for any pair of odd integers p and q we have $B' + p/q = B$. By translation invariance and subadditivity, both B and B' have positive measure. Note that

$$B + 2p/q = B \quad \text{and} \quad B' + 2p/q = B',$$

for any integer p and odd integer q. Since the set of rational numbers of the form $2p/q$ with p being an integer and q an odd integer is dense in R, both B and B' are Archimedean sets. Thus by Theorem 2

$$m(B \cap I) = m(I) = m(B' \cap I)$$

for all intervals I.

EXAMPLE 3. Following [2], construct a subset C of the set of irrational numbers S in the following manner. Well order S according to an order $<$. Let the first element of this ordering be in C. Next suppose for any y in S it has been determined whether x is in C for all $x < y$. Then let y be in C unless there is some z in C with $z < y$ and $y + z$ in Q. Hewitt and Stromberg [2] give an interesting argument showing that for all intervals I we have

$$m(C \cap I) \geqslant m(I)/2 \quad \text{and} \quad m(C' \cap I) \geqslant m(I)/2.$$

By Theorem 1, we have $\alpha(C|R) = 1 = \alpha(C'|R)$ and so

$$m(C \cap I) = m(I) = m(C' \cap I)$$

for all intervals I.

A Formula for the Sum of the Parts. We say that a subset C of a measurable set D is *completely nonmeasurable* with respect to D if for all measurable subsets E of D with positive measure the set $C \cap E$ is nonmeasurable. For such sets C and D we have

$$m(C) = m(D) = m(C' \cap D).$$

To see the truth of the last statement suppose $m(C) < m(D)$. Then there exists a measurable set D_0 with $C \subset D_0 \subset D$ and $m(C) \leqslant m(D_0) < m(D)$ and so $(D \setminus D_0) \cap C$ is empty, a contradiction. Suppose next that $m(C' \cap D) < m(D)$. Then there exists a measurable set D_0 with

$$C' \cap D \subset D_0 \subset D \quad \text{and} \quad m(C' \cap D) \leqslant m(D_0) < m(D).$$

Hence $D \setminus D_0 \subset C$, a contradiction.

We say that the three sets B, C, *and* D are a *decomposition* of a set A if the following properties hold:

(i) B is a measurable set,
(ii) C is completely nonmeasurable with respect to D,
(iii) $A = B \cup C$,
(iv) $B \cap D$ is empty.

By Zorn's lemma any set A can be uniquely decomposed (up to sets of measure zero). That is, every set can be decomposed into a "measurable part" B and a "nonmeasurable part" C. For any set A and any interval I we have the following "sum of the parts" formula, which says that the "nonmeasurable part" of A in I counts twice.

$$(*) \qquad m(A \cap I) + m(A' \cap I) = m(I) + m(D \cap I).$$

To see that (*) is true, let B, C, and D be a decomposition of A. Let $B_0 = B' \cap D'$ and $C_0 = C' \cap D$. Note that the sets B_0, C_0, and D form a decomposition of A' and that $I \subset B \cup B_0 \cup D$ and that the sets B, B_0, and D are pairwise disjoint. Next recall that for sets W, X, Y, and Z

with X and Z measurable and $W \subset X$ and $Y \subset Z$ and $X \cap Z = \varnothing$, we have $m(W \cup Y) = m(W) + m(Y)$. Hence

$$\begin{aligned} m(A \cap I) + m(A' \cap I) &= m((B \cap I) \cup (C \cap I)) + m((B_0 \cap I) \cup (C_0 \cap I)) \\ &= m(B \cap I) + m(C \cap I) + m(B_0 \cap I) + m(C_0 \cap I) \\ &= m(B \cap I) + m(D \cap I) + m(B_0 \cap I) + m(D \cap I) \\ &= m(I) + m(D \cap I). \end{aligned}$$

When A is measurable, D is a set of measure zero and (*) becomes

$$m(A \cap I) + m(A' \cap I) = m(I). \tag{1}$$

When A is as in examples 1, 2, or 3, then D can be taken to be R, and (*) becomes

$$m(A \cap I) + m(A' \cap I) = 2m(I). \tag{2}$$

A natural question is whether there exist nontrivial intermediate examples of (*). That is, does there exist a set A and an interval I such that for all nonempty open subintervals J of I, $A \cap J$ is nonmeasurable and yet neither (1) nor (2) applies? The next example gives such a set.

EXAMPLE 4. Let I be the unit interval. Let E be a set such that $\alpha(E|I) = 1 = \alpha(E'|I)$, such as the sets in examples 1, 2, and 3. Let G be a measurable subset of I such that $0 < m(G \cap J) < m(J)$ for all nonempty open subintervals J of I, such as the constructions in [1] or [4] or the existence theorem in [3]. Let $A = E \cap G$. It then follows that D can be taken to be G. Therefore

$$m(I) < m(A \cap I) + m(A' \cap I) < 2m(I) \tag{3}$$

even though both $A \cap J$ and $A' \cap J$ are nonmeasurable for all nonempty open subintervals J of I.

I wish to thank Peter Ungar for suggesting the extension of Theorem 1 and Example 2.

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NOTE ON THE CAYLEY-HAMILTON THEOREM

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The purpose of this note is to encourage students to ask "naive" questions, for they can sometimes lead to deeper insights.

The theorem asserts that if A is a square matrix with entries from a commutative ring R and if

$$\chi_A(t) = \det(tI - A) \tag{1}$$

is the characteristic polynomial of A, then $\chi_A(A) = 0$. The usual proof in linear algebra texts works only when the ring R of scalars is a field; it is based on the rational canonical form for a matrix similar to A, which reveals that the minimal polynomial of A divides the characteristic polynomial. A proof that works over an arbitrary commutative ring R is based on the formula (where I is the identity matrix)

$$(\operatorname{adj} P)P = (\det P)I = P(\operatorname{adj} P) \tag{2}$$

for any square matrix P, adj P being the adjoint matrix of P (cf. Jacobson's *Basic Algebra* I, p. 93).

Applying this formula to $P = tI - A$ over the polynomial ring $R[t]$ gives

$$(\text{adj}(tI - A))(tI - A) = \chi_A(t)I = (tI - A)\text{adj}(tI - A). \tag{3}$$

One then substitutes A for t to get $\chi_A(A) = 0$.

The naive question that prompted this note was: "Why bring in the adjoint matrix? Just substitute A for t in the definition of the characteristic polynomial, giving

$$\chi_A(A) = \det(AI - A) = \det 0 = 0."$$

After I pointed out the fallacy, another student asked: "What do you get when you substitute another matrix B for t in $\chi_A(t)$?" to which the naive answer appears to be

$$(\text{adj}(B - A))(B - A) = \chi_A(B) = (B - A)\text{adj}(B - A). \tag{4}$$

We will see that this formula is valid when B commutes with A, but need not be otherwise.

The crux of the matter is to determine which properties of substituting for the variable t remain valid when the coefficients and the substitutions come from a noncommutative ring—in this case, the ring $\mathcal{M}$ of $n \times n$ matrices over R. A matrix with polynomial entries can be viewed as a polynomial with matrix coefficients in a unique way. In the ring $\mathcal{M}[t]$ the variable t commutes with the matrix coefficients from $\mathcal{M}$. But when we substitute $B \in \mathcal{M}$ for t, B need not commute with the coefficients, and we will in general obtain different answers if we write the coefficients to the left of the powers of t than if we put them on the right before substituting. Let us agree to put the coefficients on the right, so that if

$$P = t^m P_m + t^{m-1}P_{m-1} + \cdots + P_0,$$

then by definition

$$P(B) = B^m P_m + B^{m-1}P_{m-1} + \cdots + P_0$$

for any $P \in \mathcal{M}[t]$. The operation of substituting a fixed matrix B for t is then an R-module homomorphism $\mathcal{M}[t] \to \mathcal{M}$, as is easily seen, but the difficulty is that it is *not a ring homomorphism*.

EXAMPLE. Consider $t^2 = \left(t - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)\left(t + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ and substitute $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ for t.

However, there is a formula for the image of an arbitrary product $PQ \in \mathcal{M}[t]$ under substitution:

$$(PQ)(B) = [P(B)Q](B) \tag{5}$$

(i.e., substitute B into P, multiply that by $Q \in \mathcal{M}[t]$, remembering to place the coefficients on the right of the powers of t, then substitute B into the result). We omit the straightforward calculation to verify this formula.

Let us apply this to the special case $P = tI - A$, $Q = \text{adj}(tI - A)$ from formula (3). We obtain

$$\chi_A(B) = [(B - A)Q](B)$$

and in the special case $B = A$ we get the Cayley-Hamilton theorem. Since here $PQ = QP$, we can apply (5) to QP and get

$$\chi_A(B) = BQ(B) - Q(B)A \tag{6}$$

because

$$[Q(B)P](B) = [Q(B)(tI - A)](B) = [tQ(B) - Q(B)A](B) = BQ(B) - Q(B)A.$$

In this formula, we cannot in general write $\chi_A(B) = (B - A)(Q(B))$ as in the incorrect formula (4) because of noncommutativity. Formula (6) is the general answer.

Suppose, however, that we add the assumption that B commutes with A. Comparison of coefficients of the various powers of t in equation (3) enables one to show recursively that

$Q = \operatorname{adj}(tI - A)$ is a monic polynomial of degree $n - 1$ in t whose coefficients are polynomials in the matrix A (the latter polynomials having their coefficients in the ground ring R, of course). So if B commutes with A, it also commutes with all the coefficients of Q, and we do obtain formula (4), which can be expressed as the

GENERALIZED CAYLEY-HAMILTON THEOREM. *Over an arbitrary commutative ring, if matrices A and B commute, then*

$$\chi_A(B) \equiv 0 \mod B - A$$

where $\chi_A(t)$ is the characteristic polynomial of A.

According to formula (5), the kernel of the substitution homomorphism $P \to P(B)$ of $\mathcal{M}[t]$ onto $\mathcal{M}$ is a right ideal. It follows from (5) and division by a monic polynomial that this ideal is actually a principal right ideal, generated by the matrix $tI - B$.

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A SIMPLE PROOF OF THE JENSEN-STEFFENSEN INEQUALITY

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Dedicated to Professor R. P. Boas on the Occasion of his Seventieth Birthday

In this paper we give a very simple proof of the Jensen-Steffensen inequality for convex functions.

Jensen-Steffensen Inequality: Let x and p be two n-tuples of real numbers such that x is nonincreasing, $x_i \in [a, b]$ $(1 \leqslant i \leqslant n)$, and

$$0 \leqslant S_k \leqslant S_n \ (k = 1, \ldots, n-1), \qquad S_n > 0,$$

where $S_k = \sum_{i=1}^{k} p_i \ (k = 1, \ldots, n)$. Then, for every real valued convex function f defined on $[a, b]$,

$$f\left(\frac{1}{S_n}\sum_{i=1}^{n} p_i x_i\right) \leqslant \frac{1}{S_n}\sum_{i=1}^{n} p_i f(x_i). \tag{1}$$

Note that if each p_i is positive, (1) follows easily from the definition of a convex function.

Proof. A convex function f is characterized by having a supporting line at each point, i.e.,

$$f(z) - f(c) \geqslant M(z - c), \tag{2}$$

for all z and c (where M depends on c; in fact, $M = f'(c)$ when $f'(c)$ exists, and M is any number between $f'_-(c)$ and $f'_+(c)$ at the countable set where these are different).

Using (2) we can easily obtain the following known inequalities:

$$f(z) - f(y) \geqslant M(z - y) \quad (z \geqslant y \geqslant c) \quad \text{and} \quad f(z) - f(y) \leqslant M(z - y) \quad (y \leqslant z \leqslant c), \tag{3}$$

where M is defined as above.

Let x and p satisfy the conditions of the Jensen-Steffensen Inequality. Let $\bar{x} = (1/S_n)\sum_{i=1}^{n} p_i x_i$ and let $\bar{S}_k = S_n - S_{k-1}$. Then

$$S_n(x_1 - \bar{x}) = \sum_{i=2}^{n} p_i(x_1 - x_i) = \sum_{j=2}^{n} (x_{j-1} - x_j)\bar{S}_j \geqslant 0,$$

so $\bar{x} \leqslant x_1$. Similarly,

$$S_n(\bar{x} - x_n) = \sum_{i=1}^{n-1} p_i(x_i - x_n) = \sum_{j=1}^{n-1} (x_j - x_{j+1})S_j \geqslant 0,$$

so $x_n \leqslant \bar{x} \leqslant x_1$. Let m be such that $\bar{x} \in [x_{m+1}, x_m]$. Hereafter, M is to be given its value at $c = \bar{x}$.

We can easily show that the following identity is valid:

$$(4)\quad f\left(\frac{1}{S_n}\sum_{i=1}^{n} p_i x_i\right) - \frac{1}{S_n}\sum_{i=1}^{n} p_i f(x_i) = \sum_{i=1}^{m-1} \left(M(x_i - x_{i+1}) - f(x_i) + f(x_{i+1})\right)\frac{S_i}{S_n}$$
$$+ \left(M(x_m - \bar{x}) - f(x_m) + f(\bar{x})\right)\frac{S_m}{S_n}$$
$$+ \left(f(\bar{x}) - f(x_{m+1}) - M(\bar{x} - x_{m+1})\right)\frac{\bar{S}_{m+1}}{S_n}$$
$$+ \sum_{i=m+1}^{n-1} \left(f(x_i) - f(x_{i+1}) - M(x_i - x_{i+1})\right)\frac{\bar{S}_{i+1}}{S_n}.$$

Now, using (3) and (4) we get (1).

REMARKS. (1) It is known (see for example [1] or [2]) that from the Jensen-Steffensen Inequality we can obtain some results from [3]–[7].

(2) R. P. Boas in paper [8] has proved the Jensen-Steffensen inequality using inequality (2). A similar proof of the Jensen inequality for convex functions is given, for example, in [9] (see also [8]).

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THE NUMBER OF HOMOMORPHISMS FROM Z_m INTO Z_n

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Although several abstract algebra textbooks (see, for example, [2, p. 118] and [3, p. 190]) ask their readers to determine the number of group homomorphisms from Z_m into Z_n for particular values of m and n, the authors are not aware of any which ask for a solution to the general problem nor the corresponding problem for ring homomorphisms. This is a bit surprising, since these problems are natural ones and their solutions require only elementary facts from group theory, ring theory, and number theory. In this note, we solve both of these problems.

Consider the group-homomorphism case first. It is clear that the order of the image of a group homomorphism from Z_m into Z_n must divide both m and n and, therefore, is a divisor of $\gcd(m, n)$. Also, if k is a common divisor of m and n, then Z_n has a unique subgroup of order k,

and this subgroup has $\phi(k)$ generators, where ϕ is the Euler phi function. Now, noting that in order to specify a group homomorphism from Z_m onto a subgroup H of Z_n it is necessary and sufficient to map the integer 1 to a generator of H, we see that the number of group homomorphisms from Z_m into Z_n is simply $\Sigma_{k|\gcd(m,n)}\phi(k)$. It follows from number theory [**1**, p. 85] that this sum is $\gcd(m, n)$. So, we have proved the following.

THEOREM. *The number of group homomorphisms from Z_m into Z_n is* $\gcd(m, n)$.

Next, we consider the corresponding problem for the rings. That is, we seek a formula for the number of ring homomorphisms from the ring Z_m into the ring Z_n. First, note, as in the group case, that a ring homomorphism is completely determined by its action on 1. Also, since 1 is an idempotent in Z_m, the image of 1 must be an idempotent in Z_n. Let $n = q_1^{t_1} \cdots q_s^{t_s}$ be the prime-power decomposition of n. Then, Z_n is naturally ring-isomorphic to the direct sum $Z_{q_1^{t_1}} \oplus \cdots \oplus Z_{q_s^{t_s}}$. Next, observe that any ring homomorphism from Z_m into Z_n induces a ring homomorphism from Z_m into $Z_{q_i^{t_i}}$ for $i = 1, \ldots, s$. Now, suppose a is an element of Z_n and a is a (ring) homomorphic image of 1 in Z_m. In the direct sum, let a correspond to $(a_1, a_2, \ldots, a_s)$, where each $a_i \in Z_{q_i^{t_i}}$. Then, each a_i is an idempotent of $Z_{q_i^{t_i}}$, and, therefore, each a_i is 0 or 1. This shows that there are at most 2^s ring homomorphisms from Z_m into Z_n. But, because a ring homomorphism is a group homomorphism, it must also be true that the additive order of a_i divides m. Conversely, if $(a_1, a_2, \ldots, a_s)$ is any member of the direct sum, where each $a_i = 0$ or 1 and the additive order of a_i divides m, then there is a ring homomorphism from Z_m into $Z_{q_1^{t_1}} \oplus \cdots \oplus Z_{q_s^{t_s}}$ which carries 1 to $(a_1, a_2, \ldots, a_s)$. So, the number of ring homomorphisms from Z_m into Z_n is simply the number of s-tuples which meet these two conditions. Since the (additive) order of 0 is 1 and the (additive) order of 1 in $Z_{q_i^{t_i}}$ is $q_i^{t_i}$, we may take $a_i = 0$ or 1 when $q_i^{t_i}$ divides m, and we must take $a_i = 0$ when $q_i^{t_i}$ does not divide m. But, $q_i^{t_i}$ does not divide m if and only if q_i does divide $n/\gcd(m, n)$. Thus, letting $\omega(a)$ denote the number of distinct prime divisors of the integer a, we have proved the following.

THEOREM. *The number of ring homomorphisms from Z_m into Z_n is* $2^{\omega(n)-\omega(n/\gcd(m,n))}$.

Notice that the number of group homomorphisms from Z_m into Z_n is the same as the number of group homomorphisms from Z_n into Z_m, while the corresponding statement for rings is not true.

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MISCELLANEA

122.

I asked him what kind of mathematics he did. He couldn't decide whether or not to tell me. He did mention that at Berkeley he was in a favorable position to study two of the esoteric wonders of our time, subjects only an adept might begin to penetrate. Pure mathematics and the state of California. There were no analogies from the real world that might help him explain either of these.

—Don DeLillo in *The Names*, Alfred A. Knopf, New York, 1982.

THE TEACHING OF MATHEMATICS

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MULTIPLICATIVE ANALOGUES OF SOME STATISTICS

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1. Additive and Multiplicative Averages. One of the averages grossly neglected in the teaching of statistics, as judged by the space devoted to it in text books, is the *geometric mean*.

Let us consider n *positive* numbers: $x_1, x_2, x_3, \ldots, x_n$. Their geometric mean, denoted $G(x)$, is the positive nth root of their product:

$$G(x) = \sqrt[n]{x_1 x_2 \cdots x_n} = \left(\prod_{i=1}^{n} x_i \right)^{1/n}.$$

This is the multiplicative counterpart of the arithmetic mean, the additive average that is most commonly in use:

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n} \sum_{i=1}^{n} x_i.$$

A logarithmic transformation connects the two averages, as expressed by the following equality:

$$\log G(x) = \overline{\log x},$$

that is to say, the logarithm of the geometric mean of a set of (positive) values is the arithmetic mean of their logarithms.

Naturally, each theorem concerning the arithmetic mean can be mapped to its multiplicative analogue, and vice versa.

Table 1 presents a list of (parallel) properties in both modes, the additive and the multiplicative. The first concerns the effect of transformations of the variable on the respective mean. The three following properties deal with rules connecting the means of several subsets to that of their union set ("weighted average"), and with the relation of the means of several variables to the mean of some simple function of them [**2**, pp. 14–15, and **4**, pp. 119–120]. Finally, property 5 presents a basic equality, in each mode, derived directly from the definition of the respective mean.

In what follows, "A" will stand for "additive" and "M" for "multiplicative."

Most of the commonplace measures of statistics are additive. The arithmetic mean is obtained by adding values, and measures of variability are based on differences, either squared or taken as nonnegative values. Suppose we adopt the geometric mean as our M-average; could one devise a multiplicative parallel to the variance (and possibly to other measures of variability) in order to characterize the dispersion of the values? Put differently, what is the M-measure of dispersion that is induced by the A-M isomorphism?

Each of the averages in common use (arithmetic mean, median, mode, midrange) is known to minimize some appropriate "loss-function" [**1**]. Is there any M-loss-function of interest that could be shown to be minimized by the geometric mean?

2. Extending the Analogy. The multiplicative analogue of the *difference* between an observation and an average would be the *ratio* of the respective numbers. Therefore, instead of trying to minimize errors by making some combination of the differences as close to zero as possible, one might endeavour to make some function of the ratios as close to unity as possible. Just as

TABLE 1. Analogous properties of the arithmetic mean and the geometric mean.

	Additive	Multiplicative[a]
1	$\overline{ax} = a\bar{x}$ $\overline{x+b} = \bar{x} + b$ in summary $\overline{cx+d} = c\bar{x} + d$	$G(x^a) = [G(x)]^a$ $G(bx) = bG(x)$ $G(dx^c) = d[G(x)]^c$
2	Let a set of n values x be divided into r subsets, there being n_1 values in the first, n_2 in the second, and so on ($\sum_{i=1}^{r} n_i = n$). Then: $\bar{x} = \dfrac{n_1\bar{x}_{(1)} + n_2\bar{x}_{(2)} + \cdots + n_r\bar{x}_{(r)}}{n}$ where $\bar{x}_{(i)}$ denotes the arithmetic mean of subset i.	$G(x) = \sqrt[n]{G_{(1)}^{n_1} G_{(2)}^{n_2} \cdots G_{(r)}^{n_r}}$ where $G_{(i)}$ denotes the geometric mean of subset i.
3	If $x = x_1 - x_2$, then: $\bar{x} = \bar{x}_1 - \bar{x}_2$.	If $x = x_1/x_2$, then: $G(x) = \dfrac{G(x_1)}{G(x_2)}$.
4	If $x = x_1 + x_2 + \cdots + x_k$, then: $\bar{x} = \bar{x}_1 + \bar{x}_2 + \cdots + \bar{x}_k$.	If $x = x_1 x_2 \cdots x_k$, then: $G(x) = G(x_1)G(x_2) \cdots G(x_k)$.
5	$\sum_{i=1}^{n} (x_i - \bar{x}) = 0$ i.e., the sum of the deviations from the arithmetic mean is equal to zero.	$\prod_{i=1}^{n} \dfrac{x_i}{G(x)} = 1$ i.e., the product of the ratios to the geometric mean is equal to unity.

(a) The multiplicative properties apply only to positive variables.

A-deviations, i.e., differences, denoted d, may be sorted into $d \geqslant 0$ versus $d < 0$, positive M-deviations, i.e., ratios, denoted r, would be classified into $r \geqslant 1$ versus $r < 1$.

The methods employed to eliminate negative differences are: (1-A) computing absolute values, (2-A) squaring the numbers. Obviously, the parallel M-operations (that will be numbered 1-M and 2-M, respectively) should always render the ratio greater than (or equal to) one.

The absolute value of a real number d, denoted Abs(d), is defined as:

$$\text{(1-}A\text{)} \qquad \text{Abs}(d) = \begin{cases} d & \text{if } d \geqslant 0, \\ -d & \text{if } d < 0. \end{cases}$$

Hence, it is always true that Abs(d) $\geqslant 0$.

Let r be a positive real number. The "multiplicative absolute" of r, denoted Mbs(r), will be defined by:

$$\text{(1-}M\text{)} \qquad \text{Mbs}(r) = \begin{cases} r & \text{if } r \geqslant 1, \\ \dfrac{1}{r} & \text{if } r < 1. \end{cases}$$

It follows from the definition that every positive r satisfies: Mbs(r) $\geqslant 1$, and furthermore:

$$\log[\text{Mbs}(r)] = \text{Abs}(\log r).$$

In particular, when $r = a/b$, where a and b are both positive, we obtain, Mbs(a/b) $= a/b$ for $a \geqslant b$, and Mbs(a/b) $= b/a$ for $a < b$.

The multiplicative absolute satisfies analogous relations to those of the "additive" absolute, for example,

A: For every a and b

$$\text{Abs}(a + b) \leqslant \text{Abs}(a) + \text{Abs}(b)$$

M: For every positive a and b

$$\mathrm{Mbs}(a \cdot b) \leqslant \mathrm{Mbs}(a) \cdot \mathrm{Mbs}(b).$$

The multiplicative counterpart of

(2-A) squaring a number d, is

(2-M) computing $r^{\log r}$, for any positive r.

Note that just as $d^2 \geqslant 0$, for any d, so is $r^{\log r} \geqslant 1$, for any positive r (the equality $r^{\log r} = 1$ holds for $r = 1$). Here too, a logarithmic transformation connects the two functions, namely:

$$\log(r^{\log r}) = (\log r)^2.$$

Corresponding to the equalities: $\mathrm{Abs}(x - y) = \mathrm{Abs}(y - x)$ and $(x - y)^2 = (y - x)^2$, for any x and y, we have:

$$\mathrm{Mbs}\left(\frac{a}{b}\right) = \mathrm{Mbs}\left(\frac{b}{a}\right)$$

and

$$\left(\frac{a}{b}\right)^{\log(a/b)} = \left(\frac{b}{a}\right)^{\log(b/a)},$$

for any positive a and b. Since a monotonic relationship exists between the functions $\mathrm{Mbs}(a/b)$ and $(a/b)^{\log(a/b)}$, both may serve as measures of the distance of the ratio from unity.

TABLE 2. Parallel additive and multiplicative rules concerning averages and loss-functions minimized by them.

	Additive	Multiplicative[a]
1	The absolute value of the sum of deviations is minimized by the arithmetic mean.	The multiplicative absolute of the product of ratios is minimized by the geometric mean.
	$\mathrm{Abs}\left[\sum_{i=1}^{n}(x_i - y)\right]$ is minimal for $y = \bar{x}$.	$\mathrm{Mbs}\left(\prod_{i=1}^{n}\frac{x_i}{y}\right)$ is minimal for $y = G(x)$.
2	The sum of the squared deviations is minimized by the arithmetic mean.	The product of the ratios up to the power of their logarithm is minimized by the geometric mean.
	$\sum_{i=1}^{n}(x_i - y)^2$ is minimal for $y = \bar{x}$.	$\prod_{i=1}^{n}\left(\frac{x_i}{y}\right)^{\log(x_i/y)}$ is minimal for $y = G(x)$.
3	The sum of the absolute deviations is minimized by the median.	The product of the multiplicative absolutes of the ratios is minimized by the median.
	$\sum_{i=1}^{n}\mathrm{Abs}(x_i - y)$ is minimal for $y = \mathrm{Med}(x)$.	$\prod_{i=1}^{n}\mathrm{Mbs}\left(\frac{x_i}{y}\right)$ is minimal for $y = \mathrm{Med}(x)$.

(a) The multiplicative rules refer only to positive values $x_1, x_2, \ldots, x_n$.

Suppose we wish to multiplicatively measure the loss incurred by representing a set of positive numbers $x_1, x_2, \ldots, x_n$ by some average. The definitions of the M-loss-functions can be based on the functions (1-M) and (2-M) of the ratios of the actual values to the average in question. Table 2 presents three two-fold loss-rules (an A-rule and its M-image), each of which defines a certain

loss-function and gives the average by which it is minimized. Each rule is reformulated, in the lower section of its cell, in a concise symbolic form.

A logarithmic transformation converts each of the M-rules into its isomorphic A-form.

Loss-rule 1 of Table 2 is derived directly from property 5 of Table 1, and both follow from the definitions of the respective means.

By extending the A-M analogy in rule 2 (Table 2) we obtain the multiplicative counterpart of the variance. Let $x_1, x_2, \ldots, x_n$ be n positive numbers, the multiplicative variance, denoted MV, will be defined by:

$$MV = \sqrt[n]{\prod_{i=1}^{n} \left(\frac{x_i}{G(x)}\right)^{\log(x_i/G(x))}}.$$

Note that

$$\log(MV) = \frac{1}{n}\sum_{i=1}^{n} \left(\log x_i - \overline{\log x}\right)^2 = \mathrm{Var}(\log x).$$

In conjunction with the relation $\log G(x) = \overline{\log x}$, this means that MV would be an appropriate M-measure of dispersion to accompany the geometric mean. One might perhaps use the geometric mean more readily once a matching measure of dispersion is available.

Loss-rule 3 calls for a word in favour of the median: It claims that both the sum of the absolute deviations and the product of the Mbs ratios are minimized by that average. Thus, the median is an average satisfying both additive and multiplicative criteria for reduction of losses. This is true because of the ordinal nature of the median. Unlike the geometric mean, the median satisfies:

$$\log[\mathrm{Med}(x)] = \mathrm{Med}(\log x)$$

where x is a positive variable. This means that in the A-M isomorphism, the median is mapped onto the median of the images.

It is customary to accompany the median by the mean of the absolute deviations as a measure of dispersion:

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{Abs}[x_i - \mathrm{Med}(x)].$$

However, when we use the median of n positive numbers in the "multiplicative" sense, we should better give the appropriate M-measure of dispersion:

$$\sqrt[n]{\prod_{i=1}^{n} \mathrm{Mbs}\left(\frac{x_i}{\mathrm{Med}(x)}\right)}.$$

3. Comments. The inclusion of some of the ideas on the multiplicative analogues in the teaching of statistics may be beneficial for the students beyond the mere informative value of getting acquainted with new measures. Drawing analogies between symbols and relations may strengthen our understanding of both the original concepts and their images, and may enrich the study of elementary statistics for both student and teacher. Moreover, this learning may lay the grounds for the more advanced study of the foundations of measurement [3, pp. 9, 12, 152 and 273], where measurement may be regarded as the construction of homomorphisms (scales) from empirical relational structures of interest into numerical relational structures that are useful. Understanding the additive-multiplicative analogy may enhance the (future) realization that there is a wide choice of equivalent numerical representations.

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TEACHING ABOUT THE REAL NUMBERS

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In the teaching of analysis it is usual to assume without proof that the real numbers form a complete ordered field (or an equivalent statement in more elementary terms). When a proof is taught, it usually depends on a construction of the real numbers by some method such as Dedekind cuts. I wish to explain why I consider these procedures unsatisfactory.

The idea behind them seems to be that mathematics is the study of systems subject to axioms, and these axioms may be chosen arbitrarily, provided only that they are consistent. This approach is all right in subjects like group theory: it produces valid results about groups, where a group is *defined* as a system satisfying certain axioms. But one cannot approach the real numbers in this way, because students already know (probably rather vaguely) what they are. One must therefore justify one's results from what students already know of the real numbers, and cannot use new definitions or constructions. In particular, if one constructs a complete ordered field, then one shows only that there is some complete ordered field, not that the real numbers form one.

The problem is thus to convince students that the real numbers form a complete ordered field. Now everybody believes that they form an ordered field; it is completeness that gives the trouble. Here one needs to know what students think a real number is. I think there are two possibilities: either a measurement (say of length) or a decimal. If one takes numbers as measurements, then one can justify completeness only by appealing to intuition, which may not be convincing. But if one regards real numbers as decimals, then one can give a convincing demonstration of completeness in any of its usual formulations. Thurston ([1], page 8), for instance, proves that a nonempty set of decimals bounded above has a least upper bound. As another example, I shall show that every decimal is the limit of a sequence of rationals and that every Cauchy sequence of decimals has a limit. Indeed a decimal is clearly the limit of its truncations (for instance π is the limit of the sequence $3, 3.1, 3.14, \ldots$). And if (a_n) is a Cauchy sequence of decimals, then one can construct a limit decimal

$$L = m + \frac{d_1}{10} + \frac{d_2}{100} + \cdots \quad (m, d_i \text{ integers}, 0 \leqslant d_i \leqslant 9)$$

by taking m to be the largest integer less than or equal to infinitely many members of the sequence, $m + d_1/10$ to be the largest of the numbers, $m, m + 1/10, \ldots, m + 9/10$ less than or equal to infinitely many members of the sequence, etc. For $r = 0, 1, 2, \ldots$, there is, by construction, a closed interval of length 10^{-r} containing L and infinitely many members of the sequence, from which, since the sequence is a Cauchy sequence, all but finitely many of its members lie within 2×10^{-r} of L. Thus L is a limit for the sequence.

It seems to me that some proof of this nature should be included in every analysis course.

REMARKS. 1. I have arranged this proof to avoid mentioning the fact that the same real number can have different decimal representations (such as $1 = 0.999\ldots$), as I think this fact is widely disbelieved.

2. An axiomatic treatment of the real numbers answers two questions which a professional mathematician might ask: Would one get the same real numbers from a different construction,

CENTER SECTION
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Telegraphic Reviews

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Precalculus, T(13: 1). Plane Trigonometry, Eighth Edition. Fred W. Sparks, Paul K. Rees, Charles Sparks Rees. Prentice-Hall, 1984, xiv + 331 pp, $22.95. [ISBN: 0-13-679225-1] Standard textbook on trigonometry. Changes in this edition include adding about 900 problems and reworking first two chapters. (Sixth Edition, TR, February 1972; Seventh Edition, TR, April 1977.) BK

Education, P, L*. Thinking, The Expanding Frontier. Ed: William Maxwell. Franklin Inst Pr, 1983, xv + 304 pp, $24.95. [ISBN: 0-89168-047-0] Proceedings of the International, Interdisciplinary Conference on Thinking held at the University of the South Pacific, 1982. Collection of presented papers distinguished by the wide variety of perspectives from which cognitive processes are examined. Section titles are: Some Philosophical Perspectives, Thinking and Culture, Teaching Thinking, Thinking About Teaching Thinking, Artificial Intelligence, The Symposium on Gregory Bateson's Mind and Nature: A Necessary Unity. JRG

History, S*(16), P*, L*. Geometry and Algebra in Ancient Civilizations. B.L. van der Waerden. Springer-Verlag, 1983, xii + 223 pp, $29.50. [ISBN: 0-387-12159-5] The author ventures a tentative reconstruction of a mathematical science which may have existed in the Neolithic Age and spread from Central Europe to Great Britain, to the Near East, to India, and to China. Recent discoveries lead one to question the long-held view that the history of mathematics begins with the Babylonians and Egyptians. Interesting reading. CEC

Foundations, S. 1er Congrés Català de Logica Matemàtica. Universitat Politecnica de Barcelona, 1982, 130 pp, (P).

Graph Theory, L. Graph Theory Bibliography with Two Level Key-Word Index. Gerald Berman. U of Waterloo Pr, 1983, (P) [ISBN: 0-88898-046-9]. Volume 1: Bibliography, 569 pp; Volume 2: Key-word Index and Author Index, 403 pp, (P). Lists over 15,000 books and papers in graph theory, primarily those reviewed in Mathematical Reviews, through 1981. The two-level keyword index is based on the words and phrases occurring in the titles (woe to authors fond of cryptic titles). GHM

Combinatorics, S(17-18), P. Formal Knot Theory. Louis H. Kauffman. Math. Notes No. 30. Princeton U Pr, 1983, 167 pp, $12 (P). [ISBN: 0-691-08336-3] Exploration in combinatorics and knot theory, where knots are modelled by a state polynomial. A key combinatorial result gives rise to a duality conjecture, a refinement of the Alexander polynomial, and extension of the Crowell-Murasugi theorem to the class of links. JRG

Number Theory, P. Complex Multiplication. Serge Lang. Grund. der math. Wissenschaften, B. 255. Springer-Verlag, 1983, viii + 184 pp, $42. [ISBN: 0-387-90786-6] Short but extensive treatment of fundamental results of complex multiplication. Summarizes work by Shimura, Taniyama, Serre, Tate,

etc. BK

Linear Algebra, P, L. The Collected Mathematical Papers of Leonard Eugene Dickson, Volume VI. Ed: A. Adrian Albert. Chelsea Pub, 1983, 714 pp, $55. [ISBN: 0-8284-0305-6] This volume contains forty-nine papers and one book, Algebraic Invariants. A bibliography and an index, both annotated to indicate the location of the included papers, appears at the close of the volume. CEC

Group Theory, P. Lecture Notes in Mathematics-1006: Abelian Group Theory. Ed: R. Göbel, L. Lady, A. Mader. Springer-Verlag, 1983, xvi + 771 pp, $30 (P). [ISBN: 0-387-12335-0] Proceedings of a conference held at the University of Hawaii, December 28, 1982 to January 4, 1983. Special emphasis on groups of finite rank without torsion, and on set-theoretic methods. LAS

Algebra, T(15: 1). Rings, Fields and Groups: An Introduction to Abstract Algebra. R.B.J.T. Allenby. Edward Arnold, 1983, xxvi + 294 pp, $23.50 (P). [ISBN: 0-7131-3476-3] Starts by looking at familiar examples of rings, such as Z and Q[x], then discusses rings and fields. Ends with group theory. Has historical information throughout the book. Extensive bibliography at end. BK

Finite Mathematics, T(13-15: 1), L. Finite Mathematics. Daniel Gallin. Scott Foresman, 1984, xviii + 525 pp, $23.95. [ISBN: 0-673-16048-3] A straightforward treatment of the standard elementary topics, written for the beginner, with more attention to detail than in many comparable books. Large selection of applied problems; seems carefully done. Indexes, answers. JS

Finite Mathematics, T(13: 1, 2). Finite Mathematics and Its Applications, Second Edition. Larry J. Goldstein, David I. Schneider. Prentice-Hall, 1984, xvii + 484 pp, $27.95. [ISBN: 0-13-317313-5] Changes from the First Edition (TR, April 1980) include: additional examples and exercises, a new chapter on mathematics of finance, a new section on duality theory, and a revision of the discussion on probability. JRG

Calculus, T(13: 2). Brief Calculus with Applications. Roland E. Larson, Robert P. Hostetler. DC Heath, 1983, xi + 689 pp, $26.95. [ISBN: 0-669-04803-8] Designed for students in management, social and life sciences. Initial chapter of precalculus review followed by discussion of topics common to two-term calculus sequences. Alternative edition covering one term is also available. Detailed introductory examples of applications motivate the discussion. JRG

Calculus, T(13: 1). Brief Calculus with Applications, Alternate Edition. Roland E. Larson, Robert P. Hostetler. DC Heath, 1983, x + 442 pp, $22.95. [ISBN: 0-669-06766-0] Alternate edition comprises chapters 1-6, 8 of Brief Calculus with Applications by the same authors. JRG

Calculus, T(14: 2). Mathematics for Management. Richard C. Lucking. Wiley, 1980, xix + 320 pp, $62.95. [ISBN: 0-471-27779-7] Set theory and logic, calculus, and linear algebra with applications. Emphasis is on breadth of coverage of techniques rather than depth of analysis of particular applications. JRG

Calculus, T?(13), S. Introduction to Analytic Geometry and the Calculus. H.M. Dadourian. Krieger Pub, 1983, x + 246 pp, $12.50. [ISBN: 0-89874-267-6] A bit lacking--no trigonometry, no mention of definition of limits, and only a vague reference to chain rule. However, it gives a good graphing review and nice notes on how to study math and solve problems. Would be nice for a course about the ideas of calculus. BK

Calculus, S*(13). Misteaks...and how to find them before the teacher does...A Calculus Supplement. Barry Cipra. Birkhauser Boston, 1983, xiii + 69 pp, $4.95 (P). [ISBN: 3-7643-3083-X] General techniques for deciding whether the answer to your calculus problem makes sense. Lots of advice (mostly good) and lots of illustrative examples and exercises. JRG

Real Analysis, T(15-17). Primer of Modern Analysis. Kennan T. Smith. Undergrad. Texts in Math. Springer-Verlag, 1983, xv + 446 pp, $36. [ISBN: 0-387-90797-1] A text growing out of an honors course for undergraduates. Part I thoroughly reviews elementary calculus. Part II does enough metric topology and linear algebra to discuss tangent spaces and functions on manifolds. Part III develops integration and measure theory. Exercises occur spread throughout the text to illuminate, probe proofs. Different from having problem sets. (1971 Bogden & Quigley edition, TR, February 1973.) AWR

Real Analysis, S*(16-18), P*, L*. Lecture Notes in Mathematics-999: Iterates of Maps on an Interval. Chris Preston. Springer-Verlag, 1983, 205 pp, $11 (P). [ISBN: 0-387-12322-9] Exposition for as wide an audience as possible (prerequisite: elementary real analysis) of some of the beautiful and bizarre properties of even the simplest discrete nonlinear systems. A fascinating topic for senior independent study or professional recreation, though this exposition appears to be rather formal (theorem, proof, comment, theorem...). GHM

Complex Analysis, P. Convolutions in Geometric Function Theory. Stephan Ruscheweyh. Pr U Montreal, 1982, 168 pp, $14 (P). [ISBN: 2-7606-0600-7] If $f(z) = \sum a_k z^k$ and $g(z) = \sum b_k z^k$, then the series $f*g = \sum a_k b_k z^k$ is called the Hadamard product of f and g. As f*g can be written as a convolution integral, it is also called the convolution of f and g. This monograph surveys the properties of operators $f^* : B \to A$ where A is the analytic function in the unit disc, $f \in A$ and $B \subset A$. Most results survey are less than ten years old and many first appear in this volume. PH

Differential Equations, P. Problemi della fisica matematica. Andrej N. Tichonov, Aleksandr A. Samarskij, Boris M. Budak. MIR Pub, 1982, 748 pp. An Italian translation of a very classical (1955 Russian) compendium of the differential equations of mathematical physics. JAS

Differential Equations, T, S(17-18). Gewöhnliche Differentialgleichungen. Herbert Amann. Walter de Gruyter, 1983, xi + 497 pp, DM 59. [ISBN: 3-11-009573-4] A modern and demanding introductory text on initial-value problems for ordinary differential equations. Nothing on numerical methods. JD-B

Differential Equations, T*(14: 1), S*, P, L*. Differential Equations and Their Applications (Third Edition), Short Version. Martin Braun. Springer-Verlag, 1983, x + 341 pp, $19.80. [ISBN: 0-387-90847-1] This edition features a rewritten section on singular solutions of differential equations, and a new section on Euler equations which motivates expanded treatment of singular equations. The metric system has been adopted in appropriate places. This text continues to be an important work because of the nice blend of theory and exciting applications. (First Edition, TR, December 1975; Extended Review, October 1977; Second Edition, TR, June-July 1978; Third Edition, TR, October 1983.) CEC

Differential Equations, P. Equadiff 5. Ed: Michal Greguš. Teubner-Texte zur Math., B. 47. B.G. Teubner, 1982, 396 pp, 39 M (P). Proceedings of an August 1981 conference on differential equations and their applications held in Bratislava, Czechoslovakia. LAS

Differential Equations, S(18), P, L*. Studies in Partial Differential Equations. Ed: Walter Littman. Stud. in Math., V. 23. MAA, 1982, xiii + 268 pp, $24. [ISBN: 0-88385-125-3] Five articles for "non-specialists"--i.e., those with thorough grounding in analysis but not necessarily partial differential equations--sampling current work in the context of the 200-year history of this rich subject, including boundary value problems, minimal surfaces, probabilistic methods, and propagation of singularities. LAS

Differential Equations, P.** The Method of Differential Approximation. Yu. I. Shokin. Transl: K.G. Roesner. Ser. in Computational Physics. Springer-Verlag, 1983, xiii + 296 pp, $40. [ISBN: 0-387-12225-7] Revised and expanded English version of the 1979 Russian edition. Incorporates recently obtained results triggered by Russian original. Of the 234 references, more than one-half are to papers published since 1973. For the advanced student in engineering and numerical analysis interested in differential approximation methods applied to difference schemes in the solution of equations of hyperbolic type. JK

Differential Equations, T(17: 1), S, P. Semigroups of Linear Operators and Applications to Partial Differential Equations. A. Pazy. Appl. Math. Sci., V. 44. Springer-Verlag, 1983, viii + 279 pp, $29.80. [ISBN: 0-387-90845-5] This book presents the theory of semigroups of bounded linear operators and its application to partial differential equations. Requires a knowledge of elementary functional analysis. Contains an excellent bibliography, but no exercises. CEC

Differential Equations, T(17: 1), S, P. Nonlinear Waves. Tosiya Taniuti, Katsunobu Nishihara. Transl: Tosiya Taniuti, Alan Jeffrey. Pitman Pub, 1983, xiv + 258 pp, $59.95. [ISBN: 0-273-08466-6] The nonlinear wave propagation appearing in so many fields is dealt with in a unified manner. For research workers and students who are working on theoretical problems concerning nonlinear wave propagation in science and engineering. Includes problems and a bibliography. CEC

Differential Equations, P. Lecture Notes in Mathematics-984: Propagation of Singularities for Fuchsian Operators. Antonio Bove, Jeff E. Lewis, Cesare Parenti. Springer-Verlag, 1983, 161 pp, $10.50 (P). [ISBN: 0-387-12285-0]

Numerical Analysis, T(17-18: 2). Numerical Functional Analysis. Colin W. Cryer. Oxford U Pr, 1982, 568 pp, $39. [ISBN: 0-19-853410-8] An introduction to functional analysis for numerical analysts. Covers many of the standard topics in the theory of linear spaces while emphasizing those most useful in numerical analysis. Solutions to nearly all of the exercises are provided in an appendix. AO

Numerical Analysis, T(15-16), S, P. Basic Computational Techniques for Engineers. R.A. Adey, C.A. Brebbia. Wiley, 1983, 208 pp, $24.95. [ISBN: 0-471-88970-9] This is really a text in numerical analysis and numerical methods. The problems studied are those that occur with the greatest frequency in engineering--matrix algebra, linear systems, eigenvalues, eigenvectors, and numerical integration and differentiation. Each section includes a complete program written in Fortran IV. JK

Numerical Analysis, P, L. Quadpack: A Subroutine Package for Automatic Integration. R. Piessens, et al. Ser. in Comput. Math., No. 1. Springer-Verlag, 1983, viii + 301 pp, $22 (P). [ISBN: 0-387-12553-1] QUADPACK is a collection of Fortran programs for automatic quadrature of univariate functions. This book documents the package and includes program listings. AO

Numerical Analysis, P. Lecture Notes in Mathematics-1005: Numerical Methods. Ed: V. Pereyra, A. Reinoza. Springer-Verlag, 1983, 296 pp, $15 (P). [ISBN: 0-387-12334-2] The proceedings of a workshop held in Caracas, Venezuela in June, 1982. Contains 20 papers presenting recent results on a variety of topics in numerical analysis. AO

Numerical Analysis, L. Approximations for Digital Computers. Cecil Hastings, Jr., Jeanne T. Hayward, James P. Wong, Jr. Princeton U Pr, 1955, viii + 201 pp, $25. A classic work on the numerical approximation of univariate functions (Sixth Printing, 1970). AO

Numerical Analysis, P. Numerical Treatment of Inverse Problems in Differential and Integral Equations. Ed: P. Deuflhard, E. Hairer. Progress in Sci. Comput, V. 2. Birkhauser Boston, 1983, xiii + 357 pp, $27.50. [ISBN: 3-7643-3125-9] Proceedings of a September 1982 international workshop in Heidelberg, including applications to chemistry, biology, physics, astronomy, control theory, tomography, and geophysics. LAS

Numerical Analysis, P. Lecture Notes in Computer Science-157: Direct Methods for Sparse Matrices. Ole Østerby, Zahari Zlatev. Springer-Verlag, 1983, vii + 127 pp, $8.50 (P). [ISBN: 0-387-12676-7] This monograph describes algorithms for the efficient solution of large sparse systems of linear equations using incomplete LU factorization and iterative refinement. AO

Functional Analysis, S(16-18), L. Modern Nonlinear Equations, Revision. Thomas L. Saaty. Dover Pub, 1981, 471 pp, $8.50 (P). [ISBN: 0-486-64232-1] Revised and corrected republication of a work published in 1967 by McGraw-Hill (TR, December 1967). Companion to the 1964 book Nonlinear Mathematics by Bram and Saaty. Covers operator equations, functional equations, difference equations, delay-differential equations, integral equations, integro-differential equations and stochastic differential equations. Emphasis is on methods of solution of linear and non-linear equations in function spaces. JK

Functional Analysis, S(18), P. Lecture Notes in Mathematics-995: Banach Spaces, Harmonic Analysis, and Probability Theory. Ed: R.C. Blei, S.J. Sidney. Springer-Verlag, 1983, v + 173 pp, $9.50 (P). [ISBN: 0-387-12314-8] A collection of papers printed originally as lectures at the University of Connecticut during the Special Year in Analysis, 1980-81. Broad range of topics in the general areas of Banach spaces, harmonic analysis, and probability. JS

Functional Analysis, P. Studies in Applied Mathematics. Ed: Victor Guillemin. Advances in Math. Supplementary Stud., V. 8. Academic Pr, 1983, xxv + 163 pp, $36. [ISBN: 0-12-305480-X] A volume of special papers, mostly in applied functional analysis, dedicated to Irving Segal. LAS

Analysis, T(15-16: 1), S, L. A Course in Advanced Calculus. Robert S. Borden. Elsevier Sci Pub, 1983, xii + 388 pp, $29.50. [ISBN: 0-444-00638-9] A modern treatment of classical analysis in Euclidean spaces; as such contains beautiful mathematics. Includes chapters on Forms in E^n (5), Topology (6), Measure and Integration (8). An Applications chapter (11) includes special functions and the Fourier and Laplace transforms. Numerous exercises and problems for a text at this level. Written for graduate-school-bound undergraduates and should be preceded by a theoretical treatment of one-variable calculus. Excellent preparation for graduate analysis, but definitely not for the herd. PH

Analysis, P.** The Analysis of Linear Partial Differential Operators. Lars Hörmander. Springer-Verlag, 1983. I: Distribution Theory and Fourier Analysis. Grund. der math. Wissenschaften, B. 256, ix + 391 pp, $39 [ISBN: 0-387-12104-8]; II: Differential Operators with Constant Coefficients. Grund. der math. Wissenschaften, B. 257, viii + 391 pp, $49.50. [ISBN: 0-387-12139-0] The first volume is an expanded treatment of distribution theory with special emphasis on Fourier analysis, particularly on results related to the stationary phase method and Fourier analysis of singularities. The second volume is an expanded version of Chapters III, IV, V and VII of the author's 1963 book Linear Partial Differential Equations. New chapters on convolution equations, on scattering theory and on methods from several complex variables. End-of-chapter notes and lengthy, but carefully limited, bibliography give recent sources on material presented or on closely related topics. A third volume on operators with variable coefficients is projected. A monumental work by a master. JK

Differential Geometry, T(18), P. Differential Geometry of Foliations: The Fundamental Integrability Problem. Bruce L. Reinhart. Ergebnisse der Math., No. 99. Springer-Verlag, 1983, x + 195 pp, $42. [ISBN: 0-387-12269-9] A survey of various approaches to foliations, in the context of differential geometry as a whole and the integrability problem in particular. The text-like exposition is free of the tone of jargon and loaded with passages of overview. Highly readable. RB

Differential Geometry, P. Harmonic Maps with Symmetry, Harmonic Morphisms and Deformations of Metrics. P. Baird. Res. Notes in Math., No. 87. Pitman Pub, 1983, 181 pp, $16.95 (P). [ISBN: 0-273-08603-0] Given a homotopy class of maps between two Riemannian manifolds, does a metric exist such that the homotopy class contains a harmonic representative? This, the "rendering problem," is solved for homotopy classes in $\pi_n(S^n)$ by deformations of the metrics. RB

Geometry, T*(15-17: 1, 2), S, L*. Differential Geometry and Relativity Theory, An Introduction. Richard L. Faber. Mono. & Texts in Pure & Appl. Math., V. 76. Dekker, 1983, x + 255 pp, $29.75. [ISBN: 0-8247-1749-X] Better mathematics than a physics text and better physics than a mathematics text are both presented with a prerequisite of linear algebra and multivariable calculus. The presentation is really aimed at the undergraduate mathematics major but includes a thorough presentation of relativity through the field equations, the Schwarzschild Solution, and orbits in general relativity. The geometry is relatively classical (coordinates in R^n) in its presentation but modern in its spirit. JAS

Geometry, S, P, L*. Dynamics--The Geometry of Behavior, Part 2: Chaotic Behavior. Ralph H. Abraham, Christopher D. Shaw. Visual Math. Lib.: Visimath, V. 2. Aerial Pr, 1983, xi + 139 pp, $15 (P). [ISBN: 0-942344-02-2] Continuation of an innovative presentation of dynamic phenomena: sketches of solution trajectories in two and three dimensions, supplemented with Apple disks for do-it-yourself representations. Sophisticated models of chaotic behavior (due to Poincaré, Birkhoff, Lorenz, Rössler) are treated visually, without symbolism or any trace of traditional mathematics. (Volume 1, on Periodic Behavior, was published in 1982; TR, August-September 1983.) LAS

Geometry, S*(13-14), P, L.** Lectures in Geometry. M. Postnikov. Transl: Vladimir Shokurov. MIR Pub, 1982, $8.95 each. Semester I, Analytic Geometry, 343 pp; Semester II, Linear Algebra and Differential Geometry, 319 pp. Translation of 1979 Russian edition. Faithful reproduction of fifty-six lectures delivered over a two-semester period to first-year students at Moscow State University. Thorough, but not stifling, axiomatic treatment of analytic geometry, linear algebra and elementary differential geometry of curves and surfaces in three-space. Some examples, but no exercises. Figures are good but geometric interpretation is used solely for motivational purposes. Includes systematic treatment of bivectors and of trivectors. Excellent opportunity to compare our expectations of first-year university students with those of the Russians. A very good buy. JK

Algebraic Topology, S. Sur la Topologie des Surfaces Complexes Compactes. Srinivasacharyulu Kilambi, Gottfried Barthel, Ludger Kaup. Pr U Montreal, 1982, 297 pp, $16 (P). [ISBN: 2-7606-0572-8] Contains 2 sections: (1) on the classification of certain nonsingular compact complex surfaces by Kilambi of the University of Montreal; (2) on the topology of singular compact complex surfaces by Barthel at the University of Bonn and Kaup at the University of Constance. Published under the auspices of the Mathematics Seminar of the NATO Advanced Study Institute. RJA

Topology, T(17-18: 1, 2), P. The Theory of Topological Semigroups. J.H. Carruth, J.A. Hildebrant, R.J. Koch. Pure & Appl. Math., V. 75. Dekker, 1983, vi + 244 pp, $34.75. [ISBN: 0-8247-1795-3] A thorough exposition of the elements, not a research monograph (more advanced volumes are planned). Excellent bibliography, index, but no problems or exercises. JAS

Topology, T(16-17: 1, 2), S, L. Introduction to Topology and Modern Analysis. George F. Simmons. Krieger Pub, 1983, xv + 372 pp, $23.50. [ISBN: 0-89874-551-9] Part I is a standard one-semester course in point-set topology: metric and topological spaces, separation, connectedness, compactness, approximation. For Parts II and III, covering operators and algebras of operators respectively, a good knowledge of modern algebra would be helpful. Modern algebraic topology is not introduced. A reprint of the 1963 edition--with the 1963 bibliography. PZ

Topology, T(14-15: 1, 2), S, L. Introduction to General Topology. K.D. Joshi. Halsted Pr, 1983, xii + 412 pp, $19.95. [ISBN: 0-470-27556-1] Presents the basic material "as simply as possible" for a year-long course in India. Rhetorical style, with heavy emphasis on motivation: three chapters (Aristotelian logic, preliminaries, motivation) precede the definition of a topological space. Numerous exercises with generous hints. For students lacking the maturity demanded by many texts. RB

Topology, P. Continua, Decompositions, Manifolds. Ed: R.H. Bing, William T. Eaton, Michael P. Starbird. U of Texas Pr, 1983, x + 267 pp, $30. Proceedings of the 1980 Texas Topology Symposium: continua theory, decomposition spaces, manifolds, linear isotopies. LAS

Operations Research, T(15-16: 1, 2). Optimization Methods in Operations Research and Systems Analysis, Second Edition. K.V. Mital. Halsted Pr, 1976, xvi + 328 pp, $17.95. [ISBN: 0-470-27391-7] Changes from the First Edition (TR, June-July 1978) include the addition of a chapter on integer programming and substantial revision of the chapter on nonlinear convex programming under the new title "Kuhn-Tucker Theory and Quadratic Programming." JRG

Operations Research, T(14-16: 1), S. Modeling Tomorrow's Energy System: Applications of Linear Programming. T. Owen Carroll. COMAP, 1983, ix + 66 pp, $8.95 (P). [ISBN: 0-912843-00-4] Uses linear programming and matrix algebra to discuss problems of energy resource allocation in U.S. Could be used in a semester course on linear programming. BK

Operations Research, T*(16-17: 1, 2), L*. A Guide to Simulation. Paul Bratley, Bennett L. Fox, Linus E. Schrage. Springer-Verlag, 1983, xix + 383 pp, $28. [ISBN: 0-387-90820-X] A comprehensive introduction to discrete event simulation. Assumes the reader has at least a minimal background in probability, statistics, computer programming, and data structures. AO

Optimization, P. Lecture Notes in Economics and Mathematical Systems-215: Semi-Infinite Programming and Applications. Ed: A.V. Fiacco, K.O. Kortanek. Springer-Verlag, 1983, xi + 322 pp, $20.50 (P). [ISBN: 0-387-12304-0] A collection of 20 papers presented at an international symposium held at the University of Texas at Austin in September, 1981. The papers discuss duality theory, algorithm design, problem analysis and modeling, optimality conditions, and variational principles. AO

Optimization, P. Optimization and Nonsmooth Analysis. Frank H. Clarke. Wiley, 1983, xiii + 308 pp, $34.95. [ISBN: 0-471-87504-X] Uses the generalized gradient and related geometric ideas to develop a general theory undergirding a wide variety of optimization techniques applicable in the presence of non-differentiable (nonsmooth) objective functions. A handsome first volume in a series of monographs and advanced texts to be sponsored by the Canadian Mathematical Society. AWR

Optimization, S(17-18), P. Algebraische Methoden in der ganzzahligen Optimierung. Joachim Piehler. Texte zur Math., Band 51. BG Teubner, 1983, 99 pp, 11 M (P). An account of some recent results on the use of groups, Diophantine equations and graphs in integer programming. JD-B

Optimization, T(15-16: 1), L. Discrete Optimization Algorithms with Pascal Programs. Maciej M. Sysło, Narsingh Deo, Janusz S. Kowalik. Prentice-Hall, 1983, xii + 542 pp, $37.50. [ISBN: 0-13-215509-5] Discusses algorithms for linear and integer programming, packing and covering problems, network optimization problems, and coloring and scheduling problems. Pascal programs implementing many of the algorithms are an integral part of the text. AO

Optimization, T(14-17: 1), S*, P, L**.** Linear Programming. Vašek Chvátal. WH Freeman, 1983, xiii + 478 pp, $24.95 (P); $49.50. [ISBN: 0-7167-1195-8] An innovative, attractive introduction to LP, appropriate both to beginning students and to professionals approaching LP for the first time. Chvátal defers traditional geometric and tableau methods in favor of more explicit work with "dictionaries"--the system of linear equations with both decision and slack variables. After a pedagogically sound opening chapter presenting basic theory by well-chosen examples, the text treats a variety of applications (scheduling, matrix games, network flow) and advanced techniques (Dantzig-Wolfe decomposition, the Ellipsoid method). Excellent historical notes and bibliography. LAS

Dynamical Systems, S(18), P, L. Stochastic Systems. George Adomian. Math. in Sci. & Eng., V. 169. Academic Pr, 1983, xvii + 331 pp, $47.50. [ISBN: 0-12-044370-8] An interesting venture in developing a theory of stochastic systems that more adequately models the real world of non-linearity and randomness. Assuming only a background of advanced calculus, differential equations, and linear algebra, the treatment moves through the classical development and use of Green's functions to stochastic differential equations and applications of them in such areas as biology, signal processing and filtering, and stochastic transport equations. Numerous examples, references, index. JS

Probability, P. Lecture Notes in Mathematics-986: Séminaire de Probabilités XVII, 1981/82. Ed: J. Azéma, M. Yor. Springer-Verlag, 1983, v + 507 pp, $25.50 (P). [ISBN: 0-387-12289-3] Proceedings of 1981-82 seminar at Université de Strasbourg. LAS

Probability. Educated Guessing: How to Cope in an Uncertain World. Samuel Kotz, Donna F. Stroup, Popular Stat., V. 2. Dekker, 1983, vi + 187 pp, $15.95. [ISBN: 0-8247-7000-5] An attempt to explain the mathematical concepts of probability, expectation, and information together with some of their applications to daily life. To this reviewer the exposition appears confusing, incomplete, and pedagogically unsound for the intended audience. GHM

Statistics, S(14). Basic Statistical Computing. D. Cooke, A.H. Craven, G.M. Clarke. Edward Arnold, 1982, xii + 156 pp, $13.95 (P). [ISBN: 0-7131-3441-0] Presents algorithms and programs for sorting and graphing data as well as for finding means, variances, probability functions, and for applying inference techniques and regression. Good algorithms are chosen and they are explained well. Comprehensive guide to statistical programs; a good reference for users of personal computers. MT

Statistics, T?(13: 1). Statistical Programs in FORTRAN. Ronald D. Schwartz, David T. Basso. Reston Pub, 1983, xiv + 254 pp, $16.95 (P). [ISBN: 0-8359-7096-5] Claims to "provide an innovative approach to the traditional lecture method of statistics instruction," but simply presents, without explanation or justification, Fortran programs for statistics topics from mean through estimation, testing, and regression. Programs are implementations of poor algorithms, and never use more than simple algebra. MT

Statistics, T(14: 2). Statistics for Technology: A Course in Applied Statistics. Christopher Chatfield. Chapman & Hall, 1983, 381 pp, $12.50 (P). [ISBN: 0-412-25340-2] A Third Edition of a 1970 volume (TR, November 1970); little has changed from the Second Edition (TR, October 1979). Covers typical topics of introduction to probability and statistics, from a brief section on descriptive techniques through basic probability, inference, and regression; also deals with design of experiments, quality control, and life testing. Though an applied introductory text, it is not the typical "cookbook," but treats the subjects with some sophistication. MT

Statistics, P. Proceedings of the Twenty-Eighth Conference on the Design of Experiments. US Army Research Office (P.O. Box 12211, Research Triangle Park, NC), 1982, xvii + 606 pp, (P). Papers from an October 1982 conference in Monterey, California. LAS

Statistics, P. Statistics in the Labor Market: The Role in Planning and Policy Formulation. Ed: Dennis Trewin. Lect. Notes in Stat., V. 5. Dekker, 1983, vii + 163 pp, $29.75 (P). [ISBN: 0-8247-1912-3] Seven papers from a symposium by the Statistical Society of Australia which brought together producers, analyzers and users of labor statistics. Includes analyses of unemployment benefits, technological change, migrant labor force and methodological issues concerning Australian labor market statistics. GHM

Statistics. Lecture Notes in Statistics-19: von Mises Calculus for Statistical Functionals. Luisa Turrin Fernholz. Springer-Verlag, 1983, viii + 124 pp, $9.80 (P). [ISBN: 0-387-90899-4] Places von Mises' theory for the analysis of the asymptotic behavior of nonlinear statistical functionals on a rigorous mathematical foundation using the Hadamard derivative. AO

Statistics, P, L.** The Visual Display of Quantitative Information. Edward R. Tufte. Graphics Pr, 1983, 197 pp, $34. A beautiful, brilliant book, synthesizing statistics with the history of graphics design. Profusely illustrated with examples both old and new (from 1700 to the present), this aesthetically designed volume emphasizes both graphical excellence and graphical integrity: "Graphical excellence requires telling the truth about the data." LAS

Computer Literacy, S? VISICALC for Science and Engineering. Stanley R. Trost, Charles Pomernacki. Sybex, 1983, xiii + 203 pp, $13.95 (P). [ISBN: 0-89588-096-2] Trivial applications (e.g., angle conversions, regression, lens formulae evaluation) that take advantage of VisiCalc's scientific calculation library. No examples use the distinctive spread-sheet and iterative features of VisiCalc; most examples could be done as easily (and with better understanding) by simple BASIC programs. LAS

Computer Programming, T??(13-14: 1). Structured COBOL: A Modern Approach. Henry Mullish. Harper & Row, 1983, xvi + 368 pp, $20.50 (P). [ISBN: 0-06-044652-8] An attempt to present the ideas of structured programming in the context of COBOL. The presentation is slow but readable. However, it bogs down for 44 pages in the details of punched cards and coding before describing the basic ideas of structured programming. Given Edsger Dijkstra's acid comments about COBOL, the quotation, "COBOL is a health-spa for the mind," which begins Chapter 10, makes the soundness of the book very suspect. JAS

Computer Programming, T*(13-14: 1), S. Basic-Plus for Business. Wilson T. Price. Holt, Rinehart & Winston, 1983, xii + 335 pp, (P). [ISBN: 0-03-061768-5] Written for the PDP-11 RSTS/E system with video terminals by an author who understands that BASIC is not a structured language and who makes the best of a bad situation. The emphasis is on solving business problems. The general intellectual quality is well above average for this genre. JAS

Computer Programming, T(13: 1). Basic Exercises for the Apple. Jean-Pierre Lamoitier. Sybex, 1982, xii + 251 pp, $12.95 (P). [ISBN: 0-89588-084-9] Designed to teach BASIC through graduated exercises in a variety of areas--number theory, geometry, data processing, statistics, and others. Requires pre-calculus level of mathematical sophistication. JRG

Computer Programming, T?(14-15: 1), S. Programming the Z8000. Richard Mateosian. Sybex, 1980, xii + 298 pp, $15.95 (P). [ISBN: 0-89588-032-6] Not a processor manual, but an introduction to assembly language programming using only the Z8000. Unfortunately the introductory nature of the approach makes the book hard to use as a reference yet the approach seems to assume too much while being quite elementary. JAS

Computer Programming, S?(13). Pascal for BASIC Programmers. Charles Seiter, Robert Weiss. Addison-Wesley, 1983, xii + 242 pp, $10.95 (P). [ISBN: 0-201-06577-0] An introduction to UCSD Pascal for readers familiar with BASIC. Designed primarily for self-study rather than classroom use. AO

Computer Programming, T(13: 1), S, P. VAX-BASIC. David G. Weinman, Barbara L. Kurshan. Reston Pub, 1983, xv + 266 pp, $14.95 (P). [ISBN: 0-8359-8238-6] An introduction to computer programming using VAX-11 BASIC which is virtually identical to BASIC-PLUS-2. Includes standard material on program control, functions, subroutines, strings, random numbers and files. Lots of sample programs and exercises. CEC

Computer Programming, T(13), S. Elementary FORTRAN With Scientific and Business Applications. Frank C. Lin. Reston Pub, 1983, xv + 336 pp, $15.95 (P). [ISBN: 0-8359-1696-0] A textbook on Fortran programming designed for use with students having no background in computing. Organized using the competency-based curriculum format. AO

Computer Programming, T(14: 1). Software Engineering for Small Computers: A Programmer's Companion. R.B. Coats. Reston Pub, 1982, vi + 249 pp, $16.95 (P). [ISBN: 0-8359-7025-6] After a person has completed a first course in programming, he or she has learned how to code--to write statements in some particular programming language. This text, which is intended for a second course in advanced programming, tries to teach them how to design and build complete software systems. It addresses such issues as program design, program development, testing and debugging, verification, and efficiency. It also develops a complete system as a large (40 page) case study. MS

Computer Programming, S(13). BASIC Exercises for the IBM Personal Computer. Jean-Pierre Lamoiter. Sybex, 1982, xii + 251 pp, $13.95 (P). [ISBN: 0-89588-088-1] Self-paced book on programming in BASIC on IBM personal computer. For readers with minimum scientific or technical background. BK

Data Structures, P. Optimization of Queries in Relational Databases. Yehoshua C. Sagiv. Comp. Sci.: Distributed Database Systems, No. 12. UMI Research Pr, 1981, xi + 103 pp, $34.95. [ISBN: 0-8357-1244-3] This text addresses a very advanced and special-purpose topic in the area of data base design--the foundation of queries into a relational data base that minimizes both time and cost. This topic is of importance because the relational model is the most important one in current data base systems. The material is at a very advanced level and presumes a background in both data base design and the predicate calculus. MS

Software Systems, P, L? Microprocessor Operating Systems. Ed: John Zarrella. Microcomputer Applic. Volume 1, 1981, vii + 155 pp, $12.95 (P) [ISBN: 0-935230-03-3]; Volume 2, 1982, vii + 145 pp, $12.95 (P). [ISBN: 0-935230-04-1] A collection of brief (15-20 pages each) essays on 18

different operating systems which run on microprocessors. This potentially very useful reference work is flawed by a number of technical problems. There is no common index and the essays appear to have been written in isolation, each by a devotee of the particular system. Nonetheless, the book offers a quick, non-theoretical survey with only a moderate amount of jargon. JAS

Software Systems, P, L. Computer Algebra in Applied Mathematics: An Introduction to MACSYMA. R.H. Rand. Res. Notes in Math., No. 94. Pitman Pub, 1984, 181 pp, $19.95 (P). [ISBN: 0-273-08632-4] Describes the MACSYMA symbolic manipulation package and illustrates its use by solving a variety of applied mathematics problems. A valuable introduction to the potential pedagogical and research uses of the MACSYMA system. AO

Software Systems, P. Query Processing and Data Allocation in Distributed Database Systems. P.M.G. Apers. Math. Centre Tracts, No. 156. Math Centrum, 1983, vii + 193 pp, Dfl. 26,40 (P). [ISBN: 90-6196-251-X] A thorough study, including open-ended questions, of the issues concerned with query and data allocation in situations where both users and segments of the data are distributed among many computers and data storage devices. General algorithms are discussed in a Pascal-like lingua Helvetica. JAS

Software Systems, S, P. The UNIX Operating System. Kaare Christian. Wiley, 1983, xviii + 318 pp, $24.95. [ISBN: 0-471-87542-2] A fairly thorough introduction to UNIX, in three parts: an elementary treatment of files, utilities and the editor "ed"; a more superficial treatment of advanced features (shell, C, "yacc", "lex", the kernel); and an abridged UNIX system manual with 40 of the most common commands. The gradient in prerequisite assumptions from beginning to end is very steep. LAS

Software Systems, S, P, L. Introduction to Interactive Computer Graphics. Joan E. Scott. Wiley, 1982, xi + 255 pp, $25.95. [ISBN: 0-471-05773-8] A non-technical but informative survey of the entire field of interactive computer graphics, from pixels to perspective to personnel policies. Primarily intended for users in business and industry, from programmers, to draftsmen, to managers. GHM

Software Systems, S(16-17), P. A Microprogrammed APL Implementation. Rodnay Zaks. Sybex, viii + 347 pp, $38 (P). The complete theory and design (including listings) of an interpreter for APL ("A Programming Language"), microprogrammed in only 2K ROM locations (plus some RAM for infrequently used operators). This space efficiency was gained by a careful automata-theoretic analysis of the language. Performance is claimed to rival that of large APL systems. GHM

Computer Science, T(13-15: 1), S, L. Schaum's Outline of Theory and Problems of Microprocessor Fundamentals. Roger L. Tokheim. McGraw-Hill, 1983, 378 pp, $7.95 (P). [ISBN: 0-07-064958-8] A lean P instruction-set oriented than some books. There is little material on data structures or programs longer than a single simple routine. JAS

Computer Science, T, S, P. Microcomputer Primer. Mitchell Waite, Michael Pardee. Howard W. Sams, 1980, 384 pp, $14.50 (P). [ISBN: 0-672-21653-1] This is an expanded and up-dated version of the 1976 original introducing discussion of the most popular 16-bit CPU's. It is very much hardware oriented at the chip level but its concern is more computer science than electronics. Useful as a reference but lacking exercises (laboratory or theoretical). JAS

Computer Science, S*(13-16), L*. Microcomputer Math, Second Edition. William Barden, Jr. Howard W Sams, 1982, 128 pp, $11.95 (P). [ISBN: 0-672-21927-1] All about binary arithmetic and its octal and hexadecimal representations as a tool for describing what computers really do. The sections on signed numbers, flags, logical operations, and ASCII conversions make this a very helpful resource book for gaining insights into assembly language or advanced system programming. Nicely done with a glossary and index. Unfortunately the price is a bit high to make it welcome as a widely recommended supplement to a course text. JAS

Computer Science, T(15-16: 1). File Management Techniques. Billy G. Claybrook. Wiley, 1983, xv + 247 pp, $22.95. [ISBN: 0-471-04596-9] An introduction to the design and management of file systems. Covers many common file organizations and algorithms for manipulating and maintaining these structures. AO

Computer Science, P. Lecture Notes in Computer Science-153: Graph-Grammars and Their Application to Computer Science. Ed: Hartmut Ehrig, Manfred Nagl, Grzegorz Rozenberg. Springer-Verlag, 1983, vii + 452 pp, $20.50 (P). [ISBN: 0-387-12310-5] The proceedings of an international workshop held in Haus Ohrbeck in 1982. The theory of graph grammars is an extension of standard formal language theory to deal with more general structures such as graphs. AO

Computer Science, S(16-18), P. Simulation of Computer Communication Systems. Charles H. Sauer, Edward A. MacNair. Prentice-Hall, 1983, xvi + 158 pp, $25. [ISBN: 0-13-811125-1] Presents a methodology for developing simulation models of computer communication systems using extended queueing networks. AO

Computer Science, T(15-16: 1). Recursive Descent Compiling. A.J.T. Davie. Ser. in Computers & Their Applic. Halsted Pr, 1981, 195 pp, $37.50. [ISBN: 0-470-27270-8] An introduction to compiler design in general and recursive descent compilation in particular. AO

Computer Science, T(15-16: 1). Programming Language Translation. R.E. Berry. Ser. in Computers & Their Applic. Halsted Pr, 1981, 175 pp, $44.95. [ISBN: 0-470-27305-4] An overview of the program translation process and an introduction to some of the most important techniques used in constructing compilers and assemblers. The Pascal A compiler/interpreter is used as a source of examples. AO

Computer Science, S(15-17), P, L.** Pascal--The Language and its Implementation. Ed: D.W. Barron. Ser. in Computing. Wiley, 1982, ix + 301 pp, $38.95. [ISBN: 0-471-27835-1] A collection of papers (including some previously unpublished "classics") on the design and implementation of the language Pascal. AO

Computer Science, T(17-18: 1), P, L*. Cipher Systems: The Protection of Communications. Henry Beker, Fred Piper. Wiley, 1982, 427 pp, $34.95. [ISBN: 0-471-89192-4] A broad and comprehensive introduction to cryptography. Reasonably self-contained with references to necessary background material in computer science, electronics, and statistics. AO

Computer Science, T(16-17: 1), L. The Design and Construction of Compilers. Robin Hunter. Ser. in Computing. Wiley, 1981, xii + 272 pp, $27.95. [ISBN: 0-471-28054-2] An introduction to the design and implementation of compilers for block-structured languages such as ALGOL and Pascal. Covers the topics standard in a first course. AO

Computer Science, T(16-17: 1), S, L*. The Design of Operating Systems for Small Computer Systems. Stephen H. Kaisler. Wiley, 1983, xvi + 667 pp, $32. [ISBN: 0-471-07774-7] A practical guide to the design of micro- and minicomputer operating systems. Discusses design trade-offs in detail. AO

Computer Science, T(14-16: 1, 2), P. Computer Hardware and Organization: An Introduction, Second Edition. M.E. Sloan. SRA, 1983, xiii + 514 pp, $16.13. [ISBN: 0-574-21425-9] This text takes a hierarchical look at the topic of computer organization. In the opening section it treats the hardware design level, addressing the issue of circuits, gates, and logic technologies. In the second part, it addresses the machine language level, introducing memory, I/O, and assembly language programming. Finally, the last part discusses entire computing systems, using the PDP-11, VAX-11, M68000, and IBM 370 as examples, as well as introducing the idea of computer networks. MS

Computer Science, T(16-17: 2), S, L. Introduction to the Theory of Automata. Zamir Bavel. Reston Pub, 1983, xii + 658 pp, $27.95. [ISBN: 0-8359-3271-0] An in-depth, systematic, and somewhat original presentation of automata theory. Organized around the sequential structure underlying automata, with emphasis on both the combinatorial approach and the algebraic framework of structure and homomorphism. Aimed at an audience with minimal mathematics prerequisites, though some mathematical sophistication will be helpful. RM

Computer Science, S(15-17), L*.** Lecture Notes in Computer Science-155: The Programming Language Ada Reference Manual. Springer-Verlag, 1983, ix + 330 pp, $14.50 (P). [ISBN: 0-387-12328-8] A reprint of the document ANSI/MIL-STD-1815A-1983 which defines the programming language Ada. AO

Computer Science, T(15-17: 1), L*. Fundamentals of Programming Languages, Second Edition. Ellis Horowitz. Computer Software Engin. Ser. Computer Sci Pr, 1984, xv + 446 pp, $26.95. [ISBN: 0-88175-004-2] An introduction to the theory of programming languages covering the standard topics (e.g., variables, expressions, statements, typing, scope, procedures, data types, exception handling, and concurrency) in imperative languages as well as functional, dataflow, and object-oriented programming languages. (First Edition, TR, August-September 1983.) AO

Computer Science, P. Reference Manual for the ADA Programming Language. Springer-Verlag, 1983, 330 pp, $13.50 (P). [ISBN: 0-387-90887-0] A reprint of the document ANSI/MIL-STD-1815A-1983. The content of this volume is identical with that of Volume 155 in the Lecture Notes in Computer Science series. AO

Computer Science, P. The Design of Interpreters, Compilers, and Editors for Augmented Transition Networks. Ed: Leonard Bolc. Springer-Verlag, 1983, xi + 214 pp, $29. [ISBN: 0-387-12789-5] This is a collection of four separate research papers in the area of the design of Augmented Transition Networks (ATN). ATNs are currently the most widely used method for analyzing natural languages. This book described state-of-the-art research in building and implementing ATNs. MS

Computer Science, P. Query Processing Techniques for Distributed, Relational Data Base Systems. Robert S. Epstein. Comp. Sci.: Distributed Database Systems, No. 13. UMI Research Pr, 1982, x + 93 pp, $34.95. [ISBN: 0-8357-1341-5] This is an advanced book at an extremely technical part of computer science. It addresses the issue of how to formulate a query (i.e., a retrieval request) to a database that is geographically distributed over many different computers and which uses the relational model proposed by Codd. The questions addressed include query optimization, processing strategies, and analysis of physical structures. This book presuppposes an extensive background in database design, data structures and calculus. MS

Computer Science, S(16-18), P*. Advances in Cryptology: Proceedings of Crypto 82. Ed: David Chaum, Ronald L. Rivest, Alan T. Sherman. Plenum Pr, 1983, xv + 331 pp, $45. [ISBN: 0-306-41366-3] The proceedings of a conference held at the University of California at Santa Barbara in August, 1982. The papers in this volume provide an overview of the current state of cryptologic research. AO

Computer Science, S(14). Practice Problems in Number Systems, Logic, and Boolean Algebra, Second Edition. Ed Bukstein. Howard Sams, 1982, 133 pp, $10.95 (P). [ISBN: 0-672-21451-2] A set of 64 worksheets designed to be an introduction to computer mathematics. Develops binary and octal arithmetic, Boolean algebra, and implementation of Boolean algebra in electronic grating and inverting circuits. BK

Control Theory, P. Linear Programming and Finite Markovian Control Problems. L.C.M. Kallenberg. Math. Centre Tracts, No. 148. Math Centrum, 1983, iv + 245 pp, Dfl. 33 (P). [ISBN: 90-6196-236-6] Theory and construction of algorithms for solving finite Markov decision problems, semi-Markov problems and stochastic games. Uses linear and dynamic programming, total and average reward criteria and the superharmonic property. RWN

Control Theory, P.** Lecture Notes in Control and Information Sciences-56: Identification of Continuous Dynamical Systems: The Poisson Moment Functional (PMF) Approach. Dines Chandra Saha, Ganti Prasada Rao. Springer-Verlag, 1983, ix + 158 pp, $11 (P). [ISBN: 0-387-12759-3] On the Poisson Moment Functional (PMF) Method. Timely presentation of various recent developments in the PMF area in noisy situations, in structure identification, and in models with unknown time delays and nonlinear elements. Over sixty references to papers published mostly in the past ten years. Good index. JK

Applications, L. UMAP Modules 1982, Tools for Teaching. COMAP, 1983, xii + 544 pp, $25 (P). [ISBN: 0-912843-03-9] The annual collection from the Undergraduate Mathematics and Its Applications Project. Contains 23 new modules for supplementary use in the classroom. JRG

Applications (Biology), P. Lecture Notes in Biomathematics-49: Rhythms in Biology and Other Fields of Application: Deterministic and Stochastic Approaches. Ed: M. Cosnard, J. Demongeot, A. Le Breton. Springer-Verlag, 1983, vii + 400 pp, $22 (P). [ISBN: 0-387-12302-4] Papers presented at a September 1981 conference held in Luminy, France. LAS

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-216: Fiscal Policies in a General Equilibrium Model with Persistent Unemployment. Heinz H. Müller. Springer-Verlag, 1983, vi + 92 pp, $9.50 (P). [ISBN: 0-387-12316-4]

Applications (Economics), P, L. Lecture Notes in Economics and Mathematical Systems-219: Refinements of the Nash Equilibrium Concept. Eric van Damme. Springer-Verlag, 1983, vi + 151 pp, $11.50 (P). [ISBN: 0-387-12690-2] A study of noncooperative games, the core problem being this: given a game with more than one Nash (self-enforcing) equilibrium, which one should be chosen as the solution to the game? An elegant, accessible exposition, developing the appropriate game theory concepts from scratch. GHM

Applications (Economics), S(16), P. Lecture Notes in Economics and Mathematical Systems-217: The Relation Between Final Demand and Income Distribution with Application to Japan. Christiaan Grootaert. Springer-Verlag, 1983, xiv + 105 pp, $9.50 (P). [ISBN: 0-387-12307-5] Presents a model to investigate extent to which changes in economic growth affect income distribution. BK

Applications (Economics), S(17), P. Tinbergen Lectures on Organization Theory. Martin J. Beckmann. Texts & Mono. in Econ. & Math. Syst. Springer-Verlag, 1983, xi + 176 pp, $16. [ISBN: 0-387-12646-5] An organization is defined as an association of persons using common resources to achieve well-defined but limited ends. This monograph looks at the internal structure of organizations and such questions as returns to scale, loss of control, and the economic advantage of organizations. Includes a bibliography. CEC

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-208: Econometric Decision Models. Ed: Josef Gruber. Springer-Verlag, 1983, vi + 364 pp, $22.50 (P). [ISBN: 0-387-11554-4] Papers presented at a June 1981 conference at the University of Hagen, West Germany. LAS

Applications (Engineering), T(16-18: 1, 2). Mathematical Theory of Elasticity. I.S. Sokolnikoff. Robert E. Krieger Pub, 1983, xi + 476 pp, $29.50. [ISBN: 0-89874-555-1] Reprint of second edition, first published by McGraw-Hill in 1956. Although somewhat dated, the contents constitute a readable introduction to topics in the theory of elasticity. JK

Applications (Engineering), S(17), P. Asymptotic Treatment of Chemically Reacting Systems. Ashwani K. Kapila. Appl. Math. Ser. Pitman Pub, 1983, viii + 119 pp, $32.95. [ISBN: 273-08513-1] Illustrates asymptotic treatment of problems in the field of chemically reactive systems. Chapter I covers the fundamentals of matched asymptotic expansions and the method of multiple scales. These techniques are useful in the remainder of the work in problems on chemical engineering and combustion theory. Main objective is to show that a substantial amount of information can be obtained by the judicious use of perturbation techniques. Treatment is more intuitive than rigorous. JK

Applications (Engineering), T(16-17: 1, 2), P. Mechanics of Continuous Media, Second Edition. S.C. Hunter. Ser. in Math. & Its Apppl. Halsted Pr, 1983, 640 pp, $84.95. [ISBN: 0-470-27398-4] An introductory text for a course in continuous mechanics. This edition incorporates new material on thermodynamics. AO

Applications (Engineering), P. Kinematics. Joseph Stiles Beggs. Hemisphere Pub, 1983, xvi + 223 pp, $24.50. [ISBN: 0-89116-355-7] A reference work intended for professionals in related engineering

disciplines. Final chapter devoted to sample applications of theory. AO

Applications (Engineering), P. Nonlinear Stochastic Problems. Ed: Richard S. Bucy, José M.F. Moura. NATO ASI Ser. C. D Reidel Pub, 1983, xii + 623 pp, $79.50. [ISBN: 90-277-1590-4] Invited lectures and advanced research papers presented at a May 1982 NATO Advanced Study Institute. Special emphasis on signals, control, and filtering. LAS

Applications (Medicine), P. Lecture Notes in Biomathematics-48: Tracer Kinetics and Physiologic Modeling: Theory to Practice. Ed: R.M. Lambrecht, A. Rescigno. Springer-Verlag, 1983, viii + 509 pp, $28 (P). [ISBN: 0-387-12300-8] Eleven lectures delivered at a June 1983 seminar in St. Louis, featuring statistical and analytical models of nuclear tracers in physiological and biochemical processes. LAS

Applications (Physics), P. Lecture Notes in Physics-174: A Gauge Theory of Dislocations and Disclinations. Aida Kadić, Dominic G.B. Edelen. Springer-Verlag, 1983, vii + 290 pp, $12.50 (P). [ISBN: 0-387-11977-9] The Yang-Mills universal gauge theory construction is used to erect a complete continuum theory of material bodies with dislocation and disclination fields. JAS

Applications (Physics), P. Nowcasting. Ed: K.A. Browning. Academic Pr, 1982, xiv + 256 pp, $45. [ISBN: 0-12-137760-1] The book is a collection of papers from a Nowcasting Symposium held in Hamburg in August, 1981. Nowcasting typically consists of a detailed description of current weather along with forecasts up to two hours ahead, but this book includes discussion of forecasts up to twelve hours ahead. MT

Applications (Physics), P. The Mathematical Theory of Black Holes. S. Chandrasekhar. Intern. Ser. of Mono. on Physics, No. 69. Clarendon Pr, 1983, xxi + 646 pp, $110. [ISBN: 0-19-851291-0] A comprehensive introduction to the theory of black holes. AO

Applications (Physics), P. Supersymmetry and Supergravity. Julius Wess, Jonathan Bagger. Ser. in Physics. Princeton U Pr, 1983, ix + 180 pp, $40; $12.50 (P). [ISBN: 0-691-08327-4; 0-691-08326-6] An introduction to N=1 supersymmetry and supergravity. Assumes familiarity with relativistic quantum field theory. AO

Applications (Physics), S(13-14), L. An Introduction to Error Analysis: The Study of Uncertainties in Physical Measurements. John R. Taylor. University Sci Books, 1982, xiv + 270 pp, $9.50 (P); $15. [ISBN: 0-935702-10-5; 0-935702-07-5] Lucid introduction to ideas and methods of statistical error analysis. Many examples typical of freshman and sophomore laboratory work. Derivations exhibit the typical cavalier attitude of physicists towards differentials and such subtleties, but the author intentionally chooses ease of understanding over strict rigor. GHM

Applications (Physics), P. Lecture Notes in Mathematics-1011: Scattering Theory for Many-Body Quantum Mechanical Systems--Rigorous Results. Israel Michael Sigal. Springer-Verlag, 1983, iv + 132 pp, $8 (P). [ISBN: 0-387-12672-4] Briefly summarizes the known results in N-body scattering theory and develops new results concerning asymptotic completeness. AO

Applications (Physics), T(17: 1). Radiometry and the Detection of Optical Radiation. Robert W. Boyd. Wiley, 1983, vii + 254 pp, $34.95. [ISBN: 0-471-86188-X] An introduction to the generation, transfer, and detection of optical and infrared radiation. AO

Applications (Physics), T(16-17: 1). Textbook of Dynamics, 2nd Edition. F. Chorlton. Halsted Pr, 1983, 271 pp, $49.95. [ISBN: 0-470-27407-7] A textbook on classical dynamics covering single particles, particle systems, and rigid bodies. Among the new topics in this edition are the kinematics of plate tectonics, satellite orbits, and staged rockets. AO

Applications (Physics), T(17-18: 1), S. Solitons. P.G. Drazin. London Math. Soc. Lect. Notes Ser., No. 85. Cambridge U Pr, 1983, viii + 136 pp, $15.95 (P). [ISBN: 0-521-27422-2] An introduction to the theory of solitons and the method of inverse scattering. AO

Applications (Physics), P. Functional Integrals in Quantum Field Theory and Statistical Physics. Viktor Nikolayevich Popov. Transl: J. Niederle, L. Hlavatý. Math. Physics & Appl. Math., V. 8. D Reidel Pub, 1983, viii + 299 pp, $65. [ISBN: 90-277-1471-1] Presents a method for the quantization of gauge fields using functional integrals. Applications in quantum field theory and in quantum statistical mechanics are given. AO

Reviewers

RJA: Richard J. Allen, St. Olaf; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JDB: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; PH: Paul Humke, St. Olaf; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen,

St. Olaf; MT: Michael Tveite, St. Olaf; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

Maryland-D.C.-Virginia Section

The fall meeting of the Maryland-D.C.-Virginia Section was held on November 18-19 at the University of the District of Columbia in Washington, D.C.

Invited Addresses:

"Intuitive and Counterintuitive Mathematics," by Warren Page, Editor-Elect, Two-Year College Mathematics Journal.
"Undergraduate Mathematics in China," by Florence Ashby, Montgomery College; Ron Davis, Northern Virginia Community College; and Gloria Gilmer, National Institute of Education.

Short Presentations:

"On the Classification of Quadric Surfaces," by William Sanders, James Madison University.
"Model Building Using Logistic Regression Analysis Methods--A Look at Some Case Studies," by James Maher, AT&T-Bell Telephone Labs.
"Fuzzy Sets and Computer Vision," by Bao-Ting Lerner, U.S. Naval Academy.
"The 203 Groups of Order 504," by James Alonso, University of the District of Columbia.
"Compound Perfect Squares and the Method of Trivial Compounding," by Caren Diefenderfer, Virginia Polytechnic Institute and State University.
"Problem Solving and Construction of Representations," by Carolyn Maher, Rutgers, The State University.
"One Parameter Groups of Homeomorphisms of the Real Line," by Andrew Vogt, Georgetown University.
"A More Successful Precalculus Sequence at an Open Admissions University," by Eleanor Green Jones, Norfolk State University.
"Solving the n x n x n Rubik's Cube," by Robb Koether, Hampden-Sydney College.
"Polyhedral Evolution," by Jeff Hrdlicka.
"But Who Teaches the Teachers?" by Ely Dorsey, Lockheed Engineering and Management Services Company.
"Good Number Sense and 'Expert' Statistics," by Richard Eisenman, Chaney Station, Inc.
"LIBASE: A User's Data Base for Retrieving Mathematical and Scientific Bibliography," by John Hays, Naval Research Laboratory.
"Some Simple Exercises in Complex Analysis," by John Milcetich, University of the District of Columbia.
"The Logistic and Other Population Growth Equations," by Paul Massell, U.S. Naval Academy.
"Applications of Graph Theory to Robotics," by Carol Crawford, U.S. Naval Academy.
"Angle in Geometry," by Clifford J. Maloney.
"Evaluation of a Single Mathematics Placement Test Used to Assist Students in Selecting Entry Level Courses," by Margaret Aldrich, Montgomery College; and Elizabeth Teles, Montgomery College.
"Graphs of the Solutions to the Vibrating String Problem," by Howard Penn, U.S. Naval Academy.
"Permanent and Hadamard Products," by John Chollet, Towson State University.
"Discrete Mathematics in the First Two Years: An Update of the MAA Panel Activities," by Martha Siegel, Towson State University.
"Some New Results for the M/M/I Queue and Its Departure Process," by J.R. Hubbard, University of Richmond.

Eastern Pennsylvania and Delaware Section

The fall meeting of the Eastern Pennsylvania and Delaware Section was held at Bryn Mawr College on November 19, 1983. There were 179 registrants.

Invited Addresses:

"Nonlinear Equations and Optimization: Quasi-Newton Methods and Abstract Vector Spaces," by Michael Todd, Cornell University.
"The Classification of the Finite Simple Groups," by Walter Feit, Yale University.
"What Matrices Can Do," by Albert Wilansky, Lehigh University.
"Mathematical Reminiscences and Speculations for the Future," by Stanislaw Ulam, University of Florida.

References

1. R. Falk, Minimize your losses, Teaching Statistics, 2 (1980) 80–83.
2. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge University Press, London, 1934.
3. D. H. Krantz, R. D. Luce, P. Suppes, and A. Tversky, Foundations of Measurement (volume 1; Additive and Polynomial Representations), Academic Press, New York, 1971.
4. G. U. Yule, and M. G. Kendall, An Introduction to the Theory of Statistics, 14th ed. (Second Impression), Charles Griffin, London, 1953.

TEACHING ABOUT THE REAL NUMBERS

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In the teaching of analysis it is usual to assume without proof that the real numbers form a complete ordered field (or an equivalent statement in more elementary terms). When a proof is taught, it usually depends on a construction of the real numbers by some method such as Dedekind cuts. I wish to explain why I consider these procedures unsatisfactory.

The idea behind them seems to be that mathematics is the study of systems subject to axioms, and these axioms may be chosen arbitrarily, provided only that they are consistent. This approach is all right in subjects like group theory: it produces valid results about groups, where a group is *defined* as a system satisfying certain axioms. But one cannot approach the real numbers in this way, because students already know (probably rather vaguely) what they are. One must therefore justify one's results from what students already know of the real numbers, and cannot use new definitions or constructions. In particular, if one constructs a complete ordered field, then one shows only that there is some complete ordered field, not that the real numbers form one.

The problem is thus to convince students that the real numbers form a complete ordered field. Now everybody believes that they form an ordered field; it is completeness that gives the trouble. Here one needs to know what students think a real number is. I think there are two possibilities: either a measurement (say of length) or a decimal. If one takes numbers as measurements, then one can justify completeness only by appealing to intuition, which may not be convincing. But if one regards real numbers as decimals, then one can give a convincing demonstration of completeness in any of its usual formulations. Thurston ([1], page 8), for instance, proves that a nonempty set of decimals bounded above has a least upper bound. As another example, I shall show that every decimal is the limit of a sequence of rationals and that every Cauchy sequence of decimals has a limit. Indeed a decimal is clearly the limit of its truncations (for instance π is the limit of the sequence $3, 3.1, 3.14, \ldots$). And if (a_n) is a Cauchy sequence of decimals, then one can construct a limit decimal

$$L = m + \frac{d_1}{10} + \frac{d_2}{100} + \cdots \quad (m, d_i \text{ integers}, 0 \leqslant d_i \leqslant 9)$$

by taking m to be the largest integer less than or equal to infinitely many members of the sequence, $m + d_1/10$ to be the largest of the numbers, $m, m + 1/10, \ldots, m + 9/10$ less than or equal to infinitely many members of the sequence, etc. For $r = 0, 1, 2, \ldots$, there is, by construction, a closed interval of length 10^{-r} containing L and infinitely many members of the sequence, from which, since the sequence is a Cauchy sequence, all but finitely many of its members lie within 2×10^{-r} of L. Thus L is a limit for the sequence.

It seems to me that some proof of this nature should be included in every analysis course.

REMARKS. 1. I have arranged this proof to avoid mentioning the fact that the same real number can have different decimal representations (such as $1 = 0.999\ldots$), as I think this fact is widely disbelieved.

2. An axiomatic treatment of the real numbers answers two questions which a professional mathematician might ask: Would one get the same real numbers from a different construction,

such as binary numbers? Do decimal numbers really form an ordered field? The answers are: yes, because all complete ordered fields are isomorphic, and yes, because one can construct a complete ordered field (say by Dedekind cuts) and show that its members have properly behaved decimal expansions. But one must start with the real numbers as commonly accepted, not these advanced mathematical ideas.

3. I find the assumptions of nonstandard analysis even less obviously true than those of standard analysis. So nonstandard analysis does not strike me as a good method of proving results at an elementary stage, though it may be a good method of discovering them.

Reference

1. H. A. Thurston, Differentiation and Integration, Blackie, London and Glasgow, 1961.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS), H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

EDITOR EMERITUS: EMORY P. STARKE. COLLABORATING EDITORS: VINCENT BRUNO, FRANK S. CATER, GULBANK D. CHAKERIAN, A. M. DAWES, MICHAEL J. DIXON, UNDERWOOD DUDLEY, RICHARD A. GIBBS, CLARK GIVENS, RICHARD M. GRASSL, DOUGLAS A. HENSLEY, ISRAEL N. HERSTEIN, ROBERT H. JOHNSON, ELGIN H. JOHNSTON, MURRAY S. KLAMKIN, DANIEL J. KLEITMAN, JOSEPH D. E. KONHAUSER, FREDERICK W. LUTTMANN, MARVIN MARCUS, LOUISE E. MOSER, M. J. PELLING, C. M. REIS, J. O. SHALLIT, B. L. R. SHAWYER, EDWARD T. H. WANG, AND ALBERT WILANSKY.

*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk () indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by July 31, 1984. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3039. *Proposed by I. N. Herstein, University of Chicago.*

If G is a simple nonabelian group and ϕ an automorphism of G such that $x\phi(x) = \phi(x)x$ for all $x \in G$, prove that $\phi = 1$.

E 3040. *Proposed by J. L. Brenner, Palo Alto, CA, and W. A. Newcomb, Lawrence Livermore National Laboratory.*

Let

$$M_1 = \left[\tfrac{1}{2}\left(a_1 a_2^{\varepsilon} + a_1^{\varepsilon} a_2\right)\right]^{1/(1+\varepsilon)}, \qquad M_2 = \left[\tfrac{1}{2}\left(a_1^{\varepsilon} + a_2^{\varepsilon}\right)\right]^{1/\varepsilon},$$

where $\varepsilon \neq 0, -1$. For what values of ε is it true for all $a_1 > a_2 > 0$ that $M_1 > M_2$, and for what values is the opposite true?

SOLUTIONS OF ELEMENTARY PROBLEMS

A Third Degree Congruence with Nine Solutions

E 2763 [1979, 223; 1983, 56]. *Proposed by Lorraine L. Foster, California State University, Northridge.*

Let $f(n) = n^3 + 396n^2 - 111n + 38$. Prove that the congruence $f(n) \equiv 0 \pmod{3^a}$ has precisely 9 solutions $\pmod{3^a}$ for all integers $a \geq 5$.

Correction by the proposer to the solution by L. E. Mattics, University of South Alabama.

THEOREM. *If $s \geq 3$, then there is a d such that $f(d) \equiv 0 \pmod{3^{s+2}}$ and if $f(b) \equiv 0 \pmod{3^{s+2}}$, then $b \equiv d \pmod{3^s}$.*

Indeed, $f(n) \equiv 0 \pmod{243}$ if and only if $n \equiv 19 \pmod{27}$ and we proceed by induction assuming the theorem to be true for some $s \geq 3$. Then

$$f(d + k3^s) \equiv f(d) + k3^s f'(d) \pmod{3^{s+3}}$$

and since $d \equiv 1 \pmod 9$, we can show that $3^2 | f'(d)$, but $3^3 \nmid f'(d)$. Hence the congruence

$$f(d) + k3^s f'(d) \equiv 0 \pmod{3^{s+3}}$$

can be solved uniquely for k modulo 3. If $f(c) \equiv 0 \pmod{3^{s+3}}$, then by the induction hypothesis $c = d + a \cdot 3^s$, so $a \equiv k \pmod 3$, and $c \equiv d + k3^s \pmod{3^{s+1}}$. The induction is complete and the theorem is proved.

Now if $s \geq 5$ and $f(b) \equiv 0 \pmod{3^s}$, then

$$f(b + m3^{s-2}) \equiv f(b) + m3^{s-2} f'(b) \equiv 0 \pmod{3^s},$$

since $9 | f'(b)$. Therefore, since b is unique modulo 3^{s-2}, there are precisely nine solutions modulo 3^s to the congruence $f(n) \equiv 0 \pmod{3^s}$.

When Does a Real Quartic Have No Real Zeros?

E 2878 [1981, 290]. *Proposed by M. Slater, University of Bristol, U.K.*

When does a real quartic have no real zeros?

Solution by J.A. Brandler, Brooklyn, New York. Using the notation of Cajori, let the general quartic be written in the form

$$b_0 x^4 + 4b_1 x^3 + 6b_2 x^2 + 4b_3 x + b_4 = 0, \qquad b_0 \neq 0.$$

If we eliminate the term of degree three, we obtain

$$y^4 + 6Hy^2 + 4Gy + b_0^2 I - 3H^2 = 0,$$

where

$$y = b_0 x + b_1, \qquad G = b_0^2 b_3 - 3b_0 b_1 b_2 + 2b_1^3, \qquad H = b_0 b_2 - b_1^2, \qquad I = b_0 b_4 - 4b_1 b_3 + 3b_2^2.$$

Then the roots of the quartic are all non-real if $I^3 - 27J^2 > 0$, and if $H > 0$ or else $3b_0 J - 2HI < 0$, where $b_0^3 J = b_0^2 HI - G^2 - 4H^3$.

Also solved by W. J. Blundon (Canada), D. C. Buchthal, T. C. Craven, G. Fisher, L. Kuipers (Switzerland), R. B. Nelson, W. A. Newcomb, A. Ungar, and the proposer.

As most of the solvers pointed out, solution of the problem can be found (with detailed explanations) in several sources: W. S. Burnside and A. W. Parton, The Theory of Equations, vol. 1 (Dover, NY, 1960) 211; F. Cajori, An Introduction to the Theory of Equations (Dover, NY) 56–58. As an exercise, the conditions are given in N. Jacobson, Basic Algebra I (San Francisco: Freeman, 1974) 299; L. E. Dickson, Elementary Theory of Equations (Wiley, 1917) 45; and H. Weber, Lehrbuch der Algebra, vol. 1 (1st ed., 1894, 2nd ed., 1898). Newcomb examined the bifurcation surface.

ADVANCED PROBLEMS

Solutions of these Advanced problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by July 31, 1984. The solver's full post-office address should be on each sheet.

6454. *Proposed by the Chico Problem Group, California State University, Chico.*

Evaluate the integral

$$I(p) = \int_0^{\pi/2} \theta \tan^p(\theta)\, d\theta, \quad -2 < p < 1,$$

in terms of the digamma function.

6455. *Proposed by L. E. Mattics, University of South Alabama.*

Let $a_{i1} = (-1)^{i+1}$, $a_{ij} = i^{j-1}$ for $1 \leqslant i \leqslant p-1$, $2 \leqslant j \leqslant p-1$ where p is an odd prime. Show that $\det(a_{ij}) = 0 \pmod p$ if and only if $\sum_{k=1}^{p-1}(-1)^{k+1}k^{-1} \equiv 0 \pmod p$. Is there a prime p for which $\det(a_{ij}) = 0$?

6456. *Proposed by Robert E. Shafer, Berkeley, CA.*

For $x > 0$, define

$$\psi(x) = \lim_{n\to\infty} \log\left(n + x - \frac{1}{2}\right) - \sum_{k=0}^{n} \frac{1}{x+k}$$

and

$$\psi_2(x) = \lim_{n\to\infty} \frac{1}{2}\log^2\left(n + x - \frac{1}{2}\right) - \sum_{k=0}^{n} \frac{\log(x+k)}{x+k}.$$

(a) Show that, for all $x > 0$,

$$\psi_2(x) < \frac{1}{2}\psi^2(x) - \frac{1/24}{(x-1/2)^2 + 3/10}.$$

(b) Prove that

$$\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n} = \left(\gamma - \frac{1}{2}\log 2\right)\log 2$$

where γ is Euler's constant.

SOLUTIONS OF ADVANCED PROBLEMS

A Sum Involving the Zeta Function

6399 [1982, 602]. *Proposed by Armel Mercier, Université du Québec à Chicoutimi.*

For all integers $N \geqslant 2$ prove that

$$\sum_{n=2}^{N} \left(\frac{(-1)^{n+1}\binom{N}{n}}{n} \sum_{k=1}^{n-1} (-1)^k \binom{n}{k+1} \zeta(1+k) \right) = \log N + o(1),$$

where $\zeta(s)$ stands for the Riemann zeta function.

Solution by C. Georghiou, University of Patras, Patras, Greece. In view of the known identity

$$\sum_{n=k}^{N} \frac{(-1)^{n+k}}{n}\binom{N}{n}\binom{n}{k} = \frac{1}{k}, \quad N \geqslant k \geqslant 1,$$

(see H. W. Gould, *Combinatorial Identities*, Morgantown, West Virginia, 1972, page 6, line 1.41) the given problem amounts to showing that

$$\sum_{n=2}^{N} \frac{\zeta(n)}{n} = \log N + o(1).$$

Now

$$\begin{aligned}\sum_{n=2}^{\infty} \frac{\zeta(n)-1}{n} &= \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} \frac{1}{nm^n} = \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} \frac{1}{nm^n} \\ &= \sum_{m=2}^{\infty}\left(\log\frac{m}{m-1} - \frac{1}{m}\right) \\ &= \lim_{M\to\infty}\left(\log M - \frac{1}{2} - \frac{1}{3} - \cdots - \frac{1}{M}\right) = 1 - \gamma\end{aligned}$$

where γ is Euler's constant. Hence

$$\begin{aligned}\sum_{n=2}^{N} \frac{\zeta(n)}{n} &= \sum_{n=2}^{N} \frac{\zeta(n)-1}{n} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{N} \\ &= 1 - \gamma - 1 + \gamma + \log N + o(1) = \log N + o(1).\end{aligned}$$

Also solved by George E. Andrews, Robert Breusch, Paul S. Bruckman, James L. Hafner, Eldon Hansen, William A. Newcomb, Otto G. Ruehr, S. K. Venkatesan (India), A. Zulauf (New Zealand) and the proposer.

A Vanishing Integral

6405 [1982, 703]. *Proposed by Dragoljub Milosevic, Pranjani, Yugoslavia.*

Let $T_n(x)$ be the nth Chebyshev polynomial ($n > 1$). Prove or disprove

$$\int_{-1}^{1} \frac{x}{\sqrt{1-x^2}} T_n^3(x)\,dx < \pi\sqrt{2}/8.$$

As many readers observed, the value of the integral in question is in fact zero.

MISCELLANEA

123.

Pupils should be taught how to differentiate and how to integrate simple algebraic expressions before we attempt to teach them geometry and these other complicated things. The dreadful fear of the symbols is entirely broken down in those cases where at the beginning the teaching of the calculus is adopted. Then after the pupil has mastered those symbols you may begin geometry or anything you please.

—S. P. Thompson, 1902 (quoted in R. E. Moritz, *Miscellanea Mathematica*, New York, 1914, pp. 326–327).

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Popular Lectures on Mathematical Logic. By Wang Hao. Van Nostrand Reinhold, New York, 1981. ix + 273 pp.

JAMES BAUMGARTNER
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One of the pleasures of teaching mathematical logic is knowing that in spite of the tremendous changes in the field in the last twenty years, it is still possible to present some of the most interesting and significant results to a class of undergraduates. For example, any one-semester logic course taught in a mathematics department ought at least to reach Gödel's Completeness Theorem, which asserts that if a sentence is true in all the models of a particular theory, then it can be derived from the axioms of the theory in a finitary, formal way. On every pass through this material this seems like an amazing result, and it always raises the same hopes for the power of formal systems. Of course, these hopes are always dashed during the second semester (or later in the first, if the students are bright) with the proof of Gödel's Incompleteness Theorem. Now that Hofstadter's wonderful book [**H**] is available, this argument can even be taught to freshmen.

The same remarks apply to courses in set theory. In one semester, one can easily do all-the-set-theory-every-mathematician-needs-to-know-but-most-of-them-soon-forget, that is to say the theory of ordinal and cardinal numbers, transfinite induction and the Axiom of Choice, and still have time left at the end for something really interesting. One could treat the theory of transfinite trees, say, or models of set theory, or perhaps Silver's remarkable, and embarrassingly recent, result [**S**, **BP**] that if the generalized continuum hypothesis holds for every cardinal below $\aleph_{\omega_1}$, i.e., if $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega_1$, then it also holds at $\aleph_{\omega_1}$. If another semester of set theory is offered, then one could cover Gödel's constructible universe L and finish up with the rudiments of Cohen's theory of forcing and generic sets. Thus in one course both the consistency and the independence of the continuum hypothesis could be proved. Now this last course may be stretching things a bit; it really ought to be taught at the first-year graduate level, but bright undergraduates can easily handle it.

Twenty years ago, anyone who had attended all the courses outlined above would, if he had done all the homework, have been in a position to begin doing research in logic. Nowadays, unfortunately, that is pretty far from being the case. In set theory, for example, the notions of constructibility and forcing have proved to be so fertile that a graduate student has to spend two or three years before reaching the frontier of research. The same is true for most parts of model theory, and for the theory of recursive functions. In its classical form the latter subject is now so extremely complicated that one could believe it is approaching the limits of the human mind, and interest is shifting to generalized versions of recursion theory in which the constructible sets L become the universe of discourse, or to related subjects such as the theory of algorithms, considered part of theoretical computer science. So recursion theory becomes set theory or computer science. If there is a trend in all these parts of logic, it is that the more we understand about these areas, and the more we try to generalize them, the more everything gets tangled up. It is impossible to do generalized recursion theory or model theory without knowing set theory, and it is pretty hard to do set theory without knowing something about the constructible sets, and therefore about recursion theory. Proof theory contains elements of recursion theory (at the lowest level proof is a recursive notion), set theory (forcing is really a proof-theoretic idea), and model theory (look at the recent work on independence in Peano arithmetic).

Perhaps what all this means is that if someone asks you to explain modern mathematical logic, the most attractive strategy is to put your back up and say "No."

That was precisely what Wang Hao didn't do when he accepted an invitation in 1977 to visit China and give a series of lectures on mathematical logic. These lectures have now appeared in book form as *Popular Lectures in Mathematical Logic*, published jointly by Van Nostrand Reinhold and the Science Press of Beijing. (Wang Hao, by the way, is the same logician known widely as Hao Wang. He grew up in China but spent his professional life in the West, so for many years he reversed his name in Western fashion.)

Of course, Wang cannot possibly treat all the parts of logic in a book of fewer than three hundred pages, so he simply selects what he finds most interesting. One is reminded of a remark in his paper, "Eighty years of foundational studies" [W] published twenty-five years ago: "I shall make no futile attempt to be impartial but simply neglect those aspects of mathematical logic which do not interest me." Still, he covers a lot of ground.

Two of the six main chapters are devoted to computer science, which Wang considers to be inseparable from logic. One is devoted to the famous $P = NP$ problem, and the other discusses such topics as computers and the Chinese language (typewriting Chinese is a real problem) and the proof of the four-color theorem.

The level of sophistication increases dramatically in the chapters on model theory and set theory. Wang goes so far as to discuss ultraproducts, a kind of averaging operation for first-order mathematical structures, and the method of Ehrenfeucht and Mostowski for building infinite models of theories from a basis of logically indiscernible elements, rather like the way a vector space is built up from a basis.

The set theory chapter is the high point of the book. In the course of a long discussion of the continuum problem, Wang finds the opportunity to record some anecdotes regarding the work of Gödel, several based on personal conversations. For example, Wang remarks that in 1943 Gödel found a proof in type theory of the independence of the Axiom of Choice, but that he "developed a distaste for the work and did not enjoy continuing it," and he offers the opinion that "if he had continued with it, he would probably have got the independence of CH by 1950, and the development of set theory would have progressed faster."

There is a very nice introduction to forcing. The idea behind forcing is to produce a lot of "generic" real numbers in the black-and-white-label sense of the word. If the continuum hypothesis is true and we want to make it false, then we have to add a great many highly undistinguished reals. After all, the distinguished ones, like π and e, have already been accounted for. The machinery for producing these generic sets is rather complex in the context of set theory, but it becomes quite a bit simpler when one considers a different problem. Consider subsets of the set ω of all natural numbers. Call a subset *definable* if it can be defined (in first-order logic) using only the operations of multiplication and addition. For example, the set of even numbers and the set of prime numbers are both definable. Moreover one can ask about the definability of *sets* of subsets of ω. The set of all subsets of ω, each of which contains at least one prime number, is definable in this sense. Now a question which occurs naturally is this: Is the set of all definable subsets of ω itself a definable set of subsets of ω? (Lest you think we have just defined it, remember that only multiplication and addition may be used in the definition.) The answer turns out to be negative, and the proof makes use of forcing and generic sets in the much simpler context of arithmetic. Previously, this argument had been available only in the German language, in [J], but Wang gives part of it here. In my opinion, this is the best way yet discovered to display the intuition behind forcing.

Unlike every other book of lectures that I have ever seen, this one actually reads like a book of lectures. Usually the author fills out and corrects the proofs, adds more relevant material, and polishes the prose until one wonders how anyone could have sat through such long lectures. If Wang had tried to do that, he would have ended with something about the size of *The Handbook of Mathematical Logic* [B], which tips the scales at four pounds. Instead, Wang has given us a book meant to entice the reader to learn a little more about logic; it is a book to be dipped into, where the same topics appear repeatedly but from widely different points of view, and some of those

points of view are likely to surprise even the experts in the field. Proofs are nearly always sketchy or incomplete, but there are frequent references to other sources. My biggest complaint is that the references are not collected at the end of the book, but appear as footnotes. Furthermore, the two-and-a-half page index ought to be much more extensive in a book of this nature, but this is a minor quibble, and cannot obscure the fact that Wang has written a valuable and entertaining book.

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Anyone who has taken a course in calculus has been exposed to mathematical techniques for optimization. Suppose we wish to find the optimal value of the function $f(x)$ in the interval $[0,1]$, where "optimal" is taken to mean "largest." In one sense, this is a hopeless problem. Even if the value of $f(x)$ can be computed quickly, given x, there are an uncountable number of values to be considered for x. Fortunately, if f is sufficiently well-behaved we can greatly restrict the search. For instance, if f is differentiable, then the only relevant values of x are 0, 1, and those $x \in [0,1]$ for which the derivative of f is zero. We thus may be able to reduce our infinite optimization problem to one with only a finite number of possibilities.

Combinatorial optimization can be viewed as the study of optimization problems with only a finite number of possibilities. For some mathematicians, reducing an infinite problem to a finite one is the only significant step. The rest is "merely combinatorial," solvable in principle by exhaustive search and hence mathematically uninteresting. However, although exhaustive search is indeed of limited mathematical interest, it is also of limited *use* unless the finite number of possibilities we are exhausting is sufficiently small.

Consider the *Linear Programming* problem, a general problem that originally arose in economic planning and has since been shown to have a wide variety of other practical applications. One of its many equivalent formulations is as follows: Given an $m \times n$ integer matrix A and integer vectors b of length m and c of length n, find a (not-necessarily integer) vector $x = (x_1,\ldots,x_n)$ such that the objective function $c \cdot x$ is minimized subject to

$$Ax \geqslant b$$
$$x_i \geqslant 0, \quad 1 \leqslant i \leqslant n,$$

(or show that no such vector exists).

The fact that only a finite number of potential x's need be considered follows from a geometric argument. By convexity, one need only look at the vertices of the n-dimensional polyhedron defined by the above inequalities, and there are at most $\binom{m+n}{n}$ such vertices. However, for all

practical purposes and all sufficiently large m and n, this finite number might as well be infinite—unless some cleverer method than exhaustive search can be devised to explore the possibilities.

As another, more specific example, consider what is commonly known as the *Traveling Salesman Problem*: Given a collection $c_1, c_2, \ldots, c_n$ of *cities*, with a nonnegative distance $d(c_i, c_j)$ defined for each pair c_i, c_j of cities, find an ordering $c_{\pi(1)}, c_{\pi(2)}, \ldots, c_{\pi(n)}$ of the cities which minimizes the length

$$\sum_{i=1}^{n-1} d(c_{\pi(i)}, c_{\pi(i+1)}) + d(c_{\pi(n)}, c_{\pi(1)}).$$

of a round-trip tour. Again we have merely a finite number of possibilities to consider: this time it's "only" $n!$.

If one wants to boil the study of combinatorial optimization down to just two questions, they are these: Is it possible to find optimal solutions without using the exponential (or worse) time required by exhaustive search, and if so, how?

As we shall see, these questions are not just matters of programming technique, but involve mathematics in crucial ways. Until recently, however, they have been for the most part ignored in traditional mathematics curricula and have instead been left to two relatively new disciplines: Operations Research (OR) and Computer Science (CS). The former got its start in the 1940's and the latter did not come into its own until the 1960's. These two disciplines have approached combinatorial optimization from different directions. As illustration, I shall highlight a major success of each. One is a partial answer to the question "How?" and the other is a partial answer to "Is it possible?"

Representing Operations Research, we have the famous "simplex method," which was developed by George Dantzig in 1949 for solving the Linear Programming problem. Based on the geometric interpretation of the problem, it works by constructing a path that progresses from polyhedral vertex to adjacent polyhedral vertex, never moving in a direction that worsens the objective function, and continuing until an optimal vertex is found. By the convexity of the objective function, such a path must exist no matter what the starting point. This observation, together with some clever applications of linear algebra to generate the steps along the path and some additional tricks to avoid cycling among vertices with equal objective function values, can result in surprisingly quick routes to optimal solutions, and this happens with astonishing regularity. As a consequence, the simplex method and its variants have become some of the most widely used and successful algorithms in the history of computing.

From Computer Science we have the theory of "NP-completeness," developed by Steven Cook and Richard Karp in the early 1970's. This theory, based on mathematical techniques for demonstrating the computational equivalence of problems, deals with the inherent difficulty of problems (in combinatorial optimization and a wide variety of other fields). It provides a rigorous way of identifying those problems for which exhaustive search is, in a sense, unavoidable (the *NP-hard* problems, of which the Traveling Salesman Problem is an example), as opposed to those problems for which substantial improvements on exhaustive search can be made, i.e., problems which can be solved by algorithms that are guaranteed to run in time bounded by a polynomial (rather than exponential) function of the instance size.

Surprisingly enough, the textbook under review is the first I know of that treats both of these topics between the same covers. Not only has the mathematics curriculum ignored the successes of OR and CS, but they have, to a large extent, ignored each other! One reason for this, of course, is that although both fields are interested in combinatorial optimization, they differ as to the particular combinatorial optimization problems most central to their overall concerns. However, I think there is a more fundamental conflict between the two fields, at least as they are taught, and this concerns what is called "worst-case analysis," the evaluation of algorithms in terms of guarantees that hold for all problem instances, no matter how contrived or atypical.

It is such analysis that underlies the theory of NP-completeness, and CS researchers develop a taste for it quite early in their careers. For example, one of the first problems encountered by the computer science student is that of *sorting*: Given n items $x_1, x_2, \ldots, x_n$ with sizes $s(x_1), s(x_2), \ldots, s(x_n)$, find an ordering π such that $s(x_{\pi(1)}) \geq s(x_{\pi(2)}) \geq \cdots \geq s(x_{\pi(n)})$. Although there are again $n!$ possible orderings, here there are many simple ways to avoid exhaustive search. The desired ordering can in fact be constructed using at most $O(n \log n)$ comparison steps, a guarantee that holds even in the worst case. Moreover, no algorithm based on comparisons can do better than this, even on average, given the standard models of computation. The introductory CS course in "Data Structures" covers these results and a variety of other programming tricks that are shown to improve the worst-case behavior of algorithms and that are recommended for use in practice. From paradigms such as these, the CS student develops a respect for results that rigorously quantify the efficiency of algorithms, even if the results only concern worst-case behavior (and often this is the only type of rigorous analysis that is feasible).

The operations research student grows up with a different paradigm: the simplex method. Here the conflict arises. Despite its great success in practice, the simplex method does have one technical drawback: Its good behavior cannot be guaranteed. For each of its standard implementations, there exist instances of Linear Programming where the route constructed visits essentially all the vertices of the polyhedron, and hence offers no improvement over exhaustive search. These instances require ingenuity to construct, and do not seem to occur in practice, but they do exist. Thus "worst-case" analysis seems to be misleading as far as the simplex method is concerned, and, presumably, in general.

Such attitudes have their effect on the standard textbooks, with OR texts down-playing worst-case results, while CS texts are uncomfortable with algorithms whose good behavior cannot be explained in terms of guarantees. Fortunately, such limited attitudes are not held by all in OR and CS. Some of the best work in combinatorial optimization has been done by researchers operating on the interface between the two fields, using geometric insights to construct algorithms, and then using worst-case analysis to guarantee that the algorithms are efficient. Moreover, recent theoretical developments have begun to make worst-case analysis a bit more palatable in the OR community. First of all, NP-completeness does seem to be meaningful in practice, despite its worst-case nature. Attempts to repeat the success of the simplex method by extending it to the *Integer* Programming problem (Linear Programming with the added constraint that variables take on integer values) have proved unsuccessful, and Integer Programming, containing as it does the Traveling Salesman Problem as a special case, can be viewed as the canonical NP-hard problem. Secondly, even though the simplex method is not guaranteed to be efficient, it has recently been discovered that Linear Programming *can* be solved in polynomial time (by the much-publicized ellipsoid algorithm).

The time is clearly at hand for a reconciliation. The book under review performs the task with great insight and clarity. In addition to all the above-mentioned topics, it covers the most important of the polynomial time solvable special cases of Integer Programming: network flows, shortest paths, matching (both bipartite and nonbipartite, both unweighted and weighted), minimum spanning trees, matroid intersection, etc. These are presented with an emphasis both on the insights (geometric and otherwise) that led to the development of the algorithms, and on how efficiency can be improved by careful attention to the details of an algorithm's implementation. Also covered are many of the techniques that have been developed for coping with the intractability of the NP-hard problems, both techniques such as "cutting planes" and "branch-and-bound" for narrowing the search (without necessarily making it polynomial) and techniques for doing the best one can within polynomial time (by looking for near-optimal rather than optimal solutions). Here a different kind of worst-case analysis is possible: one can analyze how close to optimal the solution found is guaranteed to be.

The book is meant as a text for graduate students in both OR and CS, and hence does not assume the background of either. Combinatorial optimization is an important and rapidly developing field, and mathematicians wishing a self-contained introduction need look no further.

Lectures on the Theory of Algebraic Numbers. By Erich Hecke. Springer-Verlag, New York, 1981. xii + 239 pp.

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For those outside the field, the labels "number theory," "algebraic number theory," and "theory of algebraic numbers" may not be self-explanatory, and perhaps it is appropriate to offer some comments on the meaning of these terms, or on their possible meanings. One could, naively, define number theory to be the study of numbers, but for this definition to have any value, one must recognize that numbers fall into two classes: those which satisfy a nontrivial polynomial equation in one variable with rational coefficients, and those which do not. The former are called algebraic numbers, the latter, transcendental; one has therefore a distinction between algebraic number theory and transcendental number theory. Concerning transcendental number theory we shall have nothing further to say; it is not the subject of Hecke's book. We emphasize, however, that according to this derivation of the phrase, the "algebraic" in "algebraic number theory" qualifies "number," not "number theory;" thus "algebraic number theory" is simply a synonym for "the theory of algebraic numbers." In contemporary usage, on the other hand, "algebraic number theory" is often understood in a different sense: "algebraic" number theory is the theory of numbers studied using "algebraic" methods, and stands in contrast to "analytic" number theory, where analysis is employed. But however natural this distinction may be, in the end it is less important than the fact—brilliantly demonstrated in Hecke's own mathematical career, and apparent even in his introductory *Lectures*—that some of the deepest results in the theory of algebraic numbers have come from a merging of algebraic and analytic insights.

A field which consists entirely of algebraic numbers is called an algebraic number field, or simply a number field; examples include the rational number field $\mathbb{Q}$, the quadratic fields (those of the form $\mathbb{Q}(\sqrt{m})$, where m is an integer which is not a square), the cyclotomic fields (those of the form $\mathbb{Q}(\zeta)$, where ζ is a root of unity different from ± 1), and the algebraic closure of the rationals. In many contexts (and henceforth in this one) the term "algebraic number field" is reserved for extension fields of the rationals which are of finite degree; such a field necessarily consists of algebraic numbers alone. We may think of a number field K as a generalization of the rational field $\mathbb{Q}$; to pose arithmetical questions about K, we evidently need a generalization of the ordinary integers $\mathbb{Z}$. This is provided by the ring of algebraic integers in K. An algebraic integer is a number which satisfies a monic polynomial with coefficients in the ordinary integers, and it is an elementary fact that the set $\mathcal{O}$ of all algebraic integers in K forms a ring. A significant part of elementary algebraic number theory, and of Hecke's book, is concerned with establishing some basic properties of this ring.

While $\mathcal{O}$ is in general not a unique factorization domain, the fundamental theorem of algebraic number theory asserts that $\mathcal{O}$ is at least a Dedekind domain: every nonzero *ideal* has a unique factorization into prime *ideals*. This is true in particular for the ideal $p\mathcal{O}$, where p is any rational prime, and therefore we may write

$$p\mathcal{O} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_r^{e_r}$$

where $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are distinct prime ideals of $\mathcal{O}$ which are uniquely determined up to order and $e_1, \ldots, e_r$ are uniquely determined nonnegative integers. To give the decomposition of p in K means to specify the number of ideals r, the multiplicities e_j, and the residue class degrees $f_j = [\mathcal{O}/\mathfrak{p}_j : \mathbb{Z}/p\mathbb{Z}]$ (they are subject to the relation $n = e_1 f_1 + \cdots + e_r f_r$, where $n = [K:\mathbb{Q}]$). To give a law for the decomposition of primes in K means to give a general rule which describes how the decomposition of p in K varies with p. The decomposition laws for quadratic fields and for cyclotomic fields are discussed in detail in Hecke's book; they can be expressed in an elementary

way in terms of congruences on p. A search for decomposition laws in more general number fields leads to class field theory and beyond, but this is outside the scope of Hecke's introductory text.

It is tempting, although probably misleading, to suggest that the search for decomposition laws in number fields is the principal unifying theme in algebraic number theory. This suggestion, if it is worth making at all, must be accompanied by two strong caveats. The first caveat is that by now, the search for decomposition laws has largely been subsumed in the more general theory of L-functions, a development which is in no small part a consequence of Hecke's own contributions to number theory. L-functions are a class of complex analytic functions defined by Dirichlet series which constitute a vast generalization of the Riemann zeta function. The decomposition laws of primes in number fields are in a sense encoded into the very definition of these functions, but it is the L-functions themselves which are the broader and more pervasive concept. In fact, the centrality of L-functions might lead one to expand considerably one's concept of a decomposition law. Here is an example. For each prime p, let $N(p)$ be the number of solutions (in integers x, y modulo p) to the congruence

$$y^2 + y \equiv x^3 - x^2 \pmod{p}.$$

One can ask whether there is some rule or principle which describes how $N(p)$ varies with p: this would be a kind of decomposition law, although no longer in the traditional sense of the term. The answer is that such a principle exists, and that it finds an analytic expression in the analytic continuation and functional equation of a certain L-function.

The second caveat is simply this: algebraic number theory is far too vast, rich, and varied a subject to admit of any real unifying theme at all. We cannot illustrate the richness and variety here, but for the uninitiated, Hecke's *Lectures* can serve as an introduction.

A Primer of Real Functions. By Ralph P. Boas, Jr., The Mathematical Association of America, Washington, D.C., 1981 (The Carus Mathematical Monographs, Number Thirteen). ix + 232 pp.

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Undergraduate courses on advanced calculus or real analysis have been the cornerstone of the mathematics major for two generations. These courses are, at the classical end of their spectrum, rather encyclopedic in content. They combine an introduction to mathematical rigor with a large number of topics ranging from point set topology, the implicit function theorem, and integration to partial derivatives, infinite series, orthogonal series, the Fourier transform and uniform convergence. At the ultraviolet end of the spectrum these courses are narrower and more abstract. They may include set theory, Baire category, Baire functions, measure, differential forms, and integration up to, and sometimes beyond, Stokes' Theorem.

Professor Boas' book is one that is both classical and modern in content, and in addition it excites the mathematical spirit—if it exists in the reader. Yet, beautiful and pedagogically sound as Professor Boas' book is, perhaps it is not the right book for the next generation. There is no mention of computing or numerical analysis either in it or in many widely used texts. An undergraduate course in real analysis (or in functions of several variables or advanced calculus) ought to serve the needs of the computer scientist and computer oriented mathematical scientist, as well as the future mathematician. For a basic analysis course introducing mathematical rigor and satisfying these needs, the concepts of number and function require reinterpretation with

mention of the concrete and discrete and discussion of function evaluation algorithms. The constructive and algorithmic aspects of many other topics need to be taught. For example, the implicit function theorem could be followed by computation of Jacobians and Newton and/or quasi-Newton solution of systems of equations (algebraic or transcendental).

Why should the cornerstone of the mathematics curriculum be relaid? The answer is that computer usage, from number crunching to investigation of geometric phenomena via computer graphics, is entering mathematics with strength and speed. Differential geometers, logicians, algebraists, and analysts of many varieties now use the computer. For those *not* entering mathematical research and teaching careers in universities, but who use mathematics in their work, the computer and computer graphics will be even more important.

My preaching now gives way to praise of a respected colleague's work. Both the elderly and the young can enjoy Boas' *Primer*. There are gems to delight any reader. Consider the universal chord theorem: Suppose $f \in C[0, 1]$ and $f(0) = f(1) = 0$. Then there are horizontal chords of lengths $1/2, 1/3, 1/4, \ldots$, but *not* necessarily a horizontal chord of any given length that is not the reciprocal of an integer.

The serious student of monotonic convex and C^∞ functions can find most of what he or she needs in this book. The reader is challenged to find errors in "proofs," as well as examples and counterexamples. The amount of abstract material is at a minimum; almost all the mathematics presented is endowed with life by Professor Boas. His *Primer* deserves a handy spot on your bookshelf.

LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

Editor:

Professor King's query about the Nobel Prize in mathematics (this MONTHLY, vol. 90, p. 502) reminds me of the cogent remarks of an important physicist who bequeathed everything to science. H. G. J. Moseley successfully verified with x-ray techniques that the concept of atomic number is far more basic to the structural relation of chemical elements than Mendeleev's notion based on atomic weight.

Moseley, who was killed in action in the 1915 Gallipoli debacle (perhaps, in part, through Nobel's own developments) and was probably the most promising English physicist of his generation, wrote on 27 June 1915:

> ... I give and bequeath all my estate real and personal and my reversionary interests therein to the Royal Society of London to be applied to the furtherance of experimental research in Pathology Physics Physiology Chemistry or other branches of science but not in pure mathematics astronomy or any branch of science which aims merely at describing, cataloguing or systematizing.

This letter-will by Moseley can be found in J. L. Heilbron's book, *H. G. J. Moseley*, University of California Press, Berkeley, 1974, pp. 271–272.

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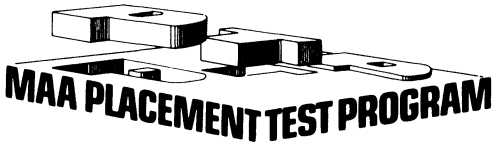
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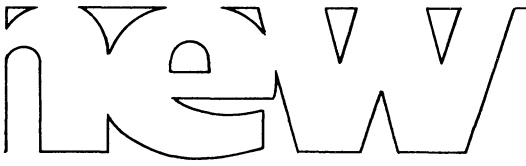
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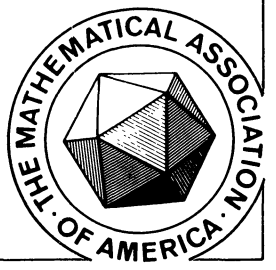
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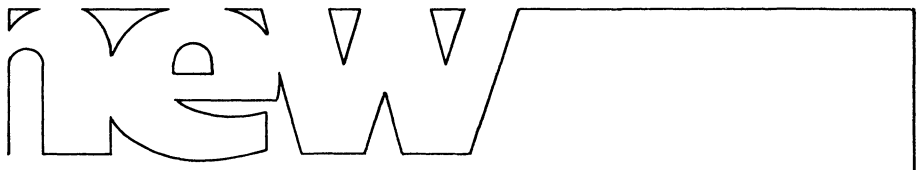
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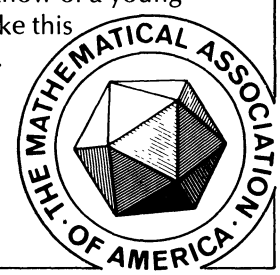
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MÖBIUS EQUIVALENCE AND EUCLIDEAN SYMMETRY

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1. Introduction. Let $\mathscr{F}$ be an arbitrary figure in Euclidean n-space and let $\mathscr{E}(\mathscr{F})$ be the full group of Euclidean symmetries of $\mathscr{F}$. Let $\mathscr{F}'$ be a second figure and let $\mathscr{E}(\mathscr{F}')$ be its full group of Euclidean symmetries. Following Kavanau [6] we shall say that $\mathscr{F}'$ belongs to the same *species* as $\mathscr{F}$ if there is a Möbius transformation h which maps $\mathscr{F}$ to $\mathscr{F}'$. If the Möbius transformation h is more general than a Euclidean similarity, $\mathscr{F}'$ may be quite different in shape from $\mathscr{F}$ and the symmetry group $\mathscr{E}(\mathscr{F}')$ quite different from the symmetry group $\mathscr{E}(\mathscr{F})$. Nevertheless there is a connection between these two groups and a major focus of our work will be to explain this connection.

By looking briefly at a few examples we can clarify the central problem and motivate several related questions. Let $\mathscr{F}_1$ be the D-shaped curve consisting of a semicircle and its diameter (see Fig. 1). Let γ be an arbitrary circle centred at one end of this diameter and let $\mathscr{F}_1^1 = \mathscr{F}_1^\gamma$ be the image of $\mathscr{F}_1$ under inversion in γ. $\mathscr{F}_1^1$ consists of two perpendicular rays emanating from a common point O just like two semiaxes of a rectangular cartesian coordinate frame. Although the curves $\mathscr{F}_1$ and $\mathscr{F}_1^1$ belong to the same species, their Euclidean symmetry groups are quite different. $\mathscr{E}(\mathscr{F}_1)$ contains just two elements: the identity and the reflection which interchanges the top and bottom of the D; $\mathscr{E}(\mathscr{F}_1^1)$ is infinite and contains the identity, the reflection which interchanges the two rays of the figure, arbitrary dilatations with centre O, and the dilative reflections which arise as products of these dilatations with the reflection mentioned above.

To introduce a third member of the species, let γ' be a circle centred at a point inside the convex hull of $\mathscr{F}_1^1$ on the axis of the reflection which interchanges the rays of $\mathscr{F}_1^1$. Then $\mathscr{F}_1^2 = \mathscr{F}_1^{1\gamma'}$ is a dented oval made up of the major arcs of two congruent perpendicular circles; the group $\mathscr{E}(\mathscr{F}_1^2)$ contains four elements: the identity, two reflections in perpendicular lines, and the product of these reflections which is a halfturn. The species determined by $\mathscr{F}_1$ has many different representatives but each has one of the three Euclidean symmetry groups already encountered. We may generalize our original question to ask how these three groups are determined by the figure $\mathscr{F}_1$.

A second example is illuminating. Consider a cycle of four congruent circles tangent in pairs at the vertices A, B, C, D of a square (see Fig. 2). Let $\mathscr{F}_2$ be the continuous curve formed by taking the major arc of the circle joining A to B, the minor arc of the circle joining B to C, the major arc of the circle joining C to D, and the minor arc of the circle joining D to A. Thus $\mathscr{F}_2$ is shaped like an hourglass and $\mathscr{E}(\mathscr{F}_2)$ consists of four elements: the identity, reflections in each of two perpendicular lines, and a halfturn about the centre O where these lines intersect. The groups $\mathscr{E}(\mathscr{F}_2)$ and $\mathscr{E}(\mathscr{F}_1^2)$ are isomorphic, but the species represented by $\mathscr{F}_2$ is essentially different from the one represented by $\mathscr{F}_1^2$ or, equivalently, by $\mathscr{F}_1$. For we shall see that the possible symmetry groups which arise as we take different representatives of the second species are the group just mentioned, the group consisting of a halfturn and the identity, the group consisting of a reflection

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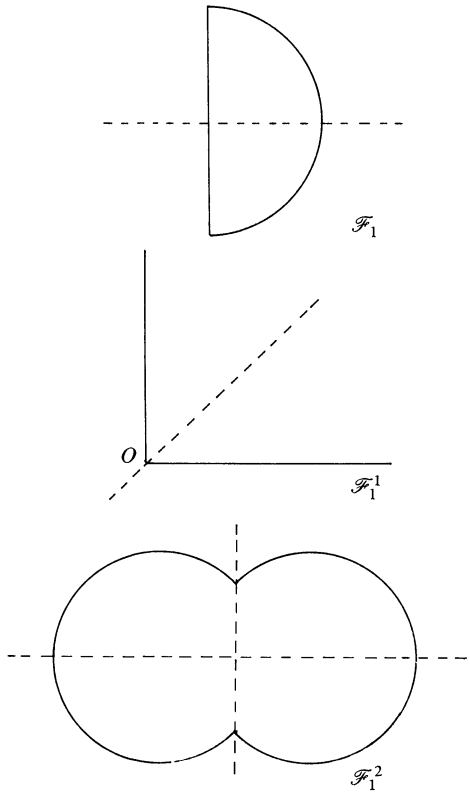


FIG. 1. Three representatives of the first species.

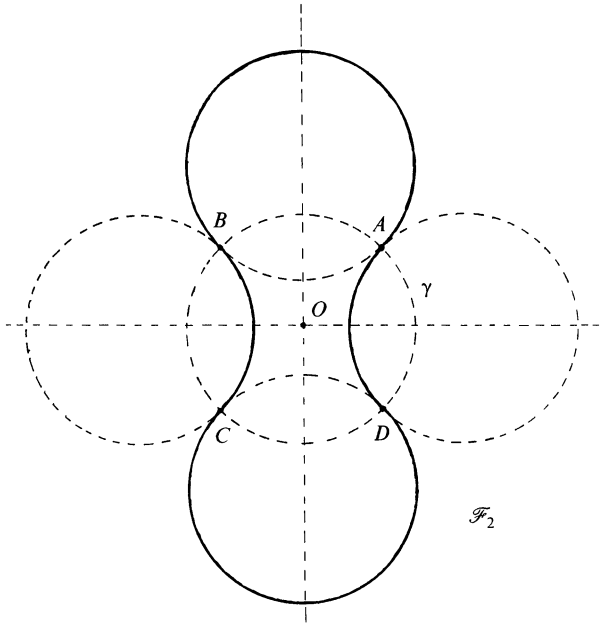


FIG. 2. The figure $\mathscr{F}_2$.

and the identity, and the trivial group containing only the identity. How can we produce representatives of this species with the symmetry groups just listed and how can we know that this list exhausts the possibilities? All these questions will be answered.

The figure $\mathscr{F}_2$ has an interesting property which actually served to motivate this whole investigation. Let γ be the circle with centre O and radius OA. Then γ passes through A, B, C, D and is perpendicular to the four circles used in the construction of $\mathscr{F}_2$. Inversion in γ fixes these circles and interchanges the arcs which form $\mathscr{F}_2$ with those omitted in the construction of $\mathscr{F}_2$. Thus $\mathscr{F}_2^\gamma$ is congruent to $\mathscr{F}_2$ and a quarter turn about O moves $\mathscr{F}_2^\gamma$ into coincidence with $\mathscr{F}_2$. In the language of J. Lee Kavanau [6], $\mathscr{F}_2$ admits a $\pi/2$ self-inversion and he asked what properties of a curve like $\mathscr{F}_2$ result from the fact that it admits this kind of symmetry.

Let us return to our general consideration of an arbitrary figure $\mathscr{F}$ in Euclidean n-space. Motivated by Kavanau's question, let us introduce $\mathscr{M}(\mathscr{F})$, the full group of Möbius symmetries of $\mathscr{F}$. When we pass to another figure $\mathscr{F}^h$ in the same species as $\mathscr{F}$, we shall find that $\mathscr{M}(\mathscr{F}^h)$ is the conjugate group $h^{-1}\mathscr{M}(\mathscr{F})h$. Thus the abstract group $\mathscr{M}(\mathscr{F})$ is an invariant of the species and we shall see that the Euclidean symmetry groups which arise in the species are isomorphic to the stabilizer subgroups of $\mathscr{M}(\mathscr{F})$. This tells us which groups can arise and shows us how to exhibit them.

If $\mathscr{M}(\mathscr{F})$ is finite, we shall see that we can represent it as an isometry group of the n-sphere in Euclidean $(n + 1)$-space. There is an intimate connection between figures representing different species and regular polytopes representing different finite isometry groups. In this correspondence the figure $\mathscr{F}_2$ is related first to a curve like the seam on a baseball or tennis ball and then to a 2-gonal antiprism. Other figures in the plane correspond to pyramids, prisms, antiprisms or the Platonic solids. This unexpected connection with a well-established theory will allow us to analyze completely the cases which occur when $n = 2$ and $\mathscr{M}(\mathscr{F})$ is finite.

An ellipse and a rectangle belong to different species but they determine the same group $\mathscr{M}$ and their symmetry theory is therefore identical (see Fig. 3). Throughout this work we shall not really be studying individual species of figures but rather collections of species or *genera* which possess the same Möbius symmetry $\mathscr{M}$. In fact it is possible to abandon figures altogether and obtain certain results for the groups themselves. These are particularly useful in connection with Möbius subgroups which are not the symmetry group of any figure. However, for reasons of space, we confine our attention in this account to symmetry groups of figures.

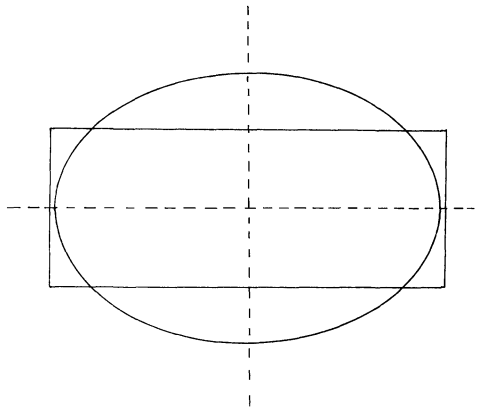


FIG. 3. An ellipse and a rectangle belong to different species but the same genus—they determine the same Möbius group $\mathscr{M}$.

The main reference is [7] and the reader can assume that results quoted without proof and without another reference are taken from there. We do not hesitate to depart from the formal

definition-theorem-proof format at points where a more discursive style seems useful. As a result, not every important statement is labelled Theorem and not every line of argument is labelled Proof. In places where it has seemed reasonable to be more formal, the distinction between theorems and propositions is that the theorems form the central chain of argument while propositions serve as lemmas or flesh out the central theme and deal with matters of more local interest.

2. Isometries, Similarities and Möbius Transformations. This section is intended primarily to fix notation and to refresh the reader's memory on fairly standard material. Results are stated in descriptive style, but full proofs arranged in careful logical format are available in [7].

We denote Euclidean n-space by $\mathbb{R}^n$ to remind the reader that although much of our discussion is carried on in the language of synthetic geometry, the n-tuples of real numbers $\vec{x} = (x_1, x_2, \ldots, x_n)$ with norm $\|\vec{x}\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ which form the basis of an analytic treatment are close at hand and can be called into play whenever we need a calculation.

2.1. *Isometries.* An *isometry* of $\mathbb{R}^n$ is a mapping f: $\mathbb{R}^n \to \mathbb{R}^n$ which preserves the distance between points. In terms of coordinates this condition reads $\|\vec{x}^f - \vec{y}^f\| = \|\vec{x} - \vec{y}\|$. It is clear that isometries must be one-to-one or injective, but it is perhaps not immediately clear that they must also be onto or surjective. However, this does turn out to be the case and it follows that the isometries of $\mathbb{R}^n$ form a group $\mathscr{I}(\mathbb{R}^n)$. This group is generated by reflections in $(n-1)$-flats and is sharply transitive on any equivalence class of mutually congruent simplexes. Any isometry can be written as the product of at most $n+1$ reflections and it is *direct* (sense-preserving) or *opposite* (sense-reversing) as the number of reflections required is even or odd. Thus the parity, if not the actual number of reflections used for a given isometry, is determined.

The product of reflections in a pair of intersecting $(n-1)$-flats is a *rotation* about their $(n-2)$-dimensional intersection through twice the dihedral angle from the first $(n-1)$-flat to the second. The product of reflections in a pair of parallel $(n-1)$-flats is a *translation* through twice the distance between the flats along a line perpendicular to them and in the direction from the first $(n-1)$-flat to the second. These descriptions show that there is some freedom in the choice of reflections factoring a given isometry and this freedom is useful in computing the product of isometries. As a two-dimensional example we mention that in computing the product of a rotation through θ about P followed by a rotation through ψ about Q we can use the line PQ for the second mirror of the first rotation and the first mirror of the second rotation. This common reflection cancels in the product and we see that the product is a rotation through $\theta + \psi$ about a third point R (or a translation in the special case $\theta + \psi = 0$).

Finally we mention that the general isometry f can be written in canonical form as the commuting product of s rotations, t reflections, and u translations, where $s \geqslant 0$, $t = 0$ or 1, $u = 0$ or 1, and $2s + t + 2u \leqslant n + 1$. If we choose coordinates appropriately and abbreviate the first $2s$ real coordinates by s complex ones $z_j = x_{2j-1} + ix_{2j}$ $(j = 1, 2, \ldots, s)$, then

$$(z_1, z_2, \ldots, z_s, x_{2s+1}, \ldots, x_n)^f = \left(e^{i\theta_1}z_1, e^{i\theta_2}z_2, \ldots, e^{i\theta_s}z_s, (-1)^t x_{2s+1}, \ldots, x_n + d\right).$$

The transformation is direct or opposite as $t = 0$ or 1 and it has a fixed point or not as $u = 0$ or 1. In two dimensions the possibilities are rotation $(s = 1)$, translation $(u = 1)$, reflection $(t = 1)$, and glide reflection $(t = 1, u = 1)$.

2.2. *Similarities.* A *similarity* with *scale factor* $k > 0$ is a mapping g: $\mathbb{R}^n \to \mathbb{R}^n$ which alters all distances by a factor of k; in terms of coordinates the definition reads $\|\vec{x}^g - \vec{y}^g\| = k\|\vec{x} - \vec{y}\|$. If $k = 1$, we have an isometry and if $k \neq 1$, we have a *proper similarity*. An example of a proper similarity is the *dilatation* with centre $\vec{x}_0$ and scale factor k given by $\vec{x}^\delta - \vec{x}_0 = k(\vec{x} - \vec{x}_0)$. When similarities are composed, their scale factors multiply, so if δ is an arbitrary dilatation with the same scale factor as g, then $\delta^{-1}g = f$ is an isometry and $g = \delta f$.

The last equation shows that similarities are bijections and hence they form a group $\mathscr{E}(\mathbb{R}^n)$.

This group is sharply transitive on any equivalence class of mutually similar simplexes. The mapping

$$s : \mathscr{E}(\mathbb{R}^n) \to \mathbb{R}^+$$

to scale factors is a homomorphism with kernel $\mathscr{I}(\mathbb{R}^n)$. There is some debate whether $\mathscr{E}(\mathbb{R}^n)$ or $\mathscr{I}(\mathbb{R}^n)$ should be considered as the group defining Euclidean geometry in the sense of Klein, but it will be natural for us to give this honour to $\mathscr{E}(\mathbb{R}^n)$ and our notation has been chosen accordingly.

Every proper similarity has a unique fixed point and can be written in canonical form as the commuting product of a dilatation and an isometry which share this common fixed point. Thus with appropriate coordinates centred at the fixed point, the most general proper similarity g has the form

$$(z_1, z_2, \ldots, z_s, x_{2s+1}, \ldots, x_n)^g = k\left(e^{i\theta_1}z_1, e^{i\theta_2}z_2, \ldots, e^{i\theta_s}z_s, (-1)^t x_{2s+1}, \ldots, x_n\right).$$

It is direct or opposite as $t = 0$ or 1. When $n = 2$, the possibilities are dilatation, dilative rotation ($s = 1$), and dilative reflection ($t = 1$).

2.3. *Möbius Transformations.* In preparation for considering Möbius transformations, we extend our space $\mathbb{R}^n$ by adding a single new point ∞ to create inversive n-space $\Pi^n = \mathbb{R}^n \cup \{\infty\}$. We declare that ∞ is incident with every m-flat $1 \leqslant m \leqslant n-1$. With this convention it is natural to let the groups $\mathscr{I}(\mathbb{R}^n)$ and $\mathscr{E}(\mathbb{R}^n)$ act on Π^n by leaving ∞ fixed. In order to extend these groups to the Möbius group, we must define inversion in an $(n-1)$-sphere.

Let γ be an $(n-1)$-sphere with centre O and radius ρ. Inversion in γ is a transformation of period 2 which interchanges O and ∞ and P and P' where P' on the ray OP satisfies the distance relation $OP \cdot OP' = \rho^2$. Inversion in γ fixes m-flats through O, interchanges m-spheres through O with m-flats not through O, and interchanges m-spheres not through O with others of the same description, $1 \leqslant m \leqslant n-1$. It preserves angles; distorts distance according to the formula $A^\gamma B^\gamma = \rho^2 AB/OA \cdot OB$; and preserves the *cross ratio* $AB \cdot CD/AC \cdot BD$ of sets of four distinct points. When $n \geqslant 2$ inversion in an $(n-1)$-sphere γ and reflection in an $(n-1)$-flat γ can be given a unified definition: they each fix the points of γ and map points $P \notin \gamma$ to the second point of intersection of any two circles or lines through P perpendicular to γ.

The *Möbius group* $\mathscr{M}(\Pi^n)$ can be defined as the group of bijections of Π^n generated by reflections in $(n-1)$-flats and inversions in $(n-1)$-spheres or alternatively as the group of transformations of Π^n which preserve cross ratio. The first definition shows that $\mathscr{I}(\mathbb{R}^n) = \mathscr{I}(\Pi^n)$ is a subgroup of $\mathscr{M}(\Pi^n)$ and the second that $\mathscr{E}(\mathbb{R}^n) = \mathscr{E}(\Pi^n)$ is a subgroup of $\mathscr{M}(\Pi^n)$. The group of Möbius transformations is sharply transitive on *clusters* of $n+2$ mutually tangent $(n-1)$-spheres and any Möbius transformation can be written as the product of at most $n+2$ reflections and inversions. The transformation is direct or opposite as the number of these factors is even or odd.

In order to understand the action of a general Möbius transformation, we must first consider the product of two generators $\gamma_1\gamma_2$. The unified definition of reflection and inversion shows that if a Möbius transformation h maps an $(n-1)$-flat or $(n-1)$-sphere γ to γ^h, then it conjugates the reflection or inversion in γ to reflection or inversion in γ^h. This shows that if the sets γ_1 and γ_2 intersect or are tangent, the transformation $\gamma_1\gamma_2$ is conjugate to a rotation or a translation. Otherwise it is conjugate to a transformation in which γ_1 and γ_2 are concentric $(n-1)$-spheres with common centre O and radii ρ_1 and ρ_2 (say). Then an easy calculation shows that $\gamma_1\gamma_2$ is the dilatation with centre O and scale factor $k = (\rho_2/\rho_1)^2$. The fact that $\mathscr{M}(\Pi^n)$ includes arbitrary dilatations gives a second proof that $\mathscr{E}(\Pi^n)$ is a subgroup of $\mathscr{M}(\Pi^n)$. In fact, $\mathscr{E}(\Pi^n)$ is the full subgroup of $\mathscr{M}(\Pi^n)$ which fixes ∞: $\mathscr{E}(\Pi^n) = \mathscr{M}(\Pi^n)_\infty$, the stabilizer of ∞.

Now the general Möbius transformation is one of three different types [**8**]. First, it may be conjugate to a Euclidean isometry with nonzero translation. Second, it may be conjugate to a proper Euclidean similarity. Third, it may be the commuting product of $s \geqslant 0$ transformations each conjugate to rotations and $t = 0$ or 1 inversions, where $2s + t \leqslant n + 1$. If $2s + t \leqslant n$, this

type of transformation is conjugate to a Euclidean isometry with a fixed point. But if $2s + t = n + 1$, the best conjugacy class representative we can offer now is the commuting product of a Euclidean isometry with fixed point O and inversion in the unit $(n-1)$-sphere with centre O. For example, when $n = 2$ we have transformations of the form $z^h = e^{i\theta}/\bar{z}$. In due course we shall see that the unit sphere Σ^n in Euclidean $(n+1)$-space affords a second model of inversive n-space and Möbius transformations of the third kind are conjugate to isometries of Σ^n.

3. The Structure of a Subgroup of $\mathscr{M}(\Pi^n)$.

3.1. *Individual Transformations.* Let $h \in \mathscr{M}(\Pi^n)$. We must now give a description of h "as it stands" without allowing ourselves the luxury of conjugating it into a more convenient form. An example with $n = 2$ will serve to clarify our objective. The transformation $z^h = \rho^2/z$ is a halfturn with fixed points $z = \rho$ and $z = -\rho$ and can be conjugated to $z^{h'} = -z$. However, in the present context we must describe h in situ as the commuting product of inversion in the circle $|z| = \rho$ and reflection in the real axis.

Before proving our first theorem, it is convenient to introduce an elaboration of our notation. We shall use δ_Q^k to denote dilatation with centre Q and scale factor k, and γ_Q^ρ to denote inversion in the $(n-1)$-sphere with centre Q and radius ρ. Recall that $\delta_Q^k = \gamma_Q^{1/\sqrt{k}}\gamma_Q^1$.

THEOREM 1. *Let $h \in \mathscr{M}(\Pi^n)$. If h fixes ∞, it is a similarity of $\mathbb{R}^n$. Otherwise, there exist points P and Q of $\mathbb{R}^n$ such that $P^h = \infty$ and $\infty^h = Q$. In this case there is a unique real number $\rho > 0$ and a unique isometry f such that $h = \gamma_P^\rho f = f\gamma_Q^\rho$. The isometry f satisfies $P^f = Q$.*

Proof. (Coxeter [1].) We have already quoted the fact that $\mathscr{M}(\Pi^n)_\infty = \mathscr{E}(\Pi^n)$ and this means that if h fixes ∞, it is a similarity of $\mathbb{R}^n$.

Otherwise, $\infty^h = Q$ and the composite transformation $h\gamma_Q^1 = g$ must be a similarity. If the scale factor of g is ρ^{-2}, we may write $g = f\delta_Q^{\rho^{-2}}$, where f is an isometry. But then

$$h = f\delta_Q^{\rho^{-2}}\gamma_Q^1 = f\gamma_Q^\rho\gamma_Q^1\gamma_Q^1 = f\gamma_Q^\rho$$

as required. It follows that we may also write

$$h = f\gamma_Q^\rho f^{-1}f = \left(\gamma_Q^\rho\right)^{f^{-1}} f = \gamma_{Q^{f^{-1}}}^\rho f = \gamma_P^\rho f,$$

where the last equality is forced by considering the action of the transformation on P.

3.2. *Subgroups of $\mathscr{M}(\Pi^n)$.* Let $\mathscr{M}$ be a subgroup of $\mathscr{M}(\Pi^n)$. For example, let $\mathscr{F}$ be a subset of Π^n and let $\mathscr{M} = \mathscr{M}(\mathscr{F})$ be the full group of transformations in $\mathscr{M}(\Pi^n)$ which map $\mathscr{F}$ to itself.

We let $\mathscr{I} = \mathscr{M} \cap \mathscr{I}(\Pi^n)$ be the isometries in $\mathscr{M}$ and $\mathscr{E} = \mathscr{M} \cap \mathscr{E}(\Pi^n)$ be the similarities in $\mathscr{M}$. Thus $\mathscr{I}$ and $\mathscr{E}$ are subgroups of $\mathscr{M}$ with $\mathscr{I} \subset \mathscr{E} \subset \mathscr{M}$ and $\mathscr{E} = \mathscr{M}_\infty$ is the stabilizer of ∞ in $\mathscr{M}$. Our main objective is to decompose $\mathscr{M}$ into cosets with respect to $\mathscr{I}$ and $\mathscr{E}$ and to obtain useful expressions for the indices of these subgroups. If S is a set, we shall use $|S|$ for its cardinality.

Here, with some introductory comments, are a number of sets related to the group $\mathscr{M}$.

Let $K^* = s(\mathscr{E})$ be the image of $\mathscr{E}$ under the scale factor homomorphism and let $K = K^* \setminus \{1\}$. Since $\mathscr{E}/\mathscr{I} \cong K^*$, the index of $\mathscr{I}$ in $\mathscr{E}$ is $|K^*| = |K| + 1$. If $K \neq \varnothing$, then K must be infinite because it contains all powers of any of its elements. In this case $\mathscr{E}$ and hence also $\mathscr{M}$ must be infinite. If $k \in K$, we write $\mathscr{E}^k$ for the coset $s^{-1}(k)$.

Let Γ be the set of centres of inversion used in the canonical factorization of the elements in $\mathscr{M}$ described in Theorem 1. Then

$$\begin{aligned}\Gamma &= \{P : \exists h \in \mathscr{M}, \rho \in \mathbb{R}^+, f \in \mathscr{I}(\Pi^n) \text{ with } h = \gamma_P^\rho f\} \\ &= \{\infty^h : h \in \mathscr{M} \setminus \mathscr{E}\}.\end{aligned}$$

The second equality follows from Theorem 1 since we showed there that the point P in $h = \gamma_P^\rho f$ is just $\infty^{h^{-1}}$.

For any $P \in \Gamma$ let

$$R_P = \{\rho \in \mathbb{R}^+ : \exists h \in \mathscr{M}, f \in \mathscr{I}(\Pi^n) \text{ with } h = \gamma_P^\rho f\}.$$

We shall see that $|R_P|$ is independent of P and equal to $|K^*|$.

THEOREM 2. *If $\mathscr{M} = \mathscr{M}(\mathscr{F})$ the inversions in $\{\gamma_P^\rho : P \in \Gamma, \rho \in R_P\}$ are precisely the inversions which transform $\mathscr{F}$ to a congruent figure.*

Proof. Let γ be an inversion. Then $\mathscr{F}^\gamma$ is congruent to $\mathscr{F}$ if and only if there is an isometry f such that $\mathscr{F}^{\gamma f} = \mathscr{F}$. We note in passing that $\mathscr{F}^\gamma$ coincides with $\mathscr{F}$ if and only if f can be chosen to be the identity. Now $\mathscr{F}^{\gamma f} = \mathscr{F}$ if and only if $\gamma f \in \mathscr{M}$ and, because the canonical factorization of Theorem 1 is unique, this holds if and only if

$$\gamma \in \{\gamma_P^\rho : P \in \Gamma \text{ and } \rho \in R_P\}.$$

Because of Theorem 2, we see that the points in $\Gamma(\mathscr{F})$ are precisely the points that Kavanau [6] has referred to as poles of subspecific inversion for $\mathscr{F}$.

We turn now to a series of three propositions which will yield the information we require about coset decompositions.

PROPOSITION 1. *Let O be an arbitrary point in $\mathbb{R}^n$. Then there is an isometry f_O^k such that $\mathscr{E}^k = \delta_O^k f_O^k \mathscr{I}$.*

Proof. If $g \in \mathscr{E}^k$, then $\mathscr{E}^k = g\mathscr{I}$. We can write $g = \delta_O^k f_O^k$ and the result follows.

COROLLARY. *The coset decomposition of $\mathscr{E}$ relative to $\mathscr{I}$ can be written*

$$\mathscr{E} = \mathscr{I} \cup \left[\bigcup_{k \in K} \delta_O^k f_O^k \mathscr{I}\right]$$

and hence the index of $\mathscr{I}$ in $\mathscr{E}$ is $|K| + 1$.

Proof. Obvious.

PROPOSITION 2. *Let $P \in \Gamma$ and $\rho \in R_P$. Then there is an isometry f_P^ρ such that the elements of $\mathscr{M}$ whose canonical factorization involves γ_P^ρ are precisely those in the set $\gamma_P^\rho f_P^\rho \mathscr{I}$.*

Proof. If $h_1 = \gamma_P^\rho f_1$ and $h_2 = \gamma_P^\rho f_2$ are elements of $\mathscr{M}$, then

$$h_1^{-1} h_2 = f_1^{-1} \gamma_P^\rho \gamma_P^\rho f_2 = f_1^{-1} f_2$$

belongs to $\mathscr{M}$, hence $\mathscr{I}$ and $f_2 \in f_1 \mathscr{I}$. Thus if we write $f_1 = f_P^\rho$, h_2 has the required form.

On the other hand if $h_1 = \gamma_P^\rho f_P^\rho \in \mathscr{M}$, then any element in $h_1 \mathscr{I} = \gamma_P^\rho f_P^\rho \mathscr{I}$ belongs to $\mathscr{M}$ and has a canonical factorization involving γ_P^ρ.

COROLLARY. *The coset decomposition of $\mathscr{M}$ relative to $\mathscr{I}$ can be written*

$$\mathscr{M} = \mathscr{I} \cup \left[\bigcup_{k \in K} \delta_O^k f_O^k \mathscr{I}\right] \cup \left[\bigcup_{P \in \Gamma} \bigcup_{\rho \in R_P} \gamma_P^\rho f_P^\rho \mathscr{I}\right]$$

and hence the index of $\mathscr{I}$ in $\mathscr{M}$ is $|K| + 1 + \Sigma_{P \in \Gamma} |R_P|$.

Proof. The sets indicated are a complete set of cosets listed once because the factorization described in Theorem 1 is unique.

PROPOSITION 3. *Let $P \in \Gamma$ and $\rho \in R_P$. Then there is an isometry f_P such that the elements of $\mathscr{M}$ whose canonical factorization involves inversion with centre P are precisely those in the set $\gamma_P^\rho f_P \mathscr{E}$. The set R_P is equal to $\rho\sqrt{K^*} = \{\rho\sqrt{k} : k \in K^*\}$.*

Proof. If $h = \gamma_P^\rho f_P$ and $h' = \gamma_P^{\rho'} f'$ are elements of $\mathscr{M}$, then

$$h^{-1} h' = (f_P)^{-1} \gamma_P^\rho \gamma_P^{\rho'} f' = (f_P)^{-1} \delta_P^{(\rho'/\rho)^2} f'$$

belongs to $\mathscr{M}$ hence $\mathscr{E}$. This proves that

$$\rho' = \sqrt{k}\,\rho \quad \text{for some } k \in K^* \quad \text{hence } R_P \subset \rho\sqrt{K^*}$$

and also that

$$f' \in \gamma_P^{\rho'}\gamma_P^{\rho} f_P \mathscr{E} \quad \text{hence } h' \in \gamma_P^{\rho'}\left(\gamma_P^{\rho'}\gamma_P^{\rho} f_P \mathscr{E}\right) = \gamma_P^{\rho} f_P \mathscr{E}.$$

Conversely, any element of the form $h\mathscr{E} = \gamma_P^{\rho} f_P \mathscr{E}$ belongs to $\mathscr{M}$ and has a canonical factorization involving inversion with centre P. To show that $R_P \supset \rho\sqrt{K^*}$ we take $g \in \mathscr{E}$ with $s(g) = k$ so $g = \delta_P^k f$. Then

$$\begin{aligned}\gamma_P^{\rho} f_P g &= \gamma_P^{\rho}\left(\delta_P^k \delta_P^{k^{-1}}\right) f_P\left(\delta_P^k f\right) \\ &= \left(\gamma_P^{\rho}\delta_P^k\right)\left[\left(\delta_P^{k^{-1}} f_P \delta_P^k\right) f\right] \\ &= \gamma_P^{\rho\sqrt{k}} f'\end{aligned}$$

and $\rho' = \rho\sqrt{k}$ as required.

COROLLARY. *The coset decomposition of $\mathscr{M}$ relative to $\mathscr{E}$ can be written*

$$\mathscr{M} = \mathscr{E} \cup \left[\bigcup_{P \in \Gamma} \gamma_P^{\rho(P)} f_P \mathscr{E}\right]$$

and hence the index of $\mathscr{E}$ in $\mathscr{M}$ is $|\Gamma| + 1$.

Proof. Since $\mathscr{E} = \mathscr{M}_\infty$ the cosets of $\mathscr{E}$ are determined by the various points P which they move to ∞.

COROLLARY. *The index of $\mathscr{I}$ in $\mathscr{M}$ is* $(|K| + 1)(|\Gamma| + 1)$.

Proof. Use $|R_P| = |K| + 1$ in the corollary of Proposition 2 or use the fact that the index of $\mathscr{I}$ in $\mathscr{M}$ is the product of the index of $\mathscr{I}$ in $\mathscr{E}$ and the index of $\mathscr{E}$ in $\mathscr{M}$.

REMARK. In Propositions 2 and 3 we can take f_P^{ρ} and f_P to be the identity if and only if γ_P^{ρ} belongs to $\mathscr{M}$. It is sometimes convenient to partition Γ and write $\Gamma = \Gamma_0 \cup \Gamma_1$, where $P \in \Gamma_0$ if and only if the inversions γ_P^{ρ}, $\rho \in R_P$, belong to $\mathscr{M}$.

We now summarize some of the most useful information from this section.

THEOREM 3. *The following conditions are equivalent*:

(i) $\mathscr{E} = \mathscr{I}$;

(ii) $K = \varnothing$;

(iii) $\forall P \in \Gamma, |R_P| = 1$;

(iv) $\mathscr{M} = \mathscr{I} \cup \left[\bigcup_{P \in \Gamma_0} \gamma_P \mathscr{I}\right] \cup \left[\bigcup_{P \in \Gamma_1} \gamma_P f_P \mathscr{I}\right]$.

If $\mathscr{M}$ is finite, the conditions all hold and the index of $\mathscr{I}$ in $\mathscr{M}$ is equal to $|\Gamma| + 1$.

Proof. $\mathscr{E} = \mathscr{I} \Leftrightarrow s(\mathscr{E}) = K^* = \{1\}$
$\Leftrightarrow K = \varnothing$
$\Leftrightarrow |R_P| = |\rho\sqrt{K^*}| = 1$.

Also $\mathscr{E} = \mathscr{I} \Leftrightarrow$ the coset decomposition of $\mathscr{M}$ shows that it contains no proper similarities. These remarks prove the equivalence of conditions (i)–(iv).

If $\mathscr{M}$ is finite, it can contain no proper similarities and hence $\mathscr{E} = \mathscr{I}$. Then $K = \varnothing$ implies the index of $\mathscr{I}$ in $\mathscr{E}$ is $(|K| + 1)(|\Gamma| + 1) = |\Gamma| + 1$.

REMARK. In the introduction we mentioned a D-shaped figure $\mathscr{F}_1$ and an hourglass figure $\mathscr{F}_2$.

Both of them satisfy conditions (i)–(iv). In the case of $\mathscr{F}_1$, $\mathscr{M}(\mathscr{F}_1)$ is infinite since $\mathscr{M}(\mathscr{F}_1) \cong \mathscr{M}(\mathscr{F}_1^1)$ and the fact that $\mathscr{I}(\mathscr{F}_1)$ is finite implies that $\Gamma(\mathscr{F}_1)$ is infinite. In the case of $\mathscr{F}_2$ we shall see that $|\mathscr{M}(\mathscr{F}_2)| = 8$ and the fact that $|\mathscr{I}(\mathscr{F}_2)| = 4$ checks with our observation that $|\Gamma(\mathscr{F}_2)| = 1$.

4. Equivalent Figures and Conjugate Subgroups. We begin by introducing three equivalence relations on the set of figures $\mathscr{F} \subset \Pi^n$. Two figures $\mathscr{F}$ and $\mathscr{F}'$

(i) belong to the same *genus* if their Möbius groups are geometrically isomorphic, that is, if there is a Möbius transformation $h \in \mathscr{M}(\Pi^n)$ such that $\mathscr{M}(\mathscr{F})^h \equiv h^{-1}\mathscr{M}(\mathscr{F})h = \mathscr{M}(\mathscr{F}')$;

(ii) belong to the same *species* if there is a Möbius transformation $h \in \mathscr{M}(\Pi^n)$ such that $\mathscr{F}^h = \mathscr{F}'$.

(iii) belong to the same *subspecies* if they are similar, that is, if there is a similarity $g \in \mathscr{E}(\Pi^n)$ such that $\mathscr{F}^g = \mathscr{F}'$.

Two figures in the same subspecies belong to the same species because $\mathscr{E}(\Pi^n) \subset \mathscr{M}(\Pi^n)$ and two figures in the same species belong to the same genus because $\mathscr{M}(\mathscr{F}^h) = h^{-1}\mathscr{M}(\mathscr{F})h$ as we shall see in the proof of Theorem 4.

Starting from a basic figure $\mathscr{F}$ we can obtain the other figures in its subspecies as

$$\{\mathscr{F}^g : g \in \mathscr{E}(\Pi^n)\}$$

and the other figures in its species as

$$\{\mathscr{F}^h : h \in \mathscr{M}(\Pi^n)\},$$

but these calculations are redundant and we shall improve them below. It is harder to determine all the other figures in the same genus as $\mathscr{F}$ and we shall not pursue this question here, though it will occasionally be useful to remark that two given figures, such as the rectangle and the ellipse mentioned in the introduction, happen to belong to the same genus.

PROPOSITION 4. *Let $\mathscr{F} \subset \Pi^n$ be a figure with symmetry groups $\mathscr{M} = \mathscr{M}(\mathscr{F})$ and $\mathscr{E} = \mathscr{E}(\mathscr{F})$. Then*

(i) *the members of the same subspecies as $\mathscr{F}$ occur once in the list $\mathscr{F}^g$, where g ranges through representatives of the right cosets of $\mathscr{E}$ in $\mathscr{E}(\Pi^n)$;*

(ii) *the members of the same species as $\mathscr{F}$ occur once in the list $\mathscr{F}^h$, where h ranges through representatives of the right cosets of $\mathscr{M}$ in $\mathscr{M}(\Pi^n)$;*

(iii) *the transformations in $\mathscr{M}(\Pi^n)$ which take $\mathscr{F}$ to some figure in the same subspecies as $\mathscr{F}^h$ are precisely those in $\mathscr{M} \cdot h \cdot \mathscr{E}(\Pi^n)$.*

Proof. (i) Any figure in the same subspecies as $\mathscr{F}$ is of the form $\mathscr{F}^g$ for some $g \in \mathscr{E}(\Pi^n)$. If $\mathscr{F}^{g_1} = \mathscr{F}^{g_2}$, then $g_2 g_1^{-1}$ belongs to $\mathscr{M}$, hence $\mathscr{E}$, and it follows that $g_2 \in \mathscr{E}g_1$.

(ii) The proof is similar to the proof of (i).

(iii) If $\mathscr{F}^h$ and $\mathscr{F}^{h'}$ belong to the same subspecies, there is a similarity g such that $\mathscr{F}^{hg} = \mathscr{F}^{h'}$. Thus $h'(hg)^{-1} \in \mathscr{M}$ and $h' \in \mathscr{M} \cdot h \cdot g \subset \mathscr{M} \cdot h \cdot \mathscr{E}(\Pi^n)$. On the other hand, if $h' \in \mathscr{M} \cdot h \cdot \mathscr{E}(\Pi^n)$, there exist transformations $h_1 \in \mathscr{M}$ and $g \in \mathscr{E}(\Pi^n)$ such that $h' = h_1 hg$. But then $\mathscr{F}^{h'} = \mathscr{F}^{h_1 hg} = \mathscr{F}^{hg}$ and this proves that $\mathscr{F}^h$ and $\mathscr{F}^{h'}$ belong to the same subspecies.

If $\mathscr{F} \subset \Pi^n$ and $h \in \mathscr{M}(\Pi^n)$, then either $h \in \mathscr{E}(\Pi^n)$ and $\mathscr{F}^h$ belongs to the same subspecies as $\mathscr{F}$ or $h = \gamma f$ and $\mathscr{F}^h$ belongs to the same subspecies as $\mathscr{F}^\gamma$. Thus suitable inversions give representatives of every subspecies, except possibly that of $\mathscr{F}$, in the species determined by $\mathscr{F}$. We do not claim that every member of each subspecies can be written as $\mathscr{F}^\gamma$, only that at least one member and hence a one-parameter family of them, not necessarily distinct, can be represented this way.

Let γ_P denote inversion in an $(n-1)$-sphere with centre P and arbitrary radius. It is an easy consequence of Theorem 2 that $\mathscr{F}^{\gamma_P}$ is similar to $\mathscr{F}$ if and only if $P \in \Gamma(\mathscr{F})$, the portion of the orbit of ∞ under $\mathscr{M}(\mathscr{F})$ which lies in $\mathbb{R}^n$. We now obtain a generalization of this result which bears on part (iii) of the preceding proposition and on the remarks above.

PROPOSITION 5. *Let $\mathcal{F} \subset \Pi^n$ be a figure with Möbius symmetry group $\mathcal{M} = \mathcal{M}(\mathcal{F})$. Let $\gamma_1 = \gamma_{Q_1}$ and $\gamma_2 = \gamma_{Q_2}$ be two inversions. Then $\mathcal{F}^{\gamma_1}$ and $\mathcal{F}^{\gamma_2}$ belong to the same subspecies if and only if Q_1 and Q_2 belong to the same $\mathcal{M}$-orbit.*

Proof. Suppose $\mathcal{F}^{\gamma_1}$ and $\mathcal{F}^{\gamma_2}$ belong to the same subspecies. Then there is a similarity g such that

$$\mathcal{F}^{\gamma_1 g} = \mathcal{F}^{\gamma_2} \quad \text{or} \quad \mathcal{F}^{\gamma_1 g \gamma_2} = \mathcal{F}.$$

But this means that $\gamma_1 g \gamma_2 \in \mathcal{M}$. Since

$$Q_1^{\gamma_1 g \gamma_2} = \infty^{g\gamma_2} = \infty^{\gamma_2} = Q_2,$$

Q_1 and Q_2 belong to the same $\mathcal{M}$-orbit.

Conversely, suppose Q_1 and Q_2 belong to the same $\mathcal{M}$-orbit, i.e., suppose there exists $h \in \mathcal{M}$ such that $Q_1^h = Q_2$. Since $h\gamma_2$ and γ_1 both map Q_1 to ∞, there is a similarity g such that $h\gamma_2 = \gamma_1 g$. Thus $\mathcal{F}^{\gamma_2} = \mathcal{F}^{h\gamma_2} = \mathcal{F}^{\gamma_1 g}$ and we have shown that $\mathcal{F}^{\gamma_1}$ and $\mathcal{F}^{\gamma_2}$ belong to the same subspecies.

Proposition 5 shows that Kavanau's notion [6] of a complete set of equivalent inversion poles for $\mathcal{F}$ agrees with an $\mathcal{M}$-orbit not including ∞.

COROLLARY. *There is a one-to-one correspondence between the orbits determined by $\mathcal{M}(\mathcal{F})$ and the subspecies in the species determined by $\mathcal{F}$.*

Proof. We let $\Gamma(\mathcal{F}) \cup \{\infty\}$ correspond to the subspecies determined by $\mathcal{F}$ and the $\mathcal{M}$-orbit of a point $P \notin \Gamma(\mathcal{F})$ correspond to the subspecies determined by $\mathcal{F}^{\gamma_P}$.

Now that we have discussed the images of $\mathcal{F}$ under various Möbius transformations, it is time to discuss the most important feature of these image figures, namely their symmetry groups. Recalling that G_P denotes the subgroup of transformations in G which fix the point P, we gather together the basic facts in

THEOREM 4. *Let $\mathcal{F} \subset \Pi^n$ be a figure with symmetry groups $\mathcal{M} = \mathcal{M}(\mathcal{F})$ and $\mathcal{E} = \mathcal{E}(\mathcal{F})$. Let $h \in \mathcal{M}(\Pi^n)$ be a Möbius transformation mapping the figure $\mathcal{F}$ to $\mathcal{F}^h$ and the point P to $P^h = \infty$. [Here we allow $P = \infty$ to include the case that $h \in \mathcal{E}(\Pi^n)$.] Then*

(i) $\mathcal{E} = \mathcal{M}_\infty$;

(ii) $\mathcal{M}(\mathcal{F}^h) = h^{-1}\mathcal{M}h$;

(iii) $\mathcal{E}(\mathcal{F}^h) = h^{-1}\mathcal{M}_P h$; and

(iv) $\Gamma(\mathcal{F}^h) = P^{\mathcal{M}h} \setminus \{\infty\}$.

Proof. (i) By definition

$$\mathcal{E} = \mathcal{M} \cap \mathcal{E}(\Pi^n) = \mathcal{M} \cap \mathcal{M}(\Pi^n)_\infty = \mathcal{M}_\infty.$$

(ii) Since any transformation in $h^{-1}\mathcal{M}h$ maps $\mathcal{F}^h$ to itself, $h^{-1}\mathcal{M}h \subset \mathcal{M}(\mathcal{F}^h)$. Similarly any transformation in $h\mathcal{M}(\mathcal{F}^h)h^{-1}$ maps $\mathcal{F}$ to itself, hence $h\mathcal{M}(\mathcal{F}^h)h^{-1} \subset \mathcal{M}$ or $\mathcal{M}(\mathcal{F}^h) \subset h^{-1}\mathcal{M}h$. This proves (ii).

(iii) From (i) and (ii), $\mathcal{E}(\mathcal{F}^h) = (h^{-1}\mathcal{M}h)_\infty$. Applying the ideas in (ii) we see that

$$(h^{-1}\mathcal{M}h)_\infty = h^{-1}(\mathcal{M}_\infty h^{-1})h = h^{-1}\mathcal{M}_P h.$$

(iv) According to Theorem 2

$$\Gamma(\mathcal{F}^h) = \infty^{\mathcal{M}(\mathcal{F}^h)} \setminus \{\infty\} = \infty^{h^{-1}\mathcal{M}h} \setminus \{\infty\} = P^{\mathcal{M}h} \setminus \{\infty\}.$$

This completes the proof of Theorem 4.

In connection with part (iv) of Theorem 4 we note that the inversion poles equivalent to P for the figure $\mathcal{F}$ are in one-to-one correspondence with the poles of subspecific inversion of $\mathcal{F}^h$; cf. Kavanau [6].

COROLLARY. *Two figures in the same subspecies have geometrically isomorphic Euclidean symmetry groups.*

Proof. If $\mathscr{F}_1$ and $\mathscr{F}_2$ are in the same subspecies, there is a similarity $g \in \mathscr{E}(\Pi^n)$ such that $\mathscr{F}_1^g = \mathscr{F}_2$. Then $\mathscr{E}(\mathscr{F}_2) = g^{-1}[\mathscr{M}(\mathscr{F}_1)_\infty]g = g^{-1}\mathscr{E}(\mathscr{F}_1)g$.

COROLLARY. *There is a function (perhaps many-to-one) from the orbits determined by $\mathscr{M}(\mathscr{F})$ to the classes of geometrically isomorphic groups that arise as Euclidean symmetry in the species determined by $\mathscr{F}$.*

Proof. We let $\Gamma(\mathscr{F}) \cup \{\infty\}$ correspond to the class of $\mathscr{E}(\mathscr{F})$ and the $\mathscr{M}(\mathscr{F})$ orbit of a point $P \notin \Gamma(\mathscr{F})$ correspond to the class of $\mathscr{E}(\mathscr{F}^{\gamma_P}) = \gamma_P[\mathscr{M}(\mathscr{F})_P]\gamma_P$.

In clarification of the last corollary let me remind the reader that two subgroups G_1 and G_2 of the Möbius group $\mathscr{M}(\Pi^n)$ are *geometrically isomorphic* if they are conjugate in $\mathscr{M}(\Pi^n)$. If G_1 is the group of order 2 generated by a reflection and G_2 is the group of order 2 generated by a halfturn, then G_1 and G_2 are isomorphic but not geometrically isomorphic. On the other hand if $\mathscr{M}$ is a subgroup of $\mathscr{M}(\Pi^n)$ and P and Q are two points in the same $\mathscr{M}$-orbit, then the stabilizer subgroups $\mathscr{M}_P$ and $\mathscr{M}_Q$ are geometrically isomorphic and any element $h \in \mathscr{M}$ with $P^h = Q$ will serve to show that $\mathscr{M}_Q = h^{-1}\mathscr{M}_P h$.

When we are presented with a figure $\mathscr{F}$, the symmetry which leaps to the eye is the symmetry in $\mathscr{E}(\mathscr{F})$. The transformations in $\mathscr{M}(\mathscr{F})\backslash\mathscr{E}(\mathscr{F})$ constitute what we may call the hidden symmetry of $\mathscr{F}$. Experience tends to show that this hidden symmetry does not particularly enhance our aesthetic appreciation of the figure. But, as Theorem 4 makes clear, it does satisfy our curiosity concerning the surprisingly rich variety of visible symmetry which is present in different representatives of the same species. Once we understand that a single Möbius group $\mathscr{M}(\mathscr{F})$ is responsible for this variety, we gain the perspective of a biologist who brings various layers of tissue into sharp focus with a powerful microscope, but realizes that these layers must be considered together to properly understand the tissue.

5. Stereographic Projection and Spherical Geometry. In order to progress further with our theory of figures and Möbius subgroups, we require the assistance of a second model of inversive n-space. Let Σ^n be the unit n-sphere in $\mathbb{R}^{n+1}$,

$$\Sigma^n = \{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| = 1\}.$$

The points, m-spheres and m-flats of Π^n correspond by stereographic projection to the points and m-spheres of Σ^n, $1 \leqslant m \leqslant n-1$. Under this correspondence the action of the Möbius group $\mathscr{M}(\Pi^n)$ can be transferred to the action of an isomorphic group $\mathscr{M}(\Sigma^n)$. It turns out that all the Möbius invariants such as angle and cross ratio, which we have dealt with in Π^n, can be measured directly in Σ^n and are invariant under the Möbius group $\mathscr{M}(\Sigma^n)$. Thus nothing is lost and a certain amount of homogeneity is gained by considering Σ^n rather than Π^n as our model for inversive n-space.

We have $\Sigma^n \subset \mathbb{R}^{n+1}$. In order to describe stereographic projection, we extend $\mathbb{R}^{n+1}$ to inversive $(n+1)$-space $\Pi^{n+1} = \mathbb{R}^{n+1} \cup \{\infty\}$ and imbed our Π-model of inversive n-space, Π^n, as the n-flat $x_{n+1} = 0$. Let $N = (0,0,\ldots,0,1)$ be the "north pole" of Σ^n and let $\Gamma_N^{\sqrt{2}}$ be the n-sphere with centre N and radius $\sqrt{2}$. The inversion $\Gamma_N^{\sqrt{2}}$ is a transformation of Π^{n+1} which maps Π^n to Σ^n. It carries ∞ to N and points $P \neq \infty$ in Π^n to points $P' \neq N$ on Σ^n, P' being the point where the ray NP cuts Σ^n for a second time.

The restriction of the transformation $\Gamma_N^{\sqrt{2}}$ to Π^n is the transformation traditionally referred to as stereographic projection, but by acknowledging that it is induced by an inversion of the larger space we can very quickly deduce its important properties. For example, it is an immediate consequence of this observation that stereographic projection maps the m-flats and m-spheres of Π^n to the m-spheres of Σ^n, $1 \leqslant m \leqslant n-1$. Moreover it does this in such a way as to preserve cross ratios and angles. Any bijection from Π^n to Σ^n would allow us to transfer the action of a group from Π^n to Σ^n by means of the commutative diagram

$$\begin{array}{ccc} \Sigma^n & \to & \Sigma^n \\ \uparrow & & \uparrow \\ \Pi^n & \to & \Pi^n \end{array}$$

but we shall see that stereographic projection transfers $\mathscr{M}(\Pi^n)$ to $\mathscr{M}(\Sigma^n)$ in a particularly nice way.

Because $\mathscr{M}(\Pi^n)$ is generated by reflections and inversions, $\mathscr{M}(\Sigma^n)$ is generated by the stereographic conjugates of these transformations. If γ is an $(n-1)$-flat or $(n-1)$-sphere in Π^n, the unified definition tells us that reflection or inversion in γ fixes the points of γ and interchanges any other point P with the second meeting of any two lines or circles through P perpendicular to γ. We have seen that stereographic projection maps γ to an $(n-1)$-sphere γ' and we refer to the stereographically conjugate transformation as inversion in γ'. It follows from the properties of stereographic projection already listed that inversion in γ' fixes the points of γ' and interchanges any other point P' with the second meeting of any two circles through P' perpendicular to γ'.

To be completely honest, this intrinsic definition of inversion in γ' only applies in dimensions $n \geqslant 2$ where angles are meaningful. To cover the anomalous case $n = 1$ and to give a useful extrinsic definition of inversion in γ', we begin our analysis again. If γ is an $(n-1)$-flat or $(n-1)$-sphere in Π^n, there is a unique n-flat or n-sphere Γ in Π^{n+1} that is perpendicular to Π^n along γ. The inversion $\Gamma_N{}^{\sqrt{2}}$ maps Γ to an n-flat or n-sphere Γ' that is perpendicular to Σ^n along γ'. Since the reflection or inversion γ is induced by the reflection or inversion Γ and since the inversion $\Gamma_N{}^{\sqrt{2}}$ conjugates the reflection or inversion Γ to the reflection or inversion Γ', it follows that the inversion γ' is induced by the reflection or inversion Γ'. Thus if γ' is a great $(n-1)$-sphere, the inversion γ' is induced by reflection in the n-flat Γ' that cuts γ' from Σ^n, and if γ' is a small $(n-1)$-sphere, the inversion γ' is induced by inversion in the n-sphere Γ' that is perpendicular to Σ^n along γ'. In the second case the centre of Γ' is the vertex V of a cone of tangents to Σ^n which touch it along γ', and the inversion in γ' interchanges pairs of points where secants from V cut Σ^n. Thus there are two cases depending on whether γ' is great or small, but in either case we obtain a beautifully simple extrinsic definition of inversion in γ'.

We originally introduced inversive geometry with the help of the Π-model Π^n and this served to show its close connection with Euclidean geometry: the group $\mathscr{M}(\Pi^n)$ has subgroups $\mathscr{E}(\Pi^n)$ and $\mathscr{S}(\Pi^n)$ which are fundamental to Euclidean geometry and play a central role in the present theory. Now we have developed the Σ-model Σ^n for inversive geometry and this suggests important connections with spherical geometry: the group $\mathscr{M}(\Sigma^n)$, which is just $\mathscr{M}(\Pi^n)$ viewed differently, has a subgroup $\mathscr{S}(\Sigma^n)$ equal to the isometries of the n-sphere. This observation is the key to further development in our theory and so we turn our attention to spherical geometry.

Let O be the centre of Σ^n. If P and Q are two points on Σ^n, we can define the distance between P and Q to be equal to the length of the chord PQ or alternatively, and perhaps more naturally, to be equal to the angle $\theta = POQ = 2 \arcsin (PQ/2)$. This angle is equal to the length of the shortest curve on Σ^n joining P and Q, namely the minor arc of the great circle PQ. With either definition of distance the isometries of Σ^n, $\mathscr{S}(\Sigma^n)$, form a group isomorphic to the subgroup of $\mathscr{S}(\mathbb{R}^{n+1})$ which stabilizes O. The connection between the two groups is that $\mathscr{S}(\Sigma^n)$ is the restriction of $\mathscr{S}(\mathbb{R}^{n+1})_O$ to Σ^n. Thus $\mathscr{S}(\Sigma^n)$ is generated by reflections in n-flats through O, that is, by inversions in great $(n-1)$-spheres, and any element can be written as the product of at most $n+1$ such inversions. The most general isometry of the n-sphere is therefore equal to the commuting product of $s \geqslant 0$ rotations and $t = 0$ or 1 reflections, $2s + t \leqslant n + 1$:

$$(z_1, z_2, \ldots, z_s, x_{2s+1}, \ldots, x_{n+1})^f = \left(e^{i\theta_1} z_1, e^{i\theta_2} z_2, \ldots, e^{i\theta_s} z_s, (-1)^t x_{2s+1}, \ldots, x_{n+1}\right).$$

In section 2 we mentioned the result [**8**] that Möbius transformations belong to one of three different types: (i) those conjugate to a fixed point free isometry of $\mathbb{R}^n$; (ii) those conjugate to a proper similarity of $\mathbb{R}^n$; and (iii) those conjugate to an isometry of Σ^n. Now we quote a very important related result [**9**].

THEOREM 5. *Let $\mathcal{M} \subset \mathcal{M}(\Sigma^n)$ be a finite group of Möbius transformations. Then $\mathcal{M}$ is conjugate to a finite group of isometries of Σ^n.*

We remark that if $\mathcal{M}$ is finite, then each of its elements must be periodic and hence of type (iii). Nevertheless it is surprising that all the elements of $\mathcal{M}$ can be conjugated into isometries simultaneously. The proof of Theorem 5 involves consideration of the connection between n-dimensional inversive geometry and $(n+1)$-dimensional hyperbolic geometry. We shall not digress to describe this connection, but full details are available in [2] and [7].

In Section 4 we considered the symmetry of a figure $\mathcal{F} \subset \Pi^n$ and spoke informally about its visible symmetry $\mathcal{E}(\mathcal{F})$ and its hidden symmetry $\mathcal{M}(\mathcal{F}) \setminus \mathcal{E}(\mathcal{F})$. If $\mathcal{M}(\mathcal{F})$ is finite, Theorem 5 shows us how to bring the hidden symmetry into clear view.

PROPOSITION 6. *Let $\mathcal{F} \subset \Pi^n$ be a figure with finite Möbius group. Let $\sigma : \Pi^n \to \Sigma^n$ be stereographic projection. Then there is a figure $\mathcal{F}'$ in the same subspecies as $\mathcal{F}$ such that*

$$\mathcal{M}(\mathcal{F}'^{\sigma}) = \sigma^{-1}\mathcal{M}(\mathcal{F}')^{\sigma}$$

is a finite group of isometries of Σ^n.

Proof. Since $\mathcal{M}(\mathcal{F}^{\sigma}) = \sigma^{-1}\mathcal{M}(\mathcal{F})\sigma$ is finite, Theorem 5 guarantees that there is a Möbius transformation $h \in \mathcal{M}(\Sigma^n)$ such that $\mathcal{M}(\mathcal{F}^{\sigma h}) = h^{-1}\mathcal{M}(\mathcal{F}^{\sigma})h$ is a subgroup of $\mathcal{I}(\Sigma^n)$. There is a rotation $f \in \mathcal{I}(\Sigma^n)$ such that $N^{hf} = N$. Then $\mathcal{M}(\mathcal{F}^{\sigma hf}) = f^{-1}\mathcal{M}(\mathcal{F}^{\sigma h})f$ is also a subgroup of $\mathcal{I}(\Sigma^n)$. If we let $\mathcal{F}' = \mathcal{F}^{\sigma hf\sigma^{-1}}$, then $\mathcal{M}(\mathcal{F}'^{\sigma}) = \mathcal{M}(\mathcal{F}^{\sigma hf})$ is a subgroup of $\mathcal{I}(\Sigma^n)$ and also

$$\infty^{\sigma hf\sigma^{-1}} = N^{hf\sigma^{-1}} = N^{\sigma^{-1}} = \infty,$$

hence $\sigma hf\sigma^{-1} \in \mathcal{E}(\Pi^n)$ and $\mathcal{F}'$ lies in the same subspecies as $\mathcal{F}$.

THEOREM 6. *Let $\sigma : \Pi^n \to \Sigma^n$ be stereographic projection. Let $O = (0,0,\ldots,0)$ be the centre of Σ^n and $S = (0,0,\ldots,-1)$ be the "south pole" of Σ^n so that $O^{\sigma} = S$. Let $\mathcal{F} \subset \Pi^n$ be a figure with the property that $\mathcal{M}(\mathcal{F}^{\sigma}) \subset \mathcal{I}(\Sigma^n)$. Then*

(i) *$\mathcal{E}(\mathcal{F}) = \mathcal{I}(\mathcal{F}) = \mathcal{I}(\mathcal{F})_O$;*

(ii) *if $h \in \mathcal{M}(\mathcal{F})$ and $h = \gamma f$, then $f \in \mathcal{I}(\Pi^n)_O$;*

(iii) *for any $P \in \Gamma(\mathcal{F})$, $R_P = \{\rho(P)\}$ where $\rho(P) = \sqrt{OP^2 + 1}$; and*

(iv) *in the coset decomposition*

$$\mathcal{M} = \mathcal{I} \cup \left[\bigcup_{P \in \Gamma_0} \gamma_P \mathcal{I}\right] \cup \left[\bigcup_{P \in \Gamma_1} \gamma_P f_P \mathcal{I}\right]$$

each isometry f_P, $P \in \Gamma_1$, fixes O.

Proof. (i) Since all the transformations in $\mathcal{M} = \mathcal{M}(\mathcal{F})$ are of type (iii), $\mathcal{E} = \mathcal{I}$. Since every isometry of Σ^n which fixes N must also fix S, every isometry of $\mathcal{F}$ must fix $S^{\sigma^{-1}} = O$ and $\mathcal{I} = \mathcal{I}_0$.

(ii) If $h \in \mathcal{M} \setminus \mathcal{E}$, then h^{σ} lies in $\mathcal{I}(\Sigma^n)$ and moves N. Let γ^{σ} be the great $(n-1)$-sphere that reflects the appropriate point to N. Then $h^{\sigma} = \gamma^{\sigma} f^{\sigma}$ where $f^{\sigma} \in \mathcal{I}(\Sigma^n)_{N,S}$. But this implies that the canonical factorization of Theorem 1 is $h = \gamma f$ where $f \in \mathcal{I}(\Pi^n)_O$ and γ^{σ} is a great $(n-1)$-sphere.

(iii) Since γ^{σ} is a great $(n-1)$-sphere, it must meet the equatorial $(n-1)$-sphere $\Pi^n \cap \Sigma^n$ in a great $(n-2)$-sphere. Thus γ must meet $\Pi^n \cap \Sigma^n$ in the same great $(n-2)$-sphere. If $\gamma = \gamma_P^{\rho(P)}$, this implies that $\rho(P) = \sqrt{OP^2 + 1}$.

(iv) This follows immediately from (ii) and the fact that every isometry in $\mathcal{I}$ fixes O.

We have phrased Theorem 6 in such a way that $\mathcal{F} \subset \Pi^n$ is given. For some applications it will be useful to regard $\mathcal{F}^{\sigma} \subset \Sigma^n$ as given and to obtain $\mathcal{F}$ by projecting from Σ^n. Then other figures in the same species as $\mathcal{F}$ can be obtained by applying a reflection to Σ^n to change the points which occupy the critical N, S positions and then projecting.

6. A Second Look at the Introductory Examples. We have now developed enough machinery to

deal efficiently with specific examples. In this section we illustrate the use of this machinery by carefully treating the examples given in the introduction. But before doing this we give a concise statement of our objectives and an informal summary of the relevant information in Theorems 1–6.

We deal with subgroups $\mathscr{M} \subset \mathscr{M}(\Pi^n)$ which are of the form $\mathscr{M} = \mathscr{M}(\mathscr{F})$. Our object is to understand their stabilizer subgroups by viewing them as Euclidean groups $\mathscr{E} = \mathscr{M}_\infty$. In order to handle the groups $\mathscr{M}_P$ with $P \neq \infty$, we must pass to conjugate groups $h^{-1}\mathscr{M}h = \mathscr{M}(\mathscr{F}^h)$ with $P^h = \infty$ so that

$$h^{-1}\mathscr{M}_P h = (h^{-1}\mathscr{M}h)_\infty = \mathscr{E}(\mathscr{F}^h).$$

Different points P in the same $\mathscr{M}$-orbit have geometrically conjugate stabilizer subgroups so it is enough to sample one point from each orbit. Moreover many orbits have the property that $\mathscr{M}_P$ reduces to the identity and these may therefore be omitted from our consideration.

Any proper Möbius transformation h acting on Π^n has a canonical factorization $h = \gamma f$. When we consider a group $\mathscr{M}$, the inversions γ which occur are those in the set

$$\{\gamma_P^\rho : P \in \Gamma, \rho \in R_P\},$$

where $\Gamma = \infty^{\mathscr{M}} \setminus \{\infty\}$ and $|R_P| = |s(\mathscr{E})| \geqslant 1$. The set Γ can be partitioned $\Gamma = \Gamma_0 \cup \Gamma_1$ with $P \in \Gamma_0$ if and only if the inversions γ_P^ρ, $\rho \in R_P$, lie in $\mathscr{M}$. To show how the stabilizer group $\mathscr{E} = \mathscr{M}_\infty$ fits into the group $\mathscr{M}$, there is the coset decomposition

$$\mathscr{M} = \mathscr{E} \cup \left[\bigcup_{P \in \Gamma_0} \gamma_P \mathscr{E}\right] \cup \left[\bigcup_{P \in \Gamma_1} \gamma_P f_P \mathscr{E}\right]$$

which also shows that the index of $\mathscr{E}$ in $\mathscr{M}$ is $|\Gamma| + 1$. In a conjugate group $h^{-1}\mathscr{M}h$ with $P^h = \infty$, $\Gamma(\mathscr{F}^h) = P^{\mathscr{M}^h} \setminus \{\infty\}$.

The Σ-model Σ^n provides an alternative setting for $\mathscr{F}$ or rather $\mathscr{F}^\sigma$ that may help us to determine the group $\mathscr{M}$ and its stabilizer subgroups. This approach is useful in general but it is especially helpful when $\mathscr{M}$ is conjugate to a subgroup of the isometries of Σ^n and in particular when $\mathscr{M}$ is finite. If we do have $\mathscr{M}$ conjugate to a subgroup of $\mathscr{I}(\Sigma^n)$, then the Π^n displays of the stabilizer subgroups of $\mathscr{M}$ contain no proper similarities. For each display $\mathscr{E} = \mathscr{I}$ is a group of isometries stabilizing a point $O \neq \infty$ and each point $P \in \Gamma$ determines a unique $(n-1)$-sphere $\gamma_P^{\rho(P)} = \gamma_P$ with $\rho(P) = \sqrt{OP^2 + 1}$. The group action can be completely specified by labelling the points $P \in \Gamma_1$ with an isometry $f_P \in \mathscr{I}(\Pi^n)_o$ such that $\gamma_P f_P \in \mathscr{M}$.

6.1. *The Figure* $\mathscr{F}_1$ (see Fig. 4). The figure $\mathscr{F}_1$ can be taken to be

$$\{(x, y) : x^2 + y^2 = 1 \text{ and } x \geqslant 0 \text{ or } x = 0 \text{ and } -1 \leqslant y \leqslant 1\}.$$

Then $\mathscr{F}_1^\sigma$ is the boundary of a lune consisting of two great semicircles

$$z = 0, \quad x \geqslant 0 \quad \text{and} \quad x = 0, \quad z \leqslant 0.$$

It is clear that the group $\mathscr{M}(\mathscr{F}_1^\sigma)$ is generated by inversions in the great circle $z = -x$ and the pencil of circles $y = c$, $-1 < c < 1$ including the great circle $y = 0$. $\mathscr{M}(\mathscr{F}_1^\sigma)$ is *not* conjugate to a subgroup of $\mathscr{I}(\Sigma^2)$ but still it is easy to see that its orbits are

$$\Omega^1 = \{(0,1,0), (0,-1,0)\},$$

$$\Omega_+^2 = \{(x, y, -x) : x > 0\}, \Omega_-^2 = \{(x, y, -x) : x < 0\},$$

and the one-parameter family of essentially equivalent orbits

$$\Omega_\theta = \left\{\left(\sqrt{1-y^2}\cos(-\pi/4 \pm \theta), \quad y, \quad \sqrt{1-y^2}\sin(-\pi/4 \pm \theta)\right) : |y| < 1\right\}, \quad 0 < \theta < \pi.$$

The figure $\mathscr{F}_1^\sigma$ is a union of orbits, namely $\mathscr{F}_1^\sigma = \Omega^1 \cup \Omega_{\pi/4}$. Inversion in the great circle $z = x$ belongs to the normalizer of $\mathscr{M}(\mathscr{F}_1^\sigma)$ and serves to amalgamate the orbits Ω_+^2 and Ω_-^2 and also the orbits Ω_θ and $\Omega_{\pi-\theta}$.

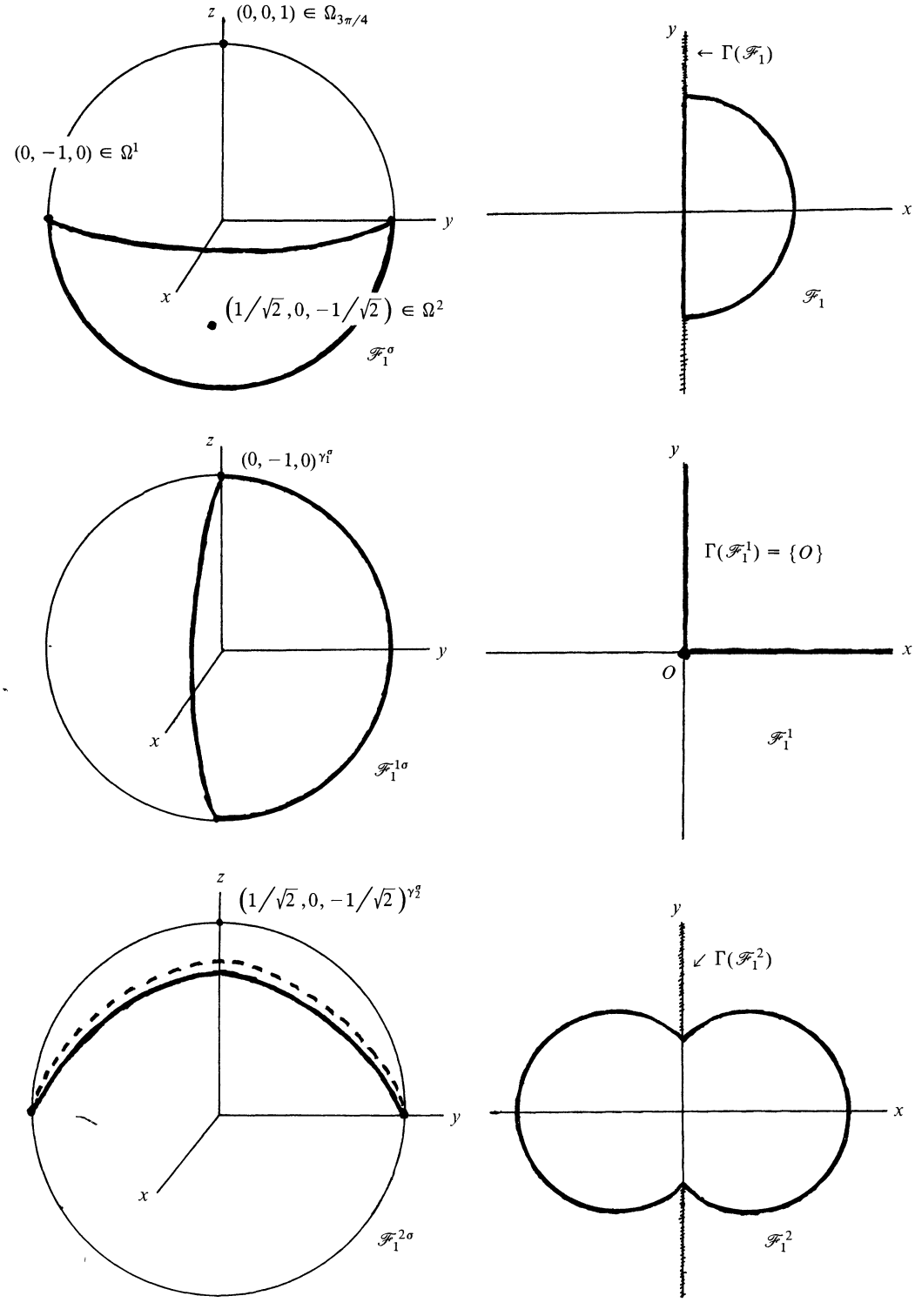
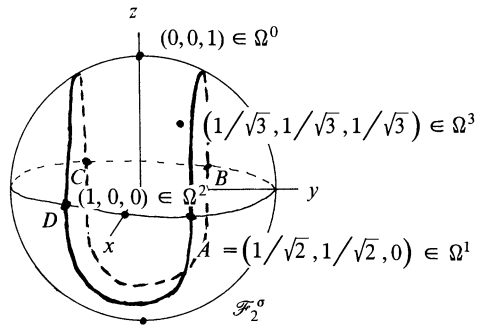
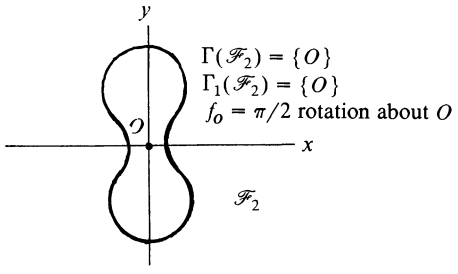
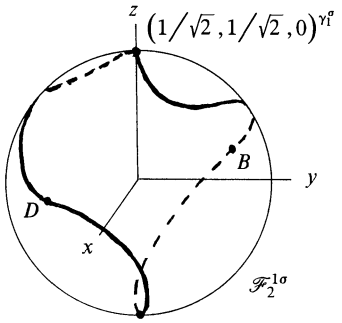
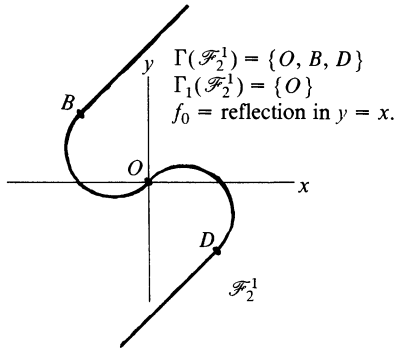


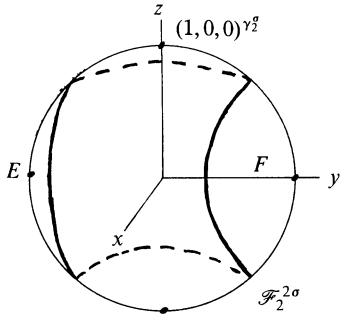
FIG. 4. Symmetry analysis of the first species.











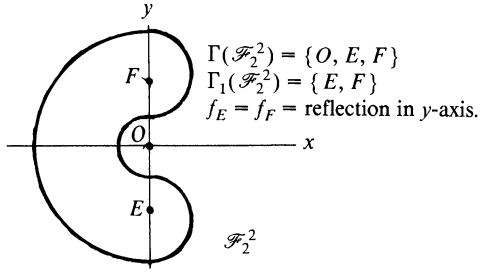
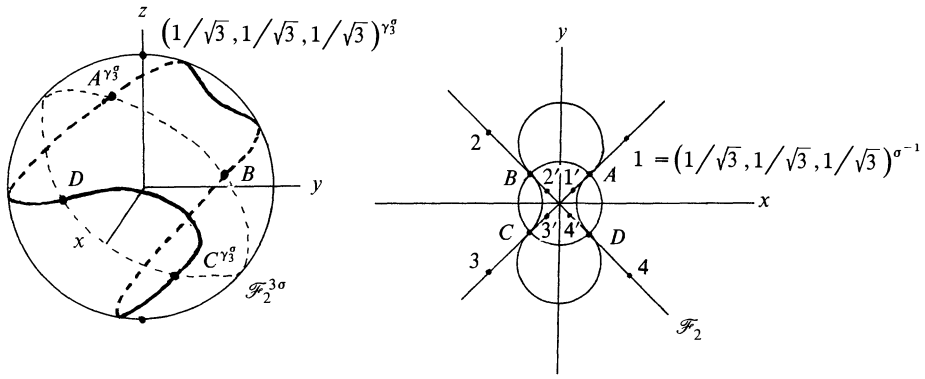
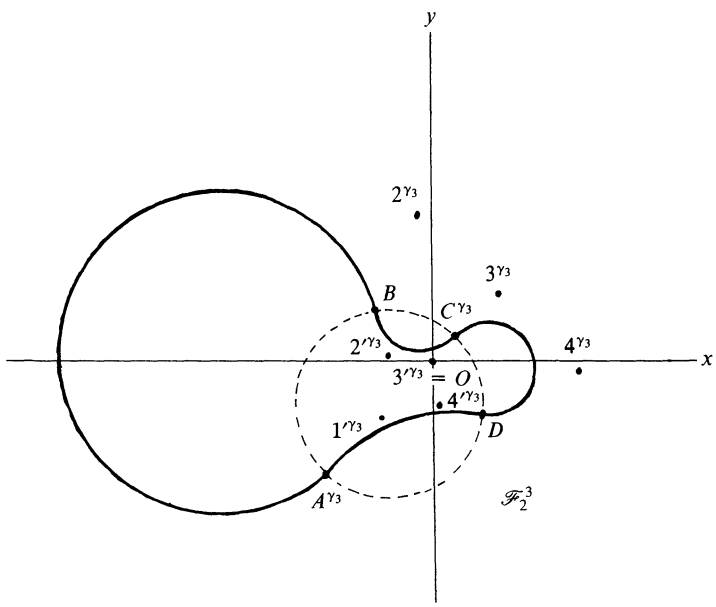


FIG. 5. Symmetry analysis of the second species (continued on p. 241).



$$\Gamma(\mathscr{F}_2^3) = \{2,3,4,1',2',3',4'\}^{\gamma_3}$$
$$\Gamma_1(\mathscr{F}_2^3) = \{3,1',2',3',4'\}^{\gamma_3}$$
$$f_3 = f_{1'} = \text{reflection in } y = x$$
$$f_{3'} = \text{reflection in } y = -x$$
$$f_{2'} = 2\pi/3 \text{ rotation about } O$$
$$f_{4'} = -2\pi/3 \text{ rotation about } O$$



Note: In calculating the isometries f_P for $P \in \Gamma_1(\mathscr{F}_2^3)$ it is very helpful to use the fact that $\{1,2,3,4,1',2',3',4'\}^{\sigma}$ are the vertices of a cube inscribed in Σ^2.

FIG. 5. Symmetry analysis of the second species (continued from p. 240).

The analysis above shows that there are just three essentially different stabilizer subgroups and these can be taken to be

$$\mathscr{M}(\mathscr{F}_1^{\sigma})_{(0,0,1)} \text{ with } (0,0,1) \in \Omega_{3\pi/4},$$
$$\mathscr{M}(\mathscr{F}_1^{\sigma})_{(0,-1,0)} \text{ with } (0,-1,0) \in \Omega^1, \text{ and}$$
$$\mathscr{M}(\mathscr{F}_1^{\sigma})_{(1/\sqrt{2},0,-1/\sqrt{2})} \text{ with } (1/\sqrt{2},0,-1/\sqrt{2}) \in \Omega_+^2.$$

The π-representations of these groups are $\mathscr{E}(\mathscr{F}_1)$, $\mathscr{E}(\mathscr{F}_1^1) = \mathscr{E}(\mathscr{F}_1^{\gamma_1})$ and $\mathscr{E}(\mathscr{F}_1^2) = \mathscr{E}(\mathscr{F}_1^{\gamma_2})$ respectively. The circles γ_i, $i = 1,2$, can be chosen conveniently so that γ_i^{σ} is the great circle which inverts the distinguished point in Ω^i to N.

In drawing $\mathscr{F}_1$, $\mathscr{F}_1^1$, and $\mathscr{F}_1^2$ we have also shown the sets $\Gamma(\mathscr{F})$. $\mathscr{E}(\mathscr{F}_1^1)$ has index 2 in $\mathscr{M}(\mathscr{F}_1^1)$ so $\Gamma(\mathscr{F}_1^1)$ is a single point $P = O$ and $K^*(\mathscr{F}_1^1) = \mathbb{R}^+$ so $R_P = \mathbb{R}^+$. For $\mathscr{F}_1$ and $\mathscr{F}_1^2$, $\mathscr{E}(\mathscr{F}) = \mathscr{I}(\mathscr{F})$ has infinite index so $\Gamma(\mathscr{F})$ is infinite but $K^*(\mathscr{F}) = \{1\}$ so for any point $P \in \Gamma |R_P| = 1$ though the formula $\rho(P) = \sqrt{\text{PA} \cdot \text{PB}}$ is not applicable and $\rho(P) = \sqrt{OP^2 - 1}$ instead.

6.2. *The Figure* $\mathscr{F}_2$ (see Fig. 5). The construction of the figure $\mathscr{F}_2$ can be based on the circle $x^2 + y^2 = 1$ with the vertices of the square taken to be

$$A = (1/\sqrt{2}, 1/\sqrt{2}), \quad B = (-1/\sqrt{2}, 1/\sqrt{2}), \quad C = (-1/\sqrt{2}, -1/\sqrt{2}), \quad D = (1/\sqrt{2}, -1/\sqrt{2}).$$

Then $\mathscr{F}_2^{\sigma}$ consists of four small semicircles on Σ^2 arranged so that their union is like the seam of a baseball:

$$y = 1/\sqrt{2}, z \geqslant 0; \quad x = -1/\sqrt{2}, \quad z \leqslant 0; \quad y = -1/\sqrt{2}, \quad z \geqslant 0; \quad x = 1/\sqrt{2}, \quad z \leqslant 0.$$

The group $\mathscr{M}(\mathscr{F}_2^{\sigma})$ *is* a subgroup of $\mathscr{I}(\Sigma^2)$ and consists of the following 8 transformations:

identity,
reflection in the plane $x = 0$,
reflection in the plane $y = 0$,
halfturn about the z-axis,
rotation through $\pi/4$ about the z-axis followed by reflection
in the plane $z = 0$ (rotatory reflection),
rotation through $-\pi/4$ about the z-axis followed by reflection
in the plane $z = 0$ (rotatory reflection),
halfturn about the axis $(1,1,0)$, and
halfturn about the axis $(-1,1,0)$.

Most orbits contain 8 points and hence stabilizing any one of their points leads to the identity. However we do have the following smaller orbits:

$$\Omega^0 = \{(0,0,1), \quad (0,0,-1)\},$$
$$\Omega^1 = \{(\pm 1/\sqrt{2}, \pm 1/\sqrt{2}, 0) : \text{all four combinations of signs to be taken}\},$$

and a one-parameter family of essentially equivalent orbits

$$\Omega_z^2 = \{(\sqrt{1-z^2}\cos(k\pi/2), \quad \sqrt{1-z^2}\sin(k\pi/2), \quad z\cos k\pi) : k = 0,1,2,3\}, \quad z \neq \pm 1.$$

The analysis above shows that there are four essentially different stabilizer subgroups and these can be taken to be:

$$\mathscr{M}(\mathscr{F}_2^{\sigma})_{(0,0,1)} \text{ with } (0,0,1) \in \Omega^0,$$
$$\mathscr{M}(\mathscr{F}_2^{\sigma})_{(1/\sqrt{2},1/\sqrt{2},0)} \text{ with } (1/\sqrt{2},1/\sqrt{2},0) \in \Omega^1,$$
$$\mathscr{M}(\mathscr{F}_2^{\sigma})_{(1,0,0)} \text{ with } (1,0,0) \in \Omega_0^2,$$
$$\mathscr{M}(\mathscr{F}_2^{\sigma})_{(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3})} \text{ with } (1/\sqrt{3},1/\sqrt{3},1/\sqrt{3}) \in \Omega^3.$$

The Π-representations of these groups are given by $\mathscr{E}(\mathscr{F}_2)$ and $\mathscr{E}(\mathscr{F}_2^i) = \mathscr{E}(\mathscr{F}_2^{\gamma_i})$, $i = 1, 2, 3$. The circles γ_i, $i = 1, 2, 3$, are determined because γ_i^σ must be the great circle that inverts the distinguished point in Ω^i to N.

In drawing the figures $\mathscr{F}_2$ and $\mathscr{F}_2^i$, $i = 1, 2, 3$, we have also shown the sets $\Gamma(\mathscr{F})$. The group $\mathscr{E}(\mathscr{F}) = \mathscr{J}(\mathscr{F})$ has index 2, 4, 4 and 8 in $\mathscr{M}(\mathscr{F})$ and accordingly $|\Gamma(\mathscr{F})| = 1$, 3, 3, and 7. Since $K^*(\mathscr{F}) = \{1\}$, there is just one circle associated with each point $P \in \Gamma(\mathscr{F})$ and the formula $\rho(P) = \sqrt{OP^2 + 1}$ does apply. The partition $\Gamma = \Gamma_0 \cup \Gamma_1$ is indicated and for each point $P \in \Gamma_1$ we have indicated an appropriate isometry f_P.

6.3. *The Genus of the Ellipse and Rectangle* (see Fig. 6). For definiteness we take $\mathscr{F}_3$ to be the rectangle with centre O. No proper Möbius transformation could leave all the edges of the rectangle as straight line segments, hence $\mathscr{M}(\mathscr{F}_3) = \mathscr{E}(\mathscr{F}_3)$, and no proper similarity could leave the diameter of $\mathscr{F}_3$ unchanged, hence $\mathscr{E}(\mathscr{F}_3) = \mathscr{J}(\mathscr{F}_3)$. The group consists of the identity, reflections in the coordinate axes, and a halfturn about the origin. There is no need to appeal to Σ^2 for help with this group and we proceed to list its essentially different orbits:

$\{\infty\}$ and $\{O\}$;
$\{A, B\}$ for any $A \neq O$ on either side of the reflection axes; and
$\{P, Q, R, S\}$ for any P on neither axis.

Inversion in any circle with centre O normalizes $\mathscr{M}(\mathscr{F}_3)$ and amalgamates the orbits $\{O\}$ and $\{\infty\}$. Thus the possible Euclidean symmetry groups are $\mathscr{E}(\mathscr{F}_3)$, $\mathscr{E}(\mathscr{F}_3^{\gamma_A})$, and $\mathscr{E}(\mathscr{F}_3^{\gamma_P})$ with orders 4, 2, and 1 respectively. The main point of interest here is to describe the groups $\mathscr{M}(\mathscr{F}_3^{\gamma_A}) = \gamma_A \mathscr{M} \gamma_A$ and $\mathscr{M}(\mathscr{F}_3^{\gamma_P}) = \gamma_P \mathscr{M} \gamma_P$ in enough detail to underline a number of points in our theory.

The inversion γ_A fixes the reflection x and transforms the reflection y to the inversion y^{γ_A}. Thus $\mathscr{M}(\mathscr{F}_3^{\gamma_A})$ consists of

identity,
reflection x,
inversion y^{γ_A}, and
halfturn $y^{\gamma_A}x$ with fixed points ∞^{γ_A} and O^{γ_A}.

In accordance with our theory,

$$\Gamma(\mathscr{F}_2^{\gamma_A}) = A^{\mathscr{M}^{\gamma_A}} \setminus \{\infty\} = \{A, B\}^{\gamma_A} \setminus \{\infty\} = B^{\gamma_A}$$

and B^{γ_A} must be the centre of y^{γ_A}. The group $\mathscr{J}(\mathscr{F}_3^{\gamma_A})$ consists of the identity and the reflection x and its complementary coset is $y^{\gamma_A} \cdot \mathscr{J}(\mathscr{F}_3^{\gamma_A})$.

The inversion γ_P transforms the reflections x and y to inversions x^{γ_P} and y^{γ_P}. Thus $\mathscr{M}(\mathscr{F}_3^{\gamma_P})$ consists of

identity,
inversions x^{γ_P} and y^{γ_P}, and
halfturn $x^{\gamma_P}y^{\gamma_P}$ with fixed points ∞^{γ_P} and O^{γ_P}.

In accordance with our theory,

$$\Gamma(\mathscr{F}_3^{\gamma_P}) = P^{\mathscr{M}^{\gamma_P}} \setminus \{\infty\} = \{P, Q, R, S\}^{\gamma_P} \setminus \{\infty\} = \{Q, R, S\}^{\gamma_P}.$$

Since Q, R, S lie on a circle through P, it follows that Q^{γ_P}, R^{γ_P}, S^{γ_P} lie on a line. The points S^{γ_P} and Q^{γ_P} must be the centres of x^{γ_P} and y^{γ_P} respectively. Since $\mathscr{J}(\mathscr{F}_3^{\gamma_P})$ reduces to the identity, the cosets are singletons. The inversions x^{γ_P} and y^{γ_P} are already in standard form and our theory tells us that the halfturn can be written as the product of inversion in a circle with centre R^{γ_P} and reflection in a line through ∞^{γ_P}. The circle must have radius $R^{\gamma_P}\infty^{\gamma_P} = R^{\gamma_P}O^{\gamma_P}$ and the line must be $\infty^{\gamma_P}O^{\gamma_P}$.

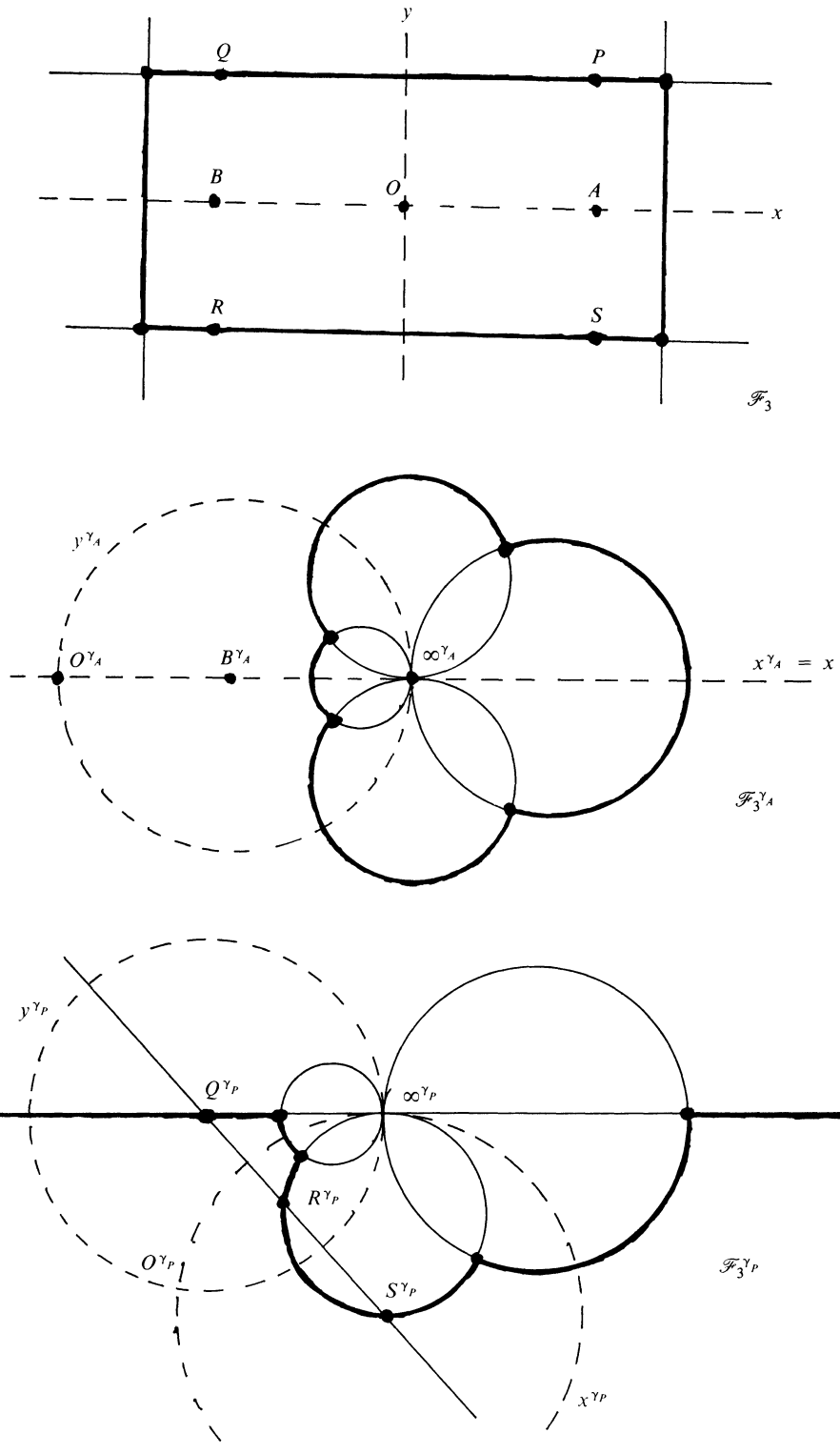


FIG. 6. Symmetry analysis for the genus of the rectangle.

7. Finite Subgroups of $\mathscr{M}(\Pi^2)$. We turn now to a complete analysis of the finite subgroups $\mathscr{M}$ of $\mathscr{M}(\Pi^2)$. Theorem 5 guarantees that these finite groups are all conjugate to finite subgroups of $\mathscr{I}(\Sigma^2)$ and complete enumerations of these finite isometry groups are given in [3], [4], and [5] as well as a number of other references. We do not attempt an independent enumeration of these groups, but we do venture to organize the list in a way that is suitable to our purposes and seems easy to remember. Then we summarize with a table which indicates for each group $\mathscr{M}$ the nontrivial stabilizer subgroups $\mathscr{M}_P$ corresponding to essentially different $\mathscr{M}$-orbits.

Before listing the groups we recall that the direct isometries of Σ^2 are rotations conjugate to

$$(z, x_3) \to (e^{i\theta}z, x_3)$$

and the opposite isometries are either reflections conjugate to

$$(x_1, x_2, x_3) \to (-x_1, x_2, x_3)$$

or rotatory reflections conjugate to

$$(z, x_3) \to (e^{i\theta}z, -x_3).$$

Groups consisting only of direct isometries are denoted G and those which include opposite isometries are denoted G^*. In the direct case the Π-representation of a stabilizer subgroup G_P must be a cyclic group c_n of order n generated by the rotation

$$z \to e^{2\pi i/n}z.$$

If opposite transformations are present, a stabilizer subgroup G_P^* may be either a cyclic group c_n of order n or a dihedral group d_n of order $2n$ generated by the rotation and reflection

$$z \to e^{2\pi i/n}z \quad \text{and} \quad z \to \bar{z}.$$

7.1. *Platonic Groups.* The first seven groups are associated with the Platonic solids: the tetrahedron, the octahedron (or its dual, the cube), and the icosahedron (or its dual, the dodecahedron). We have the rotation groups T of order 12, O of order 24, and I of order 60; and then the full symmetry groups T^* of order 24, O^* of order 48, and I^* of order 120. The groups O^* and I^* include central inversion

$$(x_1, x_2, x_3) \to (-x_1, -x_2, -x_3)$$

but the tetrahedron is not centrally symmetric and so T^* does not include this transformation. If we adjoin the central inversion to the group T, we obtain a group T_C^* of order 24 which can be regarded as the symmetry group of a cube whose faces are marked to eliminate rotations and rotatory reflections of order 4.

As a further clarification of the structure of these groups, we mention the standard isomorphisms relating them to permutation groups: $T \cong A_4$, $O \cong S_4$, $I \cong A_5$, $T^* \cong S_4$, $O^* \cong S_4 \times C_2$, $I^* \cong A_5 \times C_2$, and $T_C^* \cong A_4 \times C_2$.

7.2. *Cylindrical Frieze Groups.* Next come seven infinite families of groups associated with n-gonal pyramids, prisms or antiprisms $n \geqslant 3$. In these groups there is one rotation axis of order n which we take to be the vertical NS axis of Σ^2 and one invariant great circle which is then the usual equator of Σ^2. Since there is a perfect correspondence between these families of groups and the frieze groups of one-dimensional crystallography, we refer to them as the cylindrical frieze groups. The first five families can be generated without using rotatory reflections (the analogue of glide reflections) while the last two require these transformations. In listing these groups we adopt the crystallographers' notation in which subscripts v and h denote the presence of vertical and horizontal reflection planes.

The groups are:

(i) C_n of order n, the rotation group of the n-gonal pyramid, generated by a rotation of $2\pi/n$;

(ii) C_{nv}^* of order $2n$, the full group of the n-gonal pyramid, which is obtained by adjoining a vertical reflection to C_n;
(iii) C_{nh}^* of order $2n$, the full group of the oriented n-gonal prism, which is obtained by adjoining a horizontal reflection to C_n;
(iv) D_n of order $2n$, the rotation group of the n-gonal prism (or antiprism), which is obtained by adjoining a horizontal halfturn to C_n;
(v) D_n^* of order $4n$, the full group of the n-gonal prism, which includes all the symmetries listed above and can be generated by a horizontal reflection and two vertical reflections whose planes meet in a dihedral angle of π/n;
(vi) C_{na}^* of order $2n$, the full group of the oriented n-gonal antiprism, which is generated by a rotatory reflection with angle π/n; and
(vii) D_{na}^* of order $4n$, the full group of the n-gonal antiprism, which is obtained by adjoining a horizontal halfturn to C_{na}^*. The group D_{na}^* contains the same types of transformations as D_n^* but the axes of its horizontal halfturns lie between the mirrors of its vertical reflections rather than on them.

Since a 2-gonal antiprism is a perfectly good figure, three of the descriptions above apply without modification when $n = 2$ and hence we include the groups D_2, C_{2a}^*, and D_{2a}^* in this section. The group D_2 of order 4 is the famous four-group consisting of three mutually perpendicular halfturns and the identity. Also worthy of comment is the group D_{2a}^* of order 8. It is the symmetry group of the figure $\mathscr{F}_2$ presented in the introduction and discussed thoroughly in Section 7.

TABLE 1
Finite Subgroups of $\mathscr{M}(\Pi^2)$

	Group	Order	Stabilizer Subgroups			
	T	12	c_3	c_2		
	T^*	24	d_3	d_2	d_1	
Platonic	T_C^*	24	c_3	c_2	d_1	
Groups	O	24	c_4	c_3	c_2	
	O^*	48	d_4	d_3	d_2	d_1(2g)
	I	60	c_5	c_3	c_2	
	I^*	120	d_5	d_3	d_2	d_1
	C_n	n	c_n			
	C_{nv}^*	$2n$	d_n	$d_1^{\#}$		
Cylindrical	C_{nh}^*	$2n$	c_n	d_1		
Frieze	D_n	$2n$	c_n	$c_2^{\#}$		
Groups	D_n^*	$4n$	d_n	$d_2^{\#}$	d_{1h}	$d_{1v}^{\#}$
	C_{na}^*	$2n$	c_n			
	D_{na}^*	$4n$	d_n	c_2	d_1	
	R	2	d_1			
	H	2	c_2			
Small	C	2	—			
Groups	C_{2v}^*	4	d_2	d_1(2g)		
	C_{2h}^*	4	c_2	d_1		
	D_2^*	8	d_2(3g)	d_1(3g)		

Note: The symbols (2g) and (3g) mean 2 or 3 nonconjugate subgroups of the type indicated. The symbol # means 2 nonconjugate subgroups if n is even.

7.3. *Small Groups*. Finally we have six small groups which involve only halfturns and reflections. The symbols used to denote these groups show that they are closely related to the cylindrical frieze groups but they are degenerate and best treated separately. We have:

(i) $R = C_{1v}^* = C_{1h}^*$ of order 2, generated by a reflection;
(ii) $H = C_2 = D_1$ of order 2, generated by a halfturn;

(iii) $C = C_{1a}^*$ of order 2, generated by central inversion;
(iv) C_{2v}^* of order 4, generated by two perpendicular vertical reflections;
(v) C_{2h}^* of order 4, generated by a reflection and a perpendicular halfturn; and
(vi) D_2^* of order 8, which includes all the symmetries listed above, and can be generated by reflections in three mutually perpendicular planes.

7.4. *Table*. We complete this section with the table promised in the opening paragraph. In order to understand the Euclidean symmetry of the various transforms of a given plane figure, one has only to locate its Möbius group in Table 1 on page 246.

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CORRECTION TO "VERY BASIC LIE THEORY"

(This MONTHLY, 90 (1983) 600–623)

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I am grateful to G. D. Mostow for noticing a gap in the proof of Proposition 4. The difficulty is that it is conceivable that translating the sets $U(C, O)$ to create a homogeneous topology might create new neighborhoods of the identity. This in fact does not happen, but it must be checked.

The best procedure seems to be as follows. Our notation is as in § 1. Let $C \subseteq X$ be compact and $O \subseteq X$ be open, without requiring $C \subseteq O$. Define

$$V(C, O) = \{h \in \mathrm{Hm}(X) : h(C) \subseteq 0\}. \tag{E.1}$$

Define the compact-open topology on $\mathrm{Hm}(X)$ to be the topology generated by the $V(C, O)$ and $V(C, O)^{-1}$. Note that the requirement $h \in V(C, O)$ says h does not expand C too much, while $h \in V(C, O)^{-1}$ says h does not shrink O too much. I claim that the compact-open topology is homogeneous. Indeed, we compute for $g \in \mathrm{Hm}(X)$, and C, O as above

$$\begin{aligned} gV(C, O) &= \{gh : h(C) \subseteq O\} = \{h' : g^{-1}h'(C) \subseteq O\} \\ &= \{h' : h'(C) \subseteq g(O)\} = V(C, g(O)). \end{aligned} \tag{E.2}$$

Similarly,

$$V(C, O)g = V(g^{-1}(C), O). \tag{E.3}$$

Observe that $1_X \in V(C, O)$ if and only if $C \subseteq O$. When this happens, the set $U(C, O) = V(C, O) \cap V(C, O)^{-1}$ is a symmetric neighborhood of 1_X, and the intersections (1.10) of these sets form a base for the neighborhoods of 1_X. With these observations made, Proposition 4 should start "Since the compact-open topology is homogeneous,..." and then can proceed as written.





Two immortals, arranged here in reverse order from the usual one. (See p. 256.)

$$|m(\overline{AB})| = |m(\overbrace{f_i + g_{i-1}}^{\equiv g_i})| \text{ for } \overline{AB}| =$$

$$|m(f_i) + m(g_{i-1}) \text{ for } \overline{AB}| \geqslant$$

$$|m(f_i)| - |m(g_{i-1}) \text{ for } \overline{AB}| \geqslant$$

$$|m(f_i)| - \max_x |m(g_{i-1})| = \text{ (see fifth paragraph of this presentation)}$$

$$(5/2)^{i-1} - 2/3\left[(5/2)^{i-1} - 1\right] \approx 1/3(5/2)^{i-1} \to \infty$$

So we could have chosen (and do now choose) i large enough so $|m(\overline{AB})| > |L| + 1$, and by the Lemma (in $(*)$ take $x_1 \equiv b$):

$$\left|\frac{g(b) - g(x)}{b - x}\right| \geqslant |m(\overline{AB})| > |L| + 1,$$

contradicting $(**)$. A similar argument shows that g has no left-hand derivative at any x in $[0,1]$ Q.E.D.

Reference

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MISCELLANEA

125.

(A) The most absurd science, in my opinion, and the most capable of stifling every sort of talent, is geometry. This ridiculous science deals with surfaces, lines and points, which do not exist in Nature. One can imagine inserting a hundred thousand curves between a circle and its tangent, where in reality one could not insert the thinnest straw.

—Voltaire, *Jeannot et Colin* (1764)

(B) A geometer can prove to you that you can draw infinitely many curves between a circle and a tangent, but not a single line: your eyes and your common sense say the opposite. The geometer will answer solemnly that that's an infinity of second order....

You consult a more honest geometer, who explains the mystery. "We imagine," he says, "things that are impossible in Nature, lines that have length but no width.... No actual curve or line can pass between two actual lines that touch each other: these are just intellectual games, imaginary chimeras. True geometry is the art of measuring things that exist."

—Voltaire, *L'Homme aux quarante écus* (1768)

ANSWERS TO PHOTOS ON PAGE 248

Top, J. E. Littlewood; bottom, G. H. Hardy. The Littlewood picture is from the collection of George Pólya; the Hardy photograph was taken by R. P. Boas in about 1939.

NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

A PROBABILISTIC PROOF OF THE WEIERSTRASS APPROXIMATION THEOREM

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Among the proofs of the Weierstrass Approximation Theorem, Bernstein's proof is surely the most common. It is particularly appealing because it is a constructive proof. If f is a continuous real-valued function of the interval $[0,1]$, the sequence of Bernstein polynomials

$$B_n(f; x) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right)\binom{n}{j} x^j(1-x)^{n-j}, \quad n = 0,1,2,\ldots$$

converges uniformly to f. This fact establishes

THE WEIERSTRASS APPROXIMATION THEOREM. *If f is a continuous real-valued function on the interval $[0,1]$ and $\varepsilon > 0$, then there exists a polynomial function, p, such that $\max|f(x) - p(x)| < \varepsilon$ for $0 \leqslant x \leqslant 1$.*

Once you've seen this sequence of polynomials, the actual convergence proof is much less inspiring. However the use of some elementary probability theory can make the convergence proof a bit more interesting. This is not an original approach; it is actually a return to Bernstein's method of proof in [**1**]. Since this reference is not readily available, it is worth repeating.

The only tools that are needed are the binomial distribution and Chebyshev's Theorem. The binomial distribution is

$$b(n, x; j) = \binom{n}{j} x^j(1-x)^{n-j},$$

where n is the number of trials, x is the probability of success for any one trial, and j is the number of successes. Chebyshev's Theorem states that if a probability distribution has mean μ and standard deviation σ, the probability of obtaining a value that deviates from the mean by at least k standard deviations is at most $1/k^2$; i.e., $\Pr(|x - \mu| \geqslant k\sigma) \leqslant 1/k^2$.

To prove convergence of the Bernstein polynomials, define a sequence of families of random variables $F_{n,x}$, $0 \leqslant x \leqslant 1$ and $n = 0,1,2,\ldots$ in terms of a binomial random variable with n trials and probability of success x. If j successes occur, define the value of $F_{n,x}$ to be $f(j/n)$. Then the expected value of $F_{n,x}$ is

$$E(F_{n,x}) = \sum_{j=0}^{n} f\left(\frac{j}{n}\right) b(n, x; j) = B_n(f; x).$$

We now prove that $E(F_{n,x})$ converges uniformly to f on $[0,1]$ as $n \to \infty$.

Let $\varepsilon > 0$ be arbitrary. Since f is continuous, it is bounded on $[0,1]$; therefore there exists M such that $|f(x)| \leqslant M$ for $0 \leqslant x \leqslant 1$, which implies that $|f(x) - f(y)| \leqslant 2M$ for $0 \leqslant x \leqslant 1$. The continuity of f on $[0,1]$ also implies uniform continuity, i.e., there exists $\delta > 0$ such that if $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon/2$. Now select a positive integer k such that $2M/k^2 < \varepsilon/2$ and a positive integer N such that $k/2\sqrt{N} < \delta$. Then if $n \geqslant N$ and $0 \leqslant x \leqslant 1$,

$$|f(x) - E(F_{n,x})| = \left| \sum_{j=0}^{n} \left(f(x) - f\left(\frac{j}{n}\right)\right) b(n, x; j) \right|$$

$$\leqslant \sum_{j=0}^{n} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n, x; j)$$

$$= \sum_{\left|\frac{j}{n} - x\right| < k/2\sqrt{n}} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n, x; j)$$

$$+ \sum_{\left|\frac{j}{n} - x\right| \geqslant k/2\sqrt{n}} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n, x; j)$$

$$\leqslant \sum_{\left|\frac{j}{n} - x\right| < \delta} \left|f(x) - f\left(\frac{j}{n}\right)\right| b(n, x; j) + 2M \Pr\left(\left|\frac{j}{n} - x\right| \geqslant k/2\sqrt{n}\right).$$

The first summand is less than $\varepsilon/2$, by our choice of δ. To bound the second summand, we use the fact that the mean and standard deviation of $b(n, x; k)$ are nx and $\sqrt{nx(1-x)}$, respectively. So if

$$\left|\frac{j}{n} - x\right| \geqslant k/2\sqrt{n} \geqslant k\sqrt{\frac{x(1-x)}{n}},$$

then $|j - nx| \geqslant k\sqrt{nx(1-x)}$ and application of Chebyshev's Theorem yields

$$2M \sum_{\left|\frac{j}{n} - x\right| \geqslant k/2\sqrt{n}} b(n, x; j) \leqslant 2M/k^2 < \varepsilon/2.$$

This establishes that $|f(x) - B_n(f; x)| < \varepsilon$ for all $x \in [0,1]$ and $n \geqslant N$.

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DUALITY IN THE CLASSROOM

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This note computes the smallest solution to an underdetermined system $Ax = b$. It is a problem that puts to use five or six different topics in linear algebra, and at the same time it illustrates one new idea: the magic of duality. Perhaps the simplest example of duality, in which we can see directly that a minimum equals a maximum, is the distance between a point and a line. The minimum distance from the origin to the points on a line (in R^3) equals the maximum distance to the planes through that line. That is evident to the intuition, which is extremely good at projection, and it is the special case of our problem when the matrix A is two by three.

In general A will be an m by n matrix with $m < n$; there are more unknowns than equations. We assume that the rows are independent, so the rank is m, AA^T is invertible, and $Ax = b$ has infinitely many solutions. The problem is to minimize the Euclidean length $|x| = (x_1^2 + \cdots + x_n^2)^{1/2}$, and it can be approached in at least four ways:

(i) by introducing Lagrange multipliers and then minimizing quadratics;
(ii) by developing the duality that was intuitive above;
(iii) as an instance of the pseudo-inverse A^+ (in this case a right inverse);
(iv) through projection.

At the end we consider also the maximum norm, minimizing the largest component $|x|_\infty = \max|x_i|$, and this connects our problem to the *max flow-min cut theorem* for flows through networks.

(i) The first approach is the "automatic" way to the answer; no insight is needed. In that sense it is the most convincing illustration of the power of duality. It uses only the solution to an unconstrained problem: The minimum of $Q = \frac{1}{2}x^THx - k^Tx$ occurs where $Hx = k$, and its value $Q_{\min}$ is $-\frac{1}{2}k^TH^{-1}k$. For the proof we can set all $\partial Q/\partial x_i = 0$, which leads to $Hx = k$; or a direct comparison [**1**, p. 265] gives $Q(x) \geqslant Q(H^{-1}k)$ for every x. H is assumed symmetric and positive definite.

We begin by minimizing not $|x|$ but $\frac{1}{2}|x|^2 = \frac{1}{2}x^Tx$, and we introduce the constraint $Ax = b$ through Lagrange multipliers; there are m equations in the constraint, and therefore m multipliers $\lambda_1,\ldots,\lambda_m$. The first step is to verify that

$$\min_{Ax=b} \frac{1}{2}x^Tx = \min_x \max_\lambda \frac{1}{2}x^Tx - \lambda^T(Ax - b). \tag{1}$$

This is an easy but very useful exercise. In the maximization over λ, we get $+\infty$ if $Ax - b$ is not zero, and $\frac{1}{2}x^Tx$ if it is; therefore the "outer" minimum, which admits all x, is forced to occur where $Ax = b$. (It avoids infinity.) Now comes the crucial step; we interchange operations and do the minimization first. In our case this could easily be justified, but perhaps it can be left as magic; the effect it produces is what we want to see. The quadratic in (1) has $H = I$, so that

$$x_{\min} = k = A^T\lambda \quad \text{and } Q_{\min} = -\frac{1}{2}\lambda^TAA^T\lambda. \tag{2}$$

Now (putting back the constant term λ^Tb) we are left with a maximization:

$$\max_\lambda -\frac{1}{2}\lambda^TAA^T\lambda + \lambda^Tb. \tag{3}$$

This is the dual problem. What is important is that it is unconstrained; the equalities $Ax = b$ in the primal imply no constraints in the dual. We therefore have another quadratic Q, depending on λ instead of x, and it is to be maximized instead of minimized. After a sign change,

$$\lambda_{\max} = H^{-1}k = (AA^T)^{-1}b \quad \text{and } Q_{\max} = +\frac{1}{2}b^T(AA^T)^{-1}b. \tag{4}$$

Now the problem is solved; the smallest solution to $Ax = b$ is

$$x_{\min} = A^T\lambda_{\max} = A^T(AA^T)^{-1}b. \tag{5}$$

We check that $Ax_{\min} = b$; the least value of $\frac{1}{2}|x|^2$ is given by $Q_{\max}$.

(ii) In the second approach we work directly with $|x|$, not $\frac{1}{2}|x|^2$, and we denote the Lagrange multipliers by u instead of λ. The minimax theorem [**2**, p. 392] interchanges operations as before:

$$\min_{Ax=b} |x| = \max_u \min_x \big[|x| - u^T(Ax - b)\big]. \tag{6}$$

Setting aside u^Tb, we have to compute

$$\min_x |x| - u^TAx. \tag{7}$$

This is slightly more subtle. If the vector A^Tu has length $l > 1$, the minimum is $-\infty$; take x to be CA^Tu, and

$$|x| - u^TAx = Cl - Cl^2 \to -\infty \text{ for large } C.$$

If $l \leqslant 1$, the minimum is zero; $u^TAx \leqslant l|x|$ from the Schwarz inequality. Substituting zero into (6), duality has produced

$$\min_{Ax=b} |x| = \max_{|A^Tu|\leqslant 1} u^Tb. \tag{8}$$

It is the right hand side, which can be rewritten because of homogeneity as the maximum of $|u^Tb|/|A^Tu|$, that we want to interpret.

For the interpretation we need the distance from the origin to a plane in R^n. Every $n-1$-dimensional plane is given by a single equation $\alpha^Tx = \beta$; we meet here the special case $m = 1$ of our original problem. The answer can come from our solution (5) above, or from other and simpler approaches; the distance is just $|\beta|/|\alpha|$. (According to (5), $x_{\min}$ is $\alpha|\alpha|^{-2}\beta$, so its length is $|\beta|/|\alpha|$.) Now we see the meaning of (8). All planes through the surface $Ax = b$ are found from linear combinations of these m equations; if $Ax = b$, then for any vector u we have $u^TAx = u^Tb$. With $\alpha = A^Tu$, $\beta = u^Tb$, this is the equation of a plane through the surface. Its distance from the origin is the same ratio $|u^Tb|/|A^Tu|$ which is hiding in (8). Therefore *the minimum distance to the surface $Ax = b$ is the maximum distance to the planes through that surface.*

This confirms in R^n what was clear in R^3.

(iii) Every matrix A gives a 1-1 map from its row space $R(A^T)$ to its column space $R(A)$. Thus there is a unique x_r in the row space for which $Ax_r = b$. Furthermore, the general solution to $Ax = b$ is $x = x_r + x_n$, where x_n is in the nullspace of A. Since x_n is orthogonal to x_r [**1**, pp. 84–88], it follows from Pythagoras that x_r is the smallest solution to $Ax = b$.

The inverse map which takes b to x_r, picking out this vector among all those in the inverse image of b, is called the **pseudo-inverse** of A. We want to verify that it produces the same choice $x = A^T(AA^T)^{-1}b$ that duality did in (5). But this choice satisfies $Ax = b$, and the left-most factor A^T puts x into the row space $R(A^T)$. Therefore this x is x_r, and coincides with the pseudo-inverse A^+ applied to b.

(iv) There is another description of the smallest solution to $Ax = b$; it is the point which projects onto the origin. The projection must be taken perpendicular to the surface $Ax = 0$—the nullspace of A. This is the approach taken by Luenberger [**3**, pp. 64–69]; he gives a full treatment of the problem, with applications.

The most familiar projection is $A(A^TA)^{-1}A^T$, onto the range of a matrix A with independent columns [**1**, p. 116]. In our case it is the rows that are independent. The projection onto the row space exchanges A for A^T, and then the projection onto its orthogonal complement (the nullspace) must be

$$P = I - A^T(AA^T)^{-1}A. \tag{9}$$

But if $Ax = b$, and x is projected onto the origin, then

$$Px = x - A^T(AA^T)^{-1}Ax = 0, \quad \text{or } x = A^T(AA^T)^{-1}b.$$

This completes the circle of ideas in the Euclidean norm.

The Maximal Flow Problem

Duality becomes more crucial when there are not so many alternatives. We come immediately to such a problem by looking again for the smallest solution to $Ax = b$, but now in the norm $|x|_\infty = \max|x_i|$. We want the **minimax solution**—the point at which a coordinate cube growing from the origin first touches the surface $Ax = b$. It may not be at a vertex of the cube (the surface might be a line touching the edge of a cube), but the meeting does occur where at least $n - m + 1$ components $|x_i|$ are equal to the maximum $|x|_\infty$.

To find that point we return to method (ii) above; we minimize $|x|_\infty$ subject to $Ax = b$. The steps to (6) and (7) are the same, but now we want

$$\min_x |x|_\infty - u^TAx. \tag{10}$$

The minimum is still $-\infty$ if the vector A^Tu is too large. In fact, if $x_i = \pm 1$, chosen to match the sign of $(A^Tu)_i$, then

$$|x|_\infty - u^T Ax = 1 - \sum|(A^T u)_i|. \tag{11}$$

This last sum is the l_1 norm $|A^T u|_1$. If it exceeds unity, then (11) is negative, and by increasing to $x_i = \pm C$ the quantity becomes as negative as we like. If on the other hand $|A^T u|_1 \leqslant 1$, then the minimum in (10) is zero; the second term cannot exceed the first. (It is like replacing the upper bound $y^T x \leqslant |y|\,|x|$ in the Schwarz inequality by $|y|_1 |x|_\infty$; an l_∞ problem in the primal produces an l_1 problem in the dual.) Thus the duality which is parallel to (8) is

$$\min_{Ax=b} |x|_\infty = \max_{|A^T u|_1 \leqslant 1} u^T b. \tag{12}$$

This is true for any matrix A, but the right hand side is not necessarily simpler than the left. The magic of duality can sometimes fail. It does *not* fail, however, if A is the incidence matrix of a graph. In this case we can reach the max flow-min cut theorem of Ford and Fulkerson [4].

An incidence matrix has a row for each node of the graph, and a column for each edge; for the edge from node i to node j, the column has -1 in row i, $+1$ in row j, and otherwise zero. Then $Ax = b$ expresses Kirchhoff's conservation law at each of the m nodes, allowing for the disappearance of an amount b_i (or the appearance of $-b_i$) at node i. For a network with a source at node 1 and a sink at node m, each of unit strength, the vector b is $b_0 = (-1, 0, \ldots, 0, 1)^T$.

In the maximal flow problem, each edge is assigned a capacity c_{ij} which cannot be exceeded by the flow x_{ij}. The goal is to send as much as possible from source to sink, subject to the constraints of conservation and capacity:

$$\text{Maximize } v \text{ subject to } Ax = vb_0 \text{ and } |x_{ij}| \leqslant c_{ij}. \tag{13}$$

The simplest algorithm starts with an admissible flow x and looks for a path from source to sink along which additional flow is possible. If one exists, the flow is augmented. If not, there is a "cut" separating those nodes which can be reached by additional flow from those which cannot; the sink is on the far side of the cut, since otherwise the flow could be increased. Across this cut all edges are filled to capacity:

$$\text{maximal flow} = v_{\max} = \text{capacity of cut}.$$

Across any other cut the capacity must be at least this great, or the given flow could not reach the sink. Therefore it is the **minimal cut** which governs the flow. A cut is specified by a set of nodes S which contains the source but not the sink, its capacity $c(S)$ is the sum of capacities c_{ij} from nodes in S to nodes outside, and the Ford-Fulkerson *max flow-min cut theorem* is

$$\max_{\substack{Ax=vb_0 \\ |x_{ij}| \leqslant c_{ij}}} v = \min_S c(S). \tag{14}$$

This resembles (12), but only vaguely. Nevertheless it is implied by (12), provided we establish the following property of incidence matrices A: *The extreme points of the polyhedron* $|A^T u|_1 \leqslant 1$, *or more generally of* $\sum|u_j - u_i|c_{ij} \leqslant 1$, *are multiples of vectors whose components are zero or one.* Each of these vectors corresponds to a cut, and no others need to be considered on the right side of (12).

This approach, were we to complete it, would lead to (14)—but much less quickly than the algorithmic approach of augmenting flows. However it has a strength which the algorithm lacks. By concentrating attention on the extreme points of the feasible set, it suggests a generalization to *continuous flows through a domain* for which no algorithm is known. In that case the flow field is given by a vector $\sigma = (\sigma_1(x, y), \sigma_2(x, y))$; material crosses the boundary according to $\sigma \cdot n = vf(x, y)$, and in the interior it is created or destroyed with $\operatorname{div} \sigma = -vF(x, y)$. As before, the problem is to find the maximal v subject to $|\sigma| \leqslant c$. The max flow-min cut theorem still holds [5]: There is a cut across the domain, separating a set E where $u = 1$ from its complement where $u = 0$, which solves the dual problem. The maximal flow must be perpendicular to this minimal cut, and cross at full capacity $|\sigma| = c$. For the continuous problem duality gives more than a new interpretation for the distance from a point to a surface; in this infinite-dimensional case no other solution is known.

The duality between potentials at the nodes of a network and flows through the edges is developed more fully in [6].

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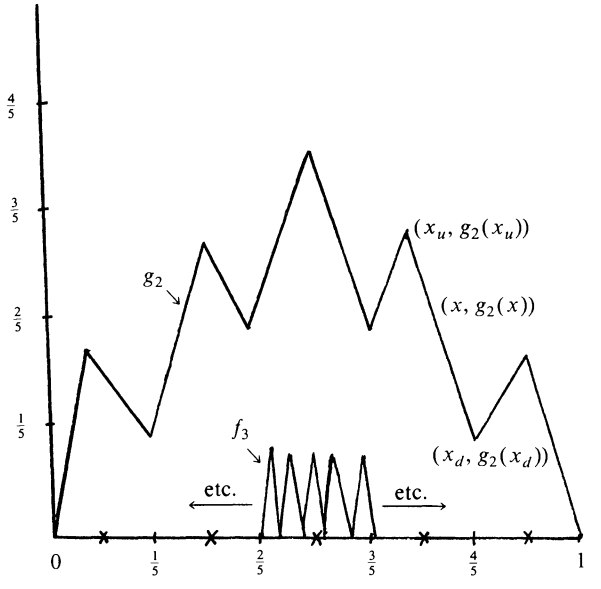
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A SIMPLE WEIERSTRASS FUNCTION

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We give perhaps one of the most easily understood constructions of a continuous, nowhere differentiable (even one-sidedly) function g on $[0,1]$. Bright students in the upper sections of elementary calculus should understand the reasoning. Our example is a new variant of one given by van der Waerden in 1930 [1].



Let the graph of $f_1 \equiv g_1$ be the union of the line segments from $(0,0)$ to $(\frac{1}{2}, \frac{1}{2})$ and from $(\frac{1}{2}, \frac{1}{2})$ to $(1,0)$. Let f_2 have five congruent teeth with height of each tooth $(\frac{1}{2})^2$. Let $g_2 \equiv f_1 + f_2$. Continue in a similar manner (see illustration); each f_i has five times as many teeth as f_{i-1} and is only half as high; define

CENTER SECTION
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Telegraphic Reviews

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General, S, L**.** Bridges to Infinity: The Human Side of Mathematics. Michael Guillen. Houghton Mifflin, 1983, 204 pp, $12.95. [ISBN: 0-87477-233-8] A superb elementary exposition of mathematical ideas--sans equations--intended to dispel fear and clarify concepts. Seventeen brief vignettes--e.g., game theory ("The Call of the Wild"), infinity ("Singular Ideas"), combinatorics ("Of War and Peace")--organized in three sections ("Fantasizing," "Compromising" and "Optimizing") illuminate mathematics with a human, often poetic light. LAS

General, S*(13-16), L. Mathematical Recreations for the Programmable Calculator. Dean Hoffman, Lee Mohler. Hayden Book, 1982, 322 pp, $14.95 (P). [ISBN: 0-8104-5188-3] Written for education and recreation. General programming techniques are introduced using flowcharts. Applications in "numbers" section include Ulam's problem, Fibonacci numbers and combinatoric, complex and recursive operations, etc. Substantial games section included. Problems given are rated for difficulty and size of calculator necessary. PB

General, T*(14: 1), S, L*. A Transition to Advanced Mathematics. Douglas Smith, Maurice Eggen, Richard St. Andre. Brooks/Cole Pub, 1983, xii + 178 pp, $21.95. [ISBN: 0-534-01249-3] After completing a currently popular "nonrigorous" calculus sequence, students are ill-prepared for courses in analysis, abstract algebra, etc., which require the writing of proofs. This book aims to bridge that gap. Chapters include logic and proof, basic set theory, relations, functions, cardinality, concepts of algebra, and real analysis. This book can be the basis of a course entitled "What every math major will need to know." TAV

Elementary, T(13: 1). Intermediate Algebra: A Guided Worktext. Irving Drooyan, Bill Rosen. Wadsworth Pub, 1983, viii + 649 pp, (P). [ISBN: 0-534-01172-1] Worktext using set terminology, "how to" approach. Useable for classes or independent study. Summary and two sample tests in each chapter. Exercise sets begin with example paralleled by "fill in the blank" sample. Answers for odd-numbered exercises and complete solutions for every other even exercise included. PB

Elementary, T(13: 1). Intermediate Algebra for College Students. Jerome E. Kaufmann. Prindle, Weber & Schmidt, 1983, ix + 543 pp. [ISBN: 0-87150-340-9] A carefully constructed text. Chapter summaries, tests, and answers to odd-numbered problems included. Two colors used effectively to distinguish reasoning (why) from technique (how) in examples. PB

Education, P, L. Undergraduate Mathematics Education in The People's Republic of China. Ed: Lynn Arthur Steen. MAA Notes No. 3. MAA, 1984, x + 99 pp, $5 (P). [ISBN: 0-88385-053-2] In June of 1983 a delegation of nearly 60 mathematicians and mathematics educators visited The People's Republic of China. This volume is a detailed report on all aspects of mathematics education in China (including applied mathematics and computer science), with sample examinations, curricula, and information on administration and special programs. JAS

History, P, L**.** Alan Turing: The Enigma. Andrew Hodges. Simon & Schuster, 1983, 587 pp, $22.50. [ISBN: 0-671-49207-1] A sympathetic, thoroughly documented, human and professional portrait of Alan Turing, from an awkward, nonconformist youth struggling with emerging homosexuality, dreaming of machines, through invention of abstract Turing machines to define "computable numbers," to triumph over the wartime code of the German Enigma machine, followed by a downward spiral to suicide. Includes numerous sections with superb elementary exposition of Turing's mathematics. LAS

History, P. History of Spatial Economic Theory. Claude Ponsard. Texts & Mono. in Econ. & Math. Syst. Springer-Verlag, 1983, xii + 238 pp, $23. [ISBN: 0-387-12802-6] A revised version of the original French edition (1958) with updated bibliography (752 references). This historical account aims to isolate and follow the development of various schools of thought concerning the importance of spatial features in economic thought, from their rather vague and anonymous beginnings to their present sophisticated mathematical form. LCL

Combinatorics, T(18), S, P*. Combinatorial Enumeration. I.P. Goulden, D.M. Jackson. Ser. in Disc. Math. Wiley, 1983, xxiv + 569 pp, $47.50. [ISBN: 0-471-86654-7] An encyclopedic account which will clearly become a standard reference in the subject. Contains a thorough study of the various combinatorial interpretations of the Lagrange inversion formula, a definitive treatment of the basic techniques of ordinary and exponential generating functions, with applications to a wide variety of enumeration problems: permutations with geometrical constraints, sequences, integer partitions, trees, maps, plane partitions, lattice paths. Exercises, with solutions, constitute a compendium of supplementary results. LCL

Number Theory, S(18), P. Lecture Notes in Mathematics-1001: The Selberg Trace Formula for PSL(2,R), Volume 2. Dennis A. Hejhal. Springer-Verlag, 1983, viii + 806 pp, $35.50 (P). [ISBN: 0-387-12323-7] A direct continuation of Volume 1 (LNM-548: TR, August-September 1977), this volume is primarily concerned with the classical trace formula for Fuchsian groups having at least one cusp. Contains a substantial list of references and several appendices. CEC

Linear Algebra, T(15-16: 2), S, L. Lineare Algebra und analytische Geometrie. Max Koecher. Grund. Math., No. 2. Springer-Verlag, 1983, xi + 286 pp, $14.80 (P). [ISBN: 0-387-12572-8] A clear, leisurely and comprehensive introduction to linear algebra over an arbitrary field, with three chapters on applications to geometry. Exercises and many historical notes. JD-B

Algebra, T(18), S, P. Commutative Group Algebras. Gregory Karpilovsky. Pure & Appl. Math., V. 78. Dekker, 1983, x + 223 pp, $39.75. [ISBN: 0-8247-1918-2] A self-contained, systematic introduction to and survey of the subject. The first part is devoted to the study of the interplay between the group-theoretic and ring-theoretic properties. The second part is concerned with the unit group and the isomorphism problem. The book ends with a discussion of several open research problems. LCL

Algebra, P. Lecture Notes in Mathematics-1004: Universal Algebra and Lattice Theory. Ed: R.S. Freese, O.C. Garcia. Springer-Verlag, 1983, vi + 308 pp, $15 (P). [ISBN: 0-387-12329-6] Proceedings of a conference in Puebla, Mexico, 1982. DD

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Algebra, P. Relative Invariants of Rings: The Commutative Theory. F. Van Oystaeyen, A. Verschoren. Pure & Appl. Math., V. 79. Dekker, 1983, x + 252 pp, $42.50. [ISBN: 0-8247-7043-9] Relative invariants (e.g., Brauer or Picard groups, parametrized by localizations of some ground ring) associated to a ring are used to study structure theory. Includes some new proofs of results of Horrocks, and applications to Noetherian integrally closed domains and algebraic geometry; discusses both graded and ungraded cases. RM

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Finite Mathematics, T*(13: 1, 2), S*, L*. Mathematics with Applications for the Management, Life, and Social Sciences, Second Edition. Howard Anton, Bernard Kolman. Academic Pr, 1982, xii + 851 pp, $27. [ISBN: 0-12-059561-3] Includes the fundamentals of finite mathematics and an intuitive introduction to calculus. A readable text with a variety of realistic and relevant applications. This edition contains many new exercises; new sections on limits at infinity, differential equations, curve sketching, and improper integrals; and new chapters on applications of integration and calculus of several variables. CEC

Calculus, T(13: 3). Technical Mathematics and Calculus. Bernard J. Rice, Jerry D. Strange. Prindle, Weber & Schmidt, 1983, xiii + 986 pp. [ISBN: 0-87150-376-X] A "utilitarian" approach to algebra, trigonometry and calculus, beginning with detailed coverage of signed numbers and fractions and concluding with Fourier series and Laplace transforms. An unrelenting cookbook for which "some capability in elementary algebra is desirable but not required." Over 7200 exercises, many keyed to calculators, many marked with icons to signal fields of applications. LAS

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exercises and examples worked in detail. Gets to applications early, before detailed development of differentiation techniques. Logarithms and exponentials introduced early. Has chapters on differential equations, series, and probability. (First Edition, TR, February 1977; Second Edition, TR, December 1980.) RM

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Differential Equations, S(17-18), P. Differential-Difference Equations: Applications and Numerical Problems. Ed: L. Collatz, G. Meinardus, W. Wetterling. ISNM 62. Birkhauser Boston, 1983, 196 pp, $24.95. [ISBN: 3-7643-1499-0] Manuscripts of papers presented at an Oberwolfach conference, June 1982. JD-B

Differential Equations, P. First Order Elliptic Systems: A Function Theoretic Approach. Robert P. Gilbert, James L. Buchanan. Math. in Sci. & Eng., V. 163. Academic Pr, 1983, xi + 281 pp, $45. [ISBN: 0-12-283280-9] First half explores the similarities between solutions to elliptic first order partial differential equations and analytic functions. Second half investigates systems in higher dimensions using Clifford algebras and generalizes some of the results of the first half to functions of several complex variables. BH

Differential Equations, T(18: 1). Introduction to the Fourier Transform & Pseudo-differential Operators. Bent E. Petersen. Pitman Pub, 1983, xi + 356 pp, $48. [ISBN: 0-273-08600-6] Chapter 1 gives an introduction to the theory of distributions; Chapter 2 develops the theory of the Fourier transform in this setting. The remainder of the book is devoted to pseudo-differential operators. Attractively printed book aimed at second-year graduate students. AWR

Differential Equations, T*(14-15: 1), S*, L.** Differential Equations and Mathematical Biology. D.S. Jones, B.D. Sleeman. George Allen & Unwin, 1983, xii + 339 pp, $30. [ISBN: 0-04-515001-X] For undergraduates. An amalgam of elementary differential equations and applications to biological and physical situations. Standard treatments of population models, the predator-prey problem, chemical reactions and epidemics. Novel features are a touch of the role of catastrophe theory in biological modelling and moving boundary value problems. Excellent chapters on heart physiology, nerve impulse transmission, diffusion and the growth of tumors. Exercises, almost all with answers, and brief notes with references round out this unusual textbook. JK

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Numerical Analysis, P. Improperly Posed Problems and Their Numerical Treatment. Ed: G. Hämmerlin, K.-H. Hoffmann. ISNM, V. 63. Birkhauser Boston, 1983, 264 pp, $24.95. [ISBN: 3-7643-1538-5] Contains 17 papers presented at a conference at the Mathematisches Forschingsinstitut, Oberwolfach, in September 1982. Topics include problems of regularization, parameter identification, free boundary and inverse problems in differential equations and integral equations of the first kind. Problems which appear in science, in technical fields, and in medicine are discussed, as well as general operator equations. BH

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Functional Analysis, P. Banach and Hilbert Spaces of Vector-valued Functions: Their General Theory and Applications to Holomorphy. Jacob Burbea, P. Masani. Research Notes in Math., No. 90. Pitman Pub, 1984, 111 pp, $16.95 (P). [ISBN: 0-273-08598-0] A ramification of work begun by Rovnyak, Nagy, and Foias, treatment begins with introductory theory of functions with values in Hilbert space and concludes with applications to holomorphy in Bergman and Hardy spaces. Appendix, references. JS

Functional Analysis, T(17-18: 1, 2), P. Fundamentals of the Theory of Operator Algebras, Volume I: Elementary Theory. Richard V. Kadison, John R. Ringrose. Pure & Appl. Math. Academic Pr, 1983, xv + 398 pp, $29.50. [ISBN: 0-12-393301-3] A comprehensive introduction in two volumes up to the research level. Volume I covers foundational material through elementary C*-algebra and von Neumann algebra theory. Designed as a text, or for self-study; with exercises. PZ

Analysis, P. Lecture Notes in Mathematics-994: Calderón-Zygmund Operators, Pseudo-Differential Operators and the Cauchy Integral of Calderón. Jean-Lin Journé. Springer-Verlag, 1983, vi + 128 pp, $8 (P). [ISBN: 0-387-12313-X] The Hilbert transform T on a once-differentiable bounded domain in the plane is written as an infinite sum of "commutator integral" operators T_k. The T_k are examples of Calderon-Zygmund operators (CZO's)--these are the objects of study, after three chapters of background. Those pseudodifferential operators which are CZO's are identified. CZO machinery is applied to Littlewood-Paley theory and to study of the Cauchy kernel on Lipschitz curves. PZ

Analysis, S(18), P*. Multiple Integrals in the Calculus of Variations and Nonlinear Elliptic Systems. Mariano Giaquinta. Annals of Math. Stud., No. 105. Princeton U Pr, 1983, vii + 296 pp, $35; $16.50 (P). [ISBN: 0-691-08330-4; 0-691-08331-2] Discusses "the existence and differentiability of stationary points of regular functionals in the calculus of variations." Focuses primarily on methods used to determine the regularity or partial regularity of vector-valued solutions of nonlinear elliptic systems. Extensive bibliography. BH

Analysis, P*. q-Hypergeometric Functions and Applications. Harold Exton. Ser. in Math. & Its Applic. Halsted Pr, 1983, 347 pp, $49.95. [ISBN: 0-470-27453-0] A q-hypergeometric or basic hypergeometric function is "an extension of an ordinary hypergeometric function, by the addition of an extra parameter q." First five chapters are devoted to the theoretical background of basic hypergeometric functions. Last two chapters give examples of their occurrence in number theory and other fields (mostly mathematics and physical science). Provides a unification of scattered results. Extensive bibliography. RSK

Analysis, P. Dynamical Systems II. Ed: A.R. Bednarek, L. Cesari. Academic Pr, 1982, xiii + 664 pp, $49. [ISBN: 0-12-084720-5] Lectures and contributed papers presented at the Second International Symposium on Dynamical Systems held at the University of Florida at Gainesville in February 1981. LAS

Algebraic Geometry, T(18: 2), S, P. Fundamentals of Diophantine Geometry. Serge Lang. Springer-Verlag, 1983, xviii + 370 pp, $48. [ISBN: 0-387-90837-4] Modernized and expanded version of Diophantine Geometry (1962). Language of schemes introduced with more elaboration of definitions from algebraic geometry. Several important results added, significantly in the theory of heights, with four new chapters dealing with results of Tate, Neron and Silverman on heights. Much improved. DD

Algebraic Geometry, P. Abelian Varieties. Serge Lang. Springer-Verlag, 1983, xii + 256 pp, $15 (P). [ISBN: 0-387-90875-7] A reprint of the 1959 classic; it contains a number of topics not covered elsewhere in textbook form. LCL

Algebraic Geometry, P. Lecture Notes in Mathematics-1008: Algebraic Geometry. Ed: I. Dolgachev. Springer-Verlag, 1983, 138 pp, $8 (P). [ISBN: 0-387-12337-7] Proceedings of the Third Midwest

Algebraic Geometry Conference, November 1981. DD

Algebraic Geometry, S(17-18), P. Recent Progress of Algebraic Geometry in Japan. Ed: Masayoshi Nagata. Math. Stud., No. 73. North-Holland, 1983, 213 pp, $46.75 (P). [ISBN: 0-444-86469-5] Seven survey articles with emphasis on Japanese research in algebraic geometry during the last decade. An expensive paperback. DD

Algebraic Geometry, P. Lecture Notes in Mathematics-1016: Algebraic Geometry. Ed: M. Raynaud, T. Shioda. Springer-Verlag, 1983, viii + 528 pp, $24 (P). [ISBN: 0-387-12685-6] Main portion of proceedings of a Japan-France conference in algebraic geometry, October 1982. Interesting and eclectic mix of papers. DD

Algebraic Geometry, S(17-18), P. Lecture Notes in Mathematics-1025: Homotopie Rationnelle: Modèles de Chen, Quillen, Sullivan. Daniel Tanré. Springer-Verlag, 1983, x + 211 pp, $11 (P). [ISBN: 0-387-12726-7]

Differential Geometry, P. CR Submanifolds of Kaehlerian and Sasakian Manifolds. Kentaro Yano, Masahiro Kon. Progress in Math., V. 30. Birkhauser Boston, 1983, x + 208 pp, $17.50. [ISBN: 3-7643-3119-4] A description of an active area of research with an invitation for the reader to join the activity. JAS

Differential Geometry, T(17: 1), S, P, L. Foundations of Differentiable Manifolds and Lie Groups. Frank W. Warner. Grad. Texts in Math., No. 94. Springer-Verlag, 1983, ix + 272 pp, $19.80. [ISBN: 0-387-90894-3] Except for a few editorial changes, this is a reproduction of the original Scott Foresman printing (TR, March 1972; Extended Review, August-September 1972). Begins with basics on manifolds, moves through Lie theory to integration, cohomology, deRham theorem, and the Hodge theorem. Exercises, index, bibliography. JS

Geometry, S(16-18), P, L*. Lecture Notes in Mathematics-1000: Differential Geometry in the Large. Heinz Hopf. Springer-Verlag, 1983, vii + 184 pp, $9.50 (P). [ISBN: 0-387-12004-1] Lecture notes from two courses given by Hopf but recorded by his students: a 1946 course at NYU (notes by Peter Lax) on polyhedral differential geometry, and a 1956 course at Stanford (notes by John W. Gray) on global geometry of two dimensional surfaces in three dimensional Euclidean space. Beautiful mathematics elegantly explained--fitting substance for Number 1000 of Lecture Notes in Mathematics. LAS

Topology, T(17-18: 1), L. Introduction to Differential Topology. Th. Bröcker, K. Jänich. Transl: C.B. and M.J. Thomas. Cambridge U Pr, 1982, vii + 160 pp, $27.50; $11.95 (P). [ISBN: 0-521-24135-9; 0-521-28470-8] Assumes point set topology, multivariable calculus, and some mathematical maturity. Covers embedding, isotopy, and transversality theorems and uses Sard's theorem, partitions of unity, dynamical systems and sprays as tools. Lots of non-trivial exercises in this translation of the 1973 German original. JAS

Topology, S(18), P. Modern Dimension Theory, Revised and Extended Edition. J. Nagata. Sigma Ser. in Pure Math., V. 2. Heldermann Verlag, 1983, ix + 284 pp, 68 DM (P). [ISBN: 3-88538-002-1] Gives a "brief account of modern dimension theory as it has been developed since 1941," presupposing only a basic understanding of point-set topology. Major revision of 1964 edition, especially of the chapters on non-metrizable and infinite dimensional spaces. Other topics include general and separable metric spaces, metrization, and cohomology. BH

Topology, T(14-18: 1), S. Elementary Topology: A Combinatorial and Algebraic Approach, Reprinted with Corrections. Donald W. Blackett. Academic Pr, 1982, ix + 224 pp, $14.50 (P). [ISBN: 0-12-103060-1] Intended for a one-term undergraduate course. Treats topics from one, two, and three dimensions from combinatorial and algebraic perspectives: classification of surfaces, covering surfaces, winding number, vector fields, networks, manifolds. Many helpful illustrations, diagrams, equations organized in an optimal visual manner throughout the text. Full of delightful examples and exercises that whet the appetite for topology. Bibliography. Subject index. RJA

Topology, P. Noeuds, Tresses et Singularités. C. Weber. L'Enseignement Math, 1983, 259 pp, 75 Frs. (P). Texts of 11 papers given at a seminar on the relations between the theory of isolated singularities and the theory of knots, held in Plans-Sur-Bex, Switzerland, in March 1982. JD-B

Operations Research, S*, L*. Markov Decision Processes. Paul Thie. UMAP Expos. Mono. Ser. COMAP, 1983, viii + 95 pp, $8.95 (P). [ISBN: 0-912843-04-7] A Markov decision process is a variation of the Markov chain model, where at each step a decision is made concerning the transition probabilities. This monograph serves as an introduction to this useful method of operations research. Interesting applications appear throughout the text. Assumes minimal background--some knowledge of sequences and series, some familiarity with matrices (inverse, transpose). LCL

Game Theory, S(13-16), L. Geometric Games and Their Applications. W.H. Ruckle. Research Notes in Math., No. 82. Pitman Pub, 1983, 187 pp, $19.95 (P). [ISBN: 0-273-08578-6] Intended as an introduction for anyone from a gifted high school student to a mathematician trained in other areas, this book concentrates on games of partial information. Applicability to models of search and ambush are emphasized. An interesting possibility for an honors project. AWR

Game Theory, T(18), P. Dynamic Noncooperative Game Theory. Tamer Basar, Geert Jan Olsder. Math. in Sci. & Eng., V. 160. Academic Pr, 1982, xii + 430 pp, $49.50. [ISBN: 0-12-080220-1] Extensive treatment of static and dynamic noncooperative game theory--emphasis on the interplay between dynamic information patterns and the structural properties of several types of equilibria. The first part covers dynamic games in which the strategy spaces are finite; the second part extends this to infinite dynamic games (differential games). Problems vary in difficulty from routine to open research types. LCL

Optimization, S(17-18), P. Complementary Pivoting on a Pseudomanifold Structure with Applications in the Decision Sciences. F.J. Gould, J.W. Tolle. Sigma Ser. in Appl. Math., V. 2. Heldermann Verlag, 1983, iv + 203 pp, 58 DM (P). [ISBN: 3-88538-402-7] The authors present a unified framework for understanding a number of combinatorial algorithms that have been widely used in the last decade (especially in nonlinear settings) to solve, or give approximate solutions to, optimization problems. The general theory is supplemented with examples, exercises, open questions. LCL

Optimization, P. Infinite-Dimensional Optimization and Convexity. Ivar Ekeland, Thomas Turnbull. Lect. in Math. U of Chicago Pr, 1983, viii + 166 pp, $8 (P); $16. [ISBN: 0-226-19988-6; 0-226-19987-8] Can we find a minimum for a functional defined over a feasible set? In infinite-dimensional cases such as occur in the calculus of variations (the basic example in these notes), one usually needs a convexity condition to prove existence of a solution. These notes give insight into why this is so. AWR

Optimization, P. Problem Complexity and Method Efficiency in Optimization. A.S. Nemirovsky, D.B. Yudin. Transl: E.R. Dawson. Ser. in Discrete Math. Wiley, 1983, xv + 388 pp, $49.95. [ISBN: 0-471-10345-4] From the Preface, "The authors set up their own mathematical model in order to investigate questions concerning complexity of optimization problems and efficiency of methods of solving them." Attempting to assess "laboriousness" of all possible methods which solve problems in a given family of optimization problems with given accuracy, the book deals with a high level of abstraction and is tough going. AWR

Optimization, T(17-18), S, P. Introduction to Sensitivity and Stability Analysis in Nonlinear Programming. Anthony V. Fiacco. Math. in Sci. & Eng., V. 165. Academic Pr, 1983, xii + 367 pp, $27.50. [ISBN: 0-12-254450-1]

Dynamical Systems, T(17), P. Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields. John Guckenheimer, Philip Holmes. Appl. Math. Sci., V. 42. Springer-Verlag, 1983, xvi + 453 pp, $32. [ISBN: 0-387-90819-6] An attempt to make research tools concerning "strange attractors" developed in the last 20 years available to applied scientists and to make clear to research mathematicians the needs in applied work. Emphasis on geometric and topological solutions of differential equations. Applications mainly drawn from nonlinear oscillations. AWR

Probability, T*(15-16: 1, 2), L. Stochastic Processes. Sheldon M. Ross. Wiley, 1983, viii + 309 pp, $29.95. [ISBN: 0-471-09942-2] A nicely organized text. The author assumes calculus and probability at the level of his text A First Course in Probability (TR, May 1976; Extended Review, February 1979). Using the sample path approach he develops Poisson and Markov processes, then covers random walks and martingales, finishing with stochastic order relations. Many exercises and references. A very nice text. TAV

Probability, S(17-18), P. Lecture Notes in Mathematics-1010: Chaines de Markov sur les permutations. Jacques-Edouard Dies. Springer-Verlag, 1983, ix + 226 pp, $11 (P). [ISBN: 0-387-12669-4]

Probability, P. Bayes Theory. J.A. Hartigan. Ser. in Stat. Springer-Verlag, 1983, xii + 145 pp, $16.80. [ISBN: 0-387-90883-8] A study of probability distributions which arise frequently in Bayes theory--distributions in which the total probability is infinite! The book is based on lectures given at Yale (1971-1981), and requires a measure-theoretic understanding of probability. Conclusions are derived formally from formal axioms and probabilistic assumptions. LCL

Probability, P. Handbook of Stochastic Methods for Physics, Chemistry and the Natural Sciences. C.W. Gardiner. Ser. in Synergetics, V. 13. Springer-Verlag, 1983, xix + 442 pp, $42. [ISBN: 0-387-11357-6] Written for theoretical physicists and chemists, it assumes a strong background in classical analysis. Deals entirely with Markov processes or systems which can be embedded in Markov processes. Unique feature is its emphasis on methods of approximation. RSK

Statistics, T?(18: 2), S. An Introduction to Construction and Analysis of Statistical Designs. D.G. Kabe, S.M. Shan. Papers in Pure & Appl. Math., No. 64. Queen's U, 1983, vi + 719 pp, (P). Notes based on lectures given by Professor M.C. Chakrabarti at the University of Bombay. Theoretical presentation, but illustrated with data. Reproduced from typewritten copy, it contains an incredible number of typographical errors. Also has no index. RSK

Statistics, T(14-15: 1). Statistics in Research and Development. Roland Caulcutt. Chapman & Hall, 1983, xii + 352 pp, $45. [ISBN: 0-412-23720-2] Nonstandard text, written originally for working chemists in England. The approach is problem-centered and nonmathematical. Although introductory in nature it contains much material on a number of sophisticated concepts, including multiple regression analysis, experimental design and the analysis of variance. RSK

Statistics, P. Multivariate Data Analysis in Industrial Practice. Paul J. Lewi. Chemometrics Res. Stud., No. 3. Wiley, 1982, xi + 244 pp, $31.95. [ISBN: 0-471-10466-3] Monograph dealing with applications of "dimensionality reduction and display by factorial analysis" to areas related to the pharmaceutical industry. Uses APL notation to describe the techniques. RSK

Statistics, T*(18: 1), P*. Theory of Point Estimation. E.L. Lehmann. Wiley, 1983, xii + 506 pp, $36.95. [ISBN: 0-471-05849-1] In the Wiley Series in Probability and Mathematical Statistics. Companion volume to the author's well-known Testing Statistical Hypotheses. Comprehensive treatment of point estimation in Euclidean sample spaces, covering both exact (small-sample) and large-sample theory. Contains extensive problem sets and references. RSK

Statistics, S(16-17), P*, L. Using Multivariate Statistics. Barbara G. Tabachnick, Linda S. Fidell. Harper & Row, 1983, xviii + 509 pp, $19.95. [ISBN: 0-06-042045-6] Use and interpretation of multivariate techniques, illustrated with computer output, in four main areas: measuring the degree of relationship amaong variables, measuring the significance of group differences, predicting group membership, and determining structure. Emphasizes the need to check assumptions, with suggestions for dealing with violations. Unique features include comparisons of BMDP and SPSS computer programs, "Results" sections which present conclusions in a form appropriate for a professional journal, and examples from the literature of the use of these techniques. RSK

Statistics, P. Tests for Preference. J.J. Dik. Math. Centre Tracts, No. 162. Math Centrum, 1983, vi + 155 pp, Dfl. 20,90 (P). [ISBN: 90-6196-259-5] Theoretical monograph dealing with a statistical solution to the practical problem of detecting differences in preferences between individuals when they make repeated choices. RSK

Statistics, T(15: 1, 2). A First Course in Probability and Statistics with Applications. Peggy Tang Strait. HarBrace J, 1983, xix + 581 pp, $27.95. [ISBN: 0-15-527520-8] Designed so it can be used for a one-semester course covering the basics of both probability and statistics at the post-calculus level. Topics covered are mostly standard, with an emphasis on explanations rather than proofs. RSK

Statistics, T(14-15: 1, 2). Data, Models and Statistical Analysis. R.A. Cooper, A.J. Weekes. Barnes & Noble, 1983, xv + 400 pp, $30; $19.50 (P). [ISBN: 0-389-20382-3; 0-389-20383-1] Applied text, dealing with the analysis, both univariate and multivariate, of non-experimental data. Assumes background in elementary statistics, but little mathematics. Uses computer output from real data sets to illustrate techniques and concepts, particularly the ideas of multivariate statistical analysis. Limited problem sets. RSK

Statistics, T(17-18: 1, 2), P*, L. Applied Multivariate Analysis: Using Bayesian and Frequentist Methods of Inference, Second Edition. S. James Press. Krieger Pub, 1982, xix + 600 pp, $29.50. [ISBN: 0-88275-976-0] Revision of the author's 1972 text published by Holt, Rinehart & Winston (TR, January 1973). Organization is the same but the content, references and computer program listings have been updated. RSK

Statistics, P. Lecture Notes in Statistics-20: Mathematical Learning Models--Theory and Algorithms, Proceedings of a Conference. Ed: Ulrich Herkenrath, Dieter Kalin, Walter Vogel. Springer-Verlag, 1983, xiii + 226 pp, $16 (P). [ISBN: 0-387-90913-3] Proceedings of a conference organized to bring together researchers using different approaches to mathematical learning models, bandit problems, stochastic approximation problems, etc. RM

Statistics, T(16-17: 1), S. Statistical Methods in Agriculture and Experimental Biology. R. Mead, R.N. Curnow. Chapman & Hall, 1983, xi + 335 pp, $19.95 (P); $49.95. [ISBN: 0-412-24240-0; 0-412-24230-3] Statistics for researchers in agriculture or experimental biology. Minimal mathematics prerequisites. Many numerical examples. Covers estimation, hypothesis testing, analysis of variance, regression, non-linear models, sampling, design of experiments, and discussion of the use of computer packages. RM

Statistics, P. Sequentielle Zuverlässigkeitsprüfung. Bernd Lisek, Johannes Hochschild. Teubner-Texte zur Math., B. 53. BG Teubner, 1983, 152 pp, 16 M (P). On sequential tests for reliability when distribution of times between failures is exponential. Many tables. JD-B

Statistics, T(17-18), P. Nonlinear Regression Modeling: A Unified Practical Approach. David A. Ratkowsky. Statistics, V. 48. Dekker, 1983, viii + 276 pp, $39.75. [ISBN: 0-8247-1907-7] A study of the behavior in estimation of nonlinear models--i.e., predicting response values and making inferences. It assumes a familiarity with the theory and practice of linear regression analysis. It features practical examples and real data sets, exercises and solutions, Fortran programs. LCL

Statistics, P*. Developments in Statistics, Volume 4. Ed: Paruchuri R. Krishnaiah. Academic Pr, 1983, xiv + 287 pp, $62. [ISBN: 0-12-426604-5] Invited papers reviewing the state of the art in four different areas: spectrum parameter estimation in time series analysis, identifiability and modeling in econometrics, inference on parameters of multivariate normal populations when some data is missing, and structured exploratory data analysis applied to mode of inheritance. RSK

Statistics, S(15-18), P, L.** The Chronological Annotated Bibliography of Order Statistics, Volume I: Pre-1950. H. Leon Harter. American Sci Pr, 1983, iv + 515 pp, $89.55 (P). [ISBN: 0-935950-04-4] Revision of a limited edition printed in 1978 by the U.S. Government Printing Office. Major part is

the chronological bibliography, which contains a brief summary of the contents of each of 942 pre-1950 articles, together with lists of references to earlier work and citations by later authors. Also has an author index and a subject index. Of historical interest as well. Note price. RSK

Statistics, P. Statistical Tables for Multivariate Analysis: A Handbook with References to Applications. Heinz Kres. Transl: Peter Wadsack. Ser. in Stat. Springer-Verlag, 1983, xxii + 504 pp, $56. [ISBN: 0-387-90909-5] A translation, with five additional tables, of the 1975 German edition. Contains thirty tables collected from journals and other less accessible sources. RSK

Statistics, T*(16-17: 1), S, P, L. Forecasting With Univariate Box-Jenkins Models: Concepts and Cases. Alan Pankratz. Wiley, 1983, xiv + 562 pp, $34.95. [ISBN: 0-471-09023-9] In the Wiley Series in Probability and Mathematical Statistics. Divided into two roughly equal parts. Part I presents the essential concepts underlying single-series (univariate) Box-Jenkins or ARIMA (auto-regressive integrated moving average) forecasting models. Theory is presented at a relatively elementary level with proofs in appendices. Part II contains a list of practical rules for building ARIMA models, together with a collection of 15 case studies, using real data, which are analyzed in detail. Limited problem sets. RSK

Statistics, T*(13: 1). Modern Elementary Statistics, 6th Edition. John E. Freund. Prentice-Hall, 1984, xii + 561 pp, $26.95. [ISBN: 0-13-593525-3] Revision of the author's well-known elementary text. Basic organization and coverage remain the same, except that the chapters on analysis of variance and nonparametric tests have been interchanged. Exercises have been greatly increased, and there are now review exercises and a checklist of key terms at the end of each chapter. (Fourth Edition, TR, March 1974; Fifth Edition, TR, August-Spetember 1979) RSK

Statistics, T(16-17: 1), P*. The Life Table and Its Applications. Chin Long Chiang. Krieger Pub, 1984, xix + 316 pp, $29.50. [ISBN: 0-89874-570-5] Of interest to biostatisticians and health statisticians in addition to those in actuarial science and demography. Presents both the theory and application of life table methodology from a statistical perspective, including the development of a new life table in which age intervals are random rather than predetermined. Extensive references. RSK

Statistics, P. Attribute Sampling Plans, Tables of Tests and Confidence Limits for Proportions. Robert E. Odeh, D.B. Owen. Statistics, V. 49. Dekker, 1983, xi + 368 pp, $69.75. [ISBN: 0-8247-7136-2] Complementary volume to the authors' Tables for Normal Tolerance Limits, Sampling Plans, and Screening (TR, January 1981), which assumed an underlying normal distribution. These tables assume underlying distributions are binomial, Poisson or hypergeometric. Of use to those involved in acceptance sampling or quality control. RSK

Computer Programming, T?(13). Introduction to Computer Programming, Basic Fortran IV: A Practical Approach. William J. Keys. Anaheim Pub, 1981, 379 pp, $17.95 (P) [ISBN: 0-88236-151-1]; Instructor's Guide and Answer Manual, 93 pp, (P). An old-fashioned (punch-card) approach to programming in Fortran for liberal art and business students. Examples and exercises limited to payroll calculations. LAS

Computer Programming, T???. Apple Machine Language. Robert D. Rosen. Holt, Rinehart & Winston, 1983, ix + 241 pp, $20.95 (P). [ISBN: 0-03-06336-2] Assumes a knowledge of BASIC and uses this as a framework for byte-by-byte machine programming via the Apple monitor. No use of assembler, no general presentation of the architecture or addressing modes of the 6502; ASCII is given with the high order bit set. Strictly for the hobbyist hacker. JAS

Computer Programming, T?(13-14: 1). Programming Microcomputers With PASCAL. M.D. Beer. Van Nostrand Reinhold Pub, 1982, x + 243 pp, $13.95 (P). [ISBN: 0-442-21368-9] A relaxed introduction to UCSD Pascal--especially on the Apple. Special effort is put into motivating program structure and the concluding chapter on using Pascal for device control is interesting. Printers are ignored except for a little hand waving when discussing files so only the most basic data structures are considered, and then only on an ad hoc basis. JAS

Computer Programming, T(13-14: 2). Basic Computing. Don Mittleman. HarBrace J, 1982, xiii + 430 pp, $16.95 (P). [ISBN: 0-15-504910-0] Commands available in BASIC are thoroughly explained, with examples (usually with flowcharts) of each command's use. Students are led from first putting fingers on keyboard to dealing with binary string files. Exercises included. PB

Computer Programming, T(13-14: 1), S. PASCAL Programming: A Spiral Approach. Walter S. Brainerd, Charles H. Goldberg, Jonathan L. Gross. Boyd & Fraser Pub, 1982, x + 584 pp, $17.95 (P). [ISBN: 0-87835-122-1] A carefully motivated beginner's introduction, teaching by examples (200 complete programs with full execution printout) and spiral instruction. Includes discussion and examples of recursion, character data (text analysis), files (dynamic data structures), simulation, and artificial intelligence. LAS

Computer Programming, S(13). A Guide to Programming in Apple Soft. Bruce Presley. Van Nostrand Reinhold, 1982, 214 pp, $13.50 (P). [ISBN: 0-442-25890-9] Introductory text for BASIC on the APPLE microcomputer. Standard treatment of topics. Suitable for self study. RM

Computer Programming, S(13-14). Doing Business with Pascal. Richard and Douglas Hergert. Sybex, 1983, xi + 371 pp, $17.95 (P). [ISBN: 0-89588-091-1] A discussion of how a microcomputer can be used

in a business, together with programming instructions for implementation. Emphasis on modular programming and design, with several worked-out examples. Written primarily for managers of small- or medium-sized businesses. LCL

Computer Programming, T(13). Introduction to Pascal and Computer Applications. Richard M. Jones. Allyn & Bacon, 1983, xiii + 429 pp, $24.24 (P). [ISBN: 0-205-07937-7] Programming workbook, divided into five parts--general, beginning (e.g., loops), intermediate (e.g., functions, procedures), advanced (e.g., arrays, records), applications (e.g., simulation)--and subdivided into individual lessons with appropriate exercises. Examples include common programming algorithms. LCL

Computer Programming, T(13: 1). Personal Pascal: Compiled Pascal for the IBM Personal Computer. David E. Cortesi, George W. Cherry. Reston Pub, 1984, x + 420 pp, $16.95 (P); $24.95. [ISBN: 0-8359-5522-2; 0-8359-5523-0] This is an introductory programming text describing the Pascal programming language on the IBM PC. The text assumes no previous background in programming and describes the entire standard Pascal language. It also describes language features unique to the IBM PC. MS

Data Structures, S(17-18), P. Lecture Notes in Computer Science-156: The Design of Dynamic Data Structures. Mark H. Overmars. Springer-Verlag, 1983, vii + 181 pp, $10 (P). [ISBN: 0-387-12330-X] A text based on the author's Ph.D. thesis on general techniques for construction of dynamic data structures (those which support efficient updates) for searching problems. Techniques based on static structures (e.g., balancing, partial and global rebuilding) as well as those based on properties of the search problems (e.g., decomposability). RM

Software Systems, S(14-16), L. Mastering CP/M. Alan R. Miller. Sybex, 1983, xiii + 398 pp, $15.95 (P). [ISBN: 0-89588-068-7] A how-to book, rather than a text, which covers the use of macro assemblers in the context of advanced CP/M systems programming. Presents most of the structure of the CP/M system calls and disks via a number of utility programs such as super directory programs which tell all--block allocations, user number, and the like. Requires some knowledge of assembly language programming and the availability of a good macro assembler running under CP/M. The biggest weakness is in the depth of the explanations--it's not clear how to adapt some of the procedures to a less standard CP/M environment than is assumed in the book. Well written, substantial information; good index. JAS

Computer Science, T(17-18). Introduction to Mathematical Techniques in Pattern Recognition. Harry C. Andrews. Wiley, 1972, xiii + 242 pp, $27.50. [ISBN: 0-471-03172-0] One of the first systematic introductions to the subject; by now the references are quite dated. Assumes a background in elementary linear algebra and probability theory. LCL

Computer Science, P. Lecture Notes in Computer Science-158: Foundations of Computation Theory. Ed: Marek Karpinski. Springer-Verlag, 1983, xi + 514 pp, $22 (P). [ISBN: 0-387-12689-9] Proceedings of an August 1983 international conference held at Borgholm, Sweden. LAS

Computer Science, P. Abstraction Mechanisms and Language Design. Paul N. Hilfinger. MIT Pr, 1983, xiii + 176 pp, $27.50. [ISBN: 0-262-08134-2] This monograph is the winning 1982 outstanding doctoral dissertation in the field of computer science. It was written while the author was a graduate student at Carnegie-Mellon University. The dissertation addresses the topics of data and control abstractions, and information hiding, in the area of language design. Specifically, it evaluates the quality of the abstraction mechanism of the new language called Ada. MS

Computer Science, T(17: 1), P. Computational Aspects of VLSI. Jeffrey D. Ullman. Principles of Computer Sci. Ser. Computer Sci Pr, 1984, x + 495 pp, $32.95. [ISBN: 0-914894-95-1] This text brings together two very diverse subject areas--the design and implementation of electrical circuits, and theoretical work on the analysis of algorithms. The author tries to apply the theoretical ideas to the everyday computational problems which arise in circuit design. Problems which are addressed are lower bound problems, layout algorithms, systolic algorithms, optimization issues, and VLSI languages. The text is for graduate level students in computer science and assumes an extensive background in computer science and electrical engineering. MS

Computer Science, P. Real Time Programming 1981. Ed: T. Hasegawa. Pergamon Pr, 1982, viii + 138 pp, $71. [ISBN: 0-08-030411-7] Proceedings of the 11th annual IFAC/IFIP workshop on real time programming. RM

Computer Science, P. Algorithmic Program Debugging. Ehud Y. Shapiro. MIT Pr, 1983, xvi + 232 pp, $30. [ISBN: 0-262-19218-7] Author's Ph.D. dissertation on the theoretical foundations of automatic program debugging. Develops Prolog-based interactive diagnosis algorithms which seek to identify bugs and fix them. RM

Computer Science, P. Formal Specification of Interactive Graphics Programming Languages. William R. Mallgren. MIT Pr, 1983, 269 pp, $35. [ISBN: 0-262-13191-9] This work is one of three published in the 1982 ACM Distinguished Dissertation Series. It provides a methodology for formally specifying the semantics of an interactive graphics programming language. AO

Computer Science, T(16-17: 1, 2), L*. Relational Information Systems. T.H. Merrett. Reston Pub, 1984, xvii + 507 pp, $24.95. [ISBN: 0-8359-6642-9] Discusses the design and implementation of information systems (e.g., databases) using the relational model. Presents two large-scale examples as illustrations. AO

Computer Science, S(16-17), P*. Computer Performance Modeling Handbook. Ed: Stephen S. Lavenberg. Notes & Reports in Comp. Sci. & Appl. Math., No. 4. Academic Pr, 1983, xiii + 399 pp, $45. [ISBN: 0-12-438720-9] A reference manual on the modeling of the performance of computer systems. The use of analytical results as well as simulation is discussed. Many examples are presented. AO

Computer Science, T*(15-17: 1), L*. Fundamentals of Data Structures in PASCAL. Ellis Horowitz, Sartaj Sahni. Comp. Software Eng. Ser. Computer Sci Pr, 1984, xiv + 542 pp, $26.95. [ISBN: 0-914894-94-3] An updated version of a well-known textbook: the algorithms in the text are now presented in Pascal. AO

Computer Science, P. Area-Efficient VLSI Computation. Charles Eric Leiserson. MIT Pr, 1983, x + 136 pp, $22.50. [ISBN: 0-262-12102-6] Winner of the 1982 ACM Doctoral Dissertation Award. Applies algorithm design techniques to the design of area- and time-efficient VLSI circuitry. AO

Control Theory, P. Multivariable Feedback Systems. Frank M. Callier, Charles A. Desoer. Springer-Verlag, 1982, ix + 275 pp, $36. [ISBN: 0-387-90768-8] Based on experience in teaching a first-year graduate course for control engineers, purpose is to provide a rapid introduction to control theory and multivariable feedback systems. Paying particular attention to the time-domain behavior of systems, the pace is brisk and assumes considerable mathematical background, particularly linear algebra and complex variables, as well as a previous course in control theory. Exercises, bibliography, appendices. JS

Systems Theory, T(16-17), S, P, L. Applications of Discrete and Continuous Fourier Analysis. H. Joseph Weaver. Wiley, 1983, xii + 375 pp, $34.95. [ISBN: 0-471-87115-X] An introduction to the mathematical properties and computational aspects of three analogous transformations: the Fourier series, the Fourier transform, and the discrete Fourier transform. Applications to science and engineering, with individual chapters on optics, heat, vibration, and stochastic analysis. Pleasant format with over 200 illustrations. No exercises. LCL

Systems Theory, P. Structural Theory of Distributed Systems. A.G. Butkovskiy. Transl: L.W. Longdon. Ser. in Math. & Its Applic. Halsted Pr, 1983, 314 pp, $84.95. [ISBN: 0-470-27469-7] Intended for control engineers and engineers in related areas, this book tries to give a systematic account and adaptation of the theory of structural diagrams of linear lumped systems to distributed parameter systems. AWR

Applications (Biology), S(16-18), P, L. The Radon Transform and Some of Its Applications. Stanley R. Deans. Wiley, 1983, xi + 289 pp, $34.95. [ISBN: 0-471-89804-X] An interesting and readable survey of theory and, especially, applications of the Radon transform to tomography, astronomy, and other areas. Includes basic properties, inversion theorems, relations to other transforms, and a translation of Radon's 1917 paper. Mathematical development is presented concretely, mostly in two- and three-dimensional Euclidean space. PZ

Applications (Biology), P. Lecture Notes in Biomathematics-51: Oscillations in Mathematical Biology. Ed: J.P.E. Hodgson. Springer-Verlag, 1983, 196 pp, $12.50 (P). [ISBN: 0-387-12670-8] Contains five papers based on talks given at a conference at Adelphi University in 1982 concerning the dynamics of non-linear systems arising in biology. BH

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-218: A Dynamic Theory of the Firm: Production, Finance and Investment. Paul van Loon. Springer-Verlag, 1983, vii + 191 pp, $11.50 (P). [ISBN: 0-387-12678-3] Describes the links between optimal production, finance, and investment policy; differs from other dynamic models dealing with production policy by using activity analysis in which a continuous production function is replaced with a limited number of production possibilities. AWR

Applications (Engineering), P. Optimization Methods in Structural Design. Ed: Hans Eschenauer, Niels Olhoff. Bibliographisches Institut, 1982, 452 pp, $24.95 (P). [ISBN: 3-411-01654-X] Takes its title from the Euromech-Colloquium held in October, 1982 at the University of Siegen, FR Germany. Contributions are numerous, short, generally illustrated with a specific engineering problem. AWR

Applications (Engineering), T(16-17: 1, 2), S, P. Engineering Optimization: Methods and Applications. G.V. Reklaitis, A. Ravindran, K.M. Ragsdell. Wiley, 1983, xvii + 684 pp, $39.95. [ISBN: 0-471-05579-4] Text on the practical and conceptual aspects of optimization methodology (continuous, nonlinear programming) used in engineering applications. Emphasis on model formulation, solution strategies, and comparisons of different methods. Includes transformation methods, constrained search, linearization techniques, quadratic approximation, case studies. RM

Applications (Engineering), S(17-18), P. Problèmes Mathématiques en Plasticité. Roger Temam. Gauthier-Villars, 1983, 353 pp, 230 FF. [ISBN: 2-04-015429-9]

Applications (Engineering), P. Issues in Acoustic Signal--Image Processing and Recognition. Ed: C.H. Chen. NATO ASI Ser. F. Springer-Verlag, 1983, viii + 333 pp, $48.80. [ISBN: 0-387-12192-7] Proceedings of an August 1982 NATO research workshop held in San Miniato, Italy to link seismic and underwater acoustic and signal analysis with research in pattern recognition and artificial intelligence. LAS

Applications (Engineering), T*(16-17: 1, 2), S*, L. Foundations of Boundary Layer Theory for Momentum, Heat, and Mass Transfer. Joseph A. Schetz. Prentice-Hall, 1984, xxi + 309 pp, $37.95. [ISBN: 0-13-329334-3] For majors in mechanical, aerospace, chemical, civil and ocean engineering with at least one prior course in fluid mechanics. Treats turbulence modelling and the applications of large digital computers to boundary layer problems. Mass transfer is treated in an integrated manner with momentum and heat transfer. Problems, but no answers. About 200 references. JK

Applications (Engineering), T(15-16: 1, 2). Civil Engineering Systems, Analysis and Design. Alan A. Smith, Ernest Hinton, Roland W. Lewis. Wiley, 1983, xii + 473 pp, $31.95 (P). [ISBN: 0-471-90060-5] Intended for civil-engineering undergraduates. Methods of systems analysis and design, including linear, nonlinear and dynamic programming. Network and critical path analysis. Modelling and simulation. Design analysis. Practical, problem-oriented. Appendix with Fortran subroutines for teaching purposes and as aids in solving examples in optimization and simulation. JK

Applications (Engineering), S*(17-18), P*. Optimal Shape Design for Elliptic Systems. Olivier Pironneau. Ser. in Comput. Physics. Springer-Verlag, 1984, xii + 168 pp, $36. [ISBN: 0-387-12069-6] An applications-oriented study of physical systems which can be described by an elliptic partial differential equation in which the optimal shape is found by the minimum of a single criterion function. Approach is that of the French school of applied mathematics. Blend of engineering and mathematics. Numerical algorithms along with questions of existence. For graduate students and researchers. Up-to-date references. JK

Applications (Genetics), P*. Statistical Analysis of DNA Sequence Data. Ed: B.S. Weir. Statistics, V. 47. Dekker, 1983, ix + 255 pp, $45. [ISBN: 0-8247-7032-3] Collection of nine articles written by active researchers in this new and rapidly growing field. Good set of references. RSK

Applications (Medicine), P*. Perspectives in Medical Statistics. Ed: J.F. Bithell, R. Coppi. Academic Pr, 1981, xix + 330 pp, $48. [ISBN: 0-12-102520-9] Proceedings of the European Symposium on Medical Statistics, held in Rome in September, 1980. Provides a survey of recent developments in four main areas: epidemiology, clinical and experimental medicine, clinical trials, and health monitoring and planning. Includes not only most of the papers presented but also key points of the discussions which followed. RSK

Applications (Physics), P. Quantum Theory, Groups, Fields and Particles. Ed: A.O. Barut. Math. Physics Stud. D Reidel Pub, 1983, vii + 334 pp, $45.50. [ISBN: 90-277-1552-1] A collection of expository articles in the areas of quantum theory and deformations, group representations and gauge theories, and particle dynamics. AO

Applications (Physics), P. Nonlinear Wave Processes of Deformation in Solids. J. Engelbrecht. Pitman Pub, 1983, ix + 223 pp, $55. [ISBN: 0-273-08574-3] A mathematical treatment of nonlinear deformation wave propagation in solids. AO

Applications (Physics), P. Nonlinear Deformation Waves. Ed: Uno Nigul, Jüri Engelbrecht. IUTAM Symp. Springer-Verlag, 1983, xvii + 452 pp, $42.50. [ISBN: 0-387-12216-8] Proceedings of an August 1982 conference sponsored in Tallinn, Estonian SSR, USSR, by the International Union of Theoretical and Applied Mechanics. LAS

Applications (Physics), T*(15-16: 1), S, L*. Dynamics. S. Neil Rasband. Wiley, 1983, xi + 272 pp, $32.95. [ISBN: 0-471-87398-5] Classical dynamics from a non-traditional geometric viewpoint with perturbation and Lie algebra techniques. Author presumes courses in linear algebra and mechanics. Emphasis is on the central role of high-speed computers. Sells differential geometry as the "natural" language of mechanics. Stress is on applications rather than rigor. Problems. Selected references. JK

Applications (Physics), S(18), P. Field Theory, A Modern Primer. Pierre Ramond. Frontiers in Physics, No. 51. Benjamin/Cummings, 1981, xviii + 397 pp, (P). [ISBN: 0-8053-7893-6] An introduction to perturbative field theories including regularization methods and renormalization theory. The approach is formal (no proofs) and calculational. AO

Applications (Physics), P. Non-Equilibrium Entropy and Irreversibility. Göran Lindblad. Math. Physics Stud. D Reidel Pub, 1983, ix + 166 pp, $29.50. [ISBN: 90-277-1640-4] Presents results of recent research by the author on the problem of deriving thermodynamics from microscopic dynamics using the concept of available work. AO

Applications (Physics), P. Bifurcation Theory, Mechanics and Physics. Ed: C.P. Bruter, A. Aragnol, A. Lichnerowicz. Math. & Its Applic. D Reidel Pub, 1983, xii + 388 pp, $58. [ISBN: 90-277-1631-5] 101 papers from a colloquium in Luminy, France focused on movement (mechanics) and creation (bifurcation). LAS

Applications (Quantum Theory), T(16-17), S, P, L. Finite Groups and Quantum Theory. D.B. Chesnut. Krieger Pub, 1982, xiii + 254 pp, $25. [ISBN: 0-89874-468-7] A compact and tidy introduction that will feel comfortable to students with a background in quantum mechanics. The mathematical aspects are amplified in great detail; the notation is consistent with that used in beginning courses in abstract algebra. Complete solutions to all exercises. LCL

Reviewers

RJA: Richard J. Allen, St. Olaf; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JD-B: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MT: Michael Tveite, St. Olaf; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

Northeastern Section

The annual fall meeting of the Northeastern Section was held on November 18-19, 1983 at Providence College, Providence, Rhode Island. There were 112 registrants.

Invited Addresses:

"Some Observations on Mathematics and Mathematicians," by Ivan Niven, President of MAA.
"On the Addressing Problem in Loop Switching, Or, How to Embed an Arbitrary Graph in a Squashed Cube," by Henry O. Pollak, Bell Laboratories.

Contributed Papers:

"Teaching Derivatives Through Differentials," by Jack Brin, Western New England College.
"Recent Results Concerning Fermat's Last Theorem," by Eric C. Numela, New England College.
"Data Structures in Basic," by Richard Howland, Dickinson College.
"Generating the Pythagorean Triangles Which Contain a Given Integer," by Jeffrey Smith, University of New Haven.
"Maximum Energy Estimators," by James J. Mathersele, Worcester Polytechnic Institute.
"Triangles With Lattice Point Vertices," by Robert Weldon, New England College.

Panel Discussion:

"Articulation Between Two-Year and Four-Year Colleges," by Joseph Marcus, University of Massachusetts; Robert Yahwin, Springfield Technical Community College; Kerry Grant, Southern Connecticut State University.

Computer Workshops:

"An Introduction to the Concept of a Compiled Language," by Frank Ford, Providence College; Nancy Myers, Bunker Hill Community College.
"An Introduction to Programming in PASCAL," by Mary Russell, Providence College; Nancy Myers, Bunker Hill Community College.
"Data Structures in PASCAL," by Richard Howland, Dickinson College.

Southern California Section

The fall meeting of the Southern California Section was a joint MAA-AMS meeting held at Cal Poly San Luis Obispo on November 11-12. There were about 200 people in attendance.

Invited Addresses:

"Some Peculiarities of Rational and Irrational Numbers," by Ivan Niven, University of Oregon.
"The Newton-Leibniz Controversy," by Judith Grabiner, California State College Dominguez Hills.

Panel Discussion:

"Should There Be An Undergraduate Major in Applied Mathematics," by Courtney Coleman (Moderator), Harvey Mudd College; Robert Borelli, Harvey Mudd College; Jane M. Day, San Jose State University; Edward C. Posner, Jet Propulsion Lab.

The duality between potentials at the nodes of a network and flows through the edges is developed more fully in [6].

References

1. Gilbert Strang, Linear Algebra and Its Applications, Academic Press, New York, 1980.
2. R. T. Rockafellar, Convex Analysis, Princeton University Press, Princeton, 1970.
3. David Luenberger, Optimization by Vector Space Methods, Wiley, New York, 1969.
4. L. R. Ford and D. R. Fulkerson, Flows in Networks, Princeton University Press, Princeton, 1962.
5. Gilbert Strang, Maximal flow through a domain, Mathematical Programming, 26 (1983) 123–143.
6. _____, An Introduction to Applied Mathematics, textbook in preparation.

THE TEACHING OF MATHEMATICS

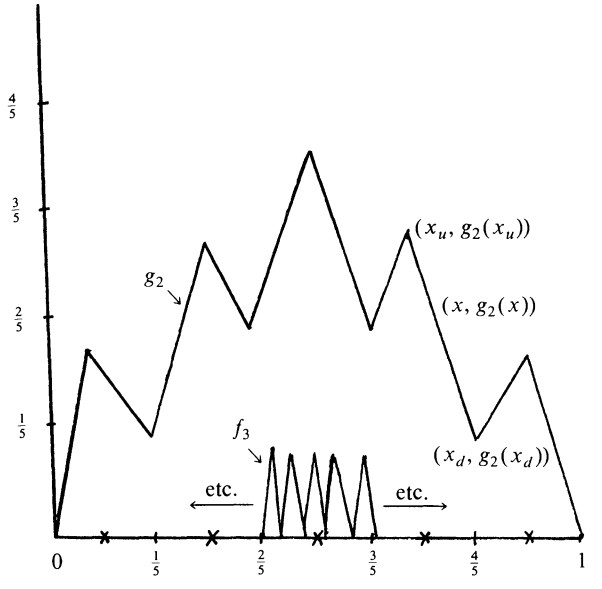
EDITED BY MARY R. WARDROP AND ROBERT F. WARDROP

Material for this department should be sent to Professor Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.

A SIMPLE WEIERSTRASS FUNCTION

DONALD P. MINASSIAN AND JOHN W. GAISSER
Department of Mathematical Sciences, Butler University, Indianapolis, IN 46208

We give perhaps one of the most easily understood constructions of a continuous, nowhere differentiable (even one-sidedly) function g on $[0,1]$. Bright students in the upper sections of elementary calculus should understand the reasoning. Our example is a new variant of one given by van der Waerden in 1930 [1].



Let the graph of $f_1 \equiv g_1$ be the union of the line segments from $(0,0)$ to $(\frac{1}{2}, \frac{1}{2})$ and from $(\frac{1}{2}, \frac{1}{2})$ to $(1,0)$. Let f_2 have five congruent teeth with height of each tooth $(\frac{1}{2})^2$. Let $g_2 \equiv f_1 + f_2$. Continue in a similar manner (see illustration); each f_i has five times as many teeth as f_{i-1} and is only half as high; define

$$g_i \equiv \sum_{j=1}^{i} f_j \equiv g_{i-1} + f_i.$$

Then $g \equiv \lim g_i$ is the desired function.

We first show that g, being the uniform limit of a sequence of continuous functions, is continuous. Fix x in $[0,1]$, and let $\varepsilon > 0$ be given. Choose i large enough so that $\sum_{j=i+1}^{\infty}(\frac{1}{2})^j < \varepsilon/3$. Then

$$|g(y) - g_i(y)| = \left| \sum_{j=i+1}^{\infty} f_j(y) \right| \leqslant \sum_{j=i+1}^{\infty} \left(\frac{1}{2}\right)^j < \varepsilon/3$$

for all y in $[0,1]$. (Hence g_i converges to g uniformly). Since g_i is continuous at x, we may pick $\delta > 0$ such that $|g_i(x') - g_i(x)| < \varepsilon/3$ for all $x' \varepsilon [0,1] \cap (x - \delta, x + \delta)$. Hence

$$\begin{aligned} |g(x') - g(x)| &\leqslant |g(x') - g_i(x')| + |g_i(x') - g_i(x)| + |g_i(x) - g(x)| \\ &\leqslant \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \end{aligned}$$

whenever $x' \in [0,1] \cap (x - \delta, x + \delta)$. So g is continuous.

The rest of this note shows g does not have a one-sided derivative anywhere.

The absolute slope of f_i is $(5/2)^{i-1}$. [Each time we halve the maximum height and quintuple the frequency. It is this combination of halving the height and quintupling the frequency that, as will be seen, makes the slopes of the partial sums below increase without bound. This is what renders this example different from van der Waerden's.] This exceeds the maximum slope of $g_{i-1} \equiv \sum_{j=1}^{i-1} f_j$, which is $\sum_{j=0}^{i-2}(5/2)^j = (2/3)[(5/2)^{i-1} - 1]$. Hence the slope of f_i dominates in forming the slope of $g_i \equiv g_{i-1} + f_i$. So, since the slope of f_i is alternately positive and negative, so is that of g_i—and over the same (equal) intervals on the x-axis (add f_3 to g_2 in the illustration to verify this for $g_3 \equiv g_2 + f_3$). Thus all abscissas for "upcorners" (teeth high points) for f_i are abscissas for "upcorners" for g_i; similarly for "downcorners." Also (see illustration), upcorners for g_{i-1} remain upcorners for g_i except that they get sharper and higher; the same holds for downcorners, except that they don't rise. Of course, g_i has five times as many upcorners as g_{i-1}.

Thus (see illustration) when we add f_3 to g_2 to form g_3, upcorner $(x_u, g_2(x_u))$ rises by at least as much as $(x, g_2(x))$ which rises by at least as much as downcorner $(x_d, g_2(x_d))$, which rises not at all. So, considering slopes of secants, we see that

$$\left| \frac{g_3(x_u) - g_3(x)}{x_u - x} \right| \geqslant \left| \frac{g_2(x_u) - g_2(x)}{x_u - x} \right|$$

which also holds if "d" replaces "u". This continues to hold for the *same* x_u, x, x_d as we increase the index i in g_i, so it holds in the limit g; i.e., secant slopes can only get worse:

LEMMA. *If* $(x_1, g_i(x_1))$ *and* $(x_2, g_i(x_2))$ *are adjacent corners for* <u>any</u> *particular* g_i, *then for* <u>any</u> *point* $(x, g_i(x))$ *on the joining line segment*:

$$(*) \qquad \left| \frac{g(x_1) - g(x)}{x_1 - x} \right| \geqslant \left| \frac{g_i(x_1) - g_i(x)}{x_1 - x} \right|$$

and the same holds if x_2 replaces x_1.

We are nearly done. Suppose for some x in $[0,1]$, g had a right-hand derivative, say L. Then for all b in some δ-interval to the right of x:

$$(**) \qquad L - 1 < \text{secant slope } \frac{g(b) - g(x)}{b - x} < L + 1.$$

Now as i increases, eventually there is g_i and $a \leqslant x < b < x + \delta$ where $A \equiv (a, g_i(a))$ and $B \equiv (b, g_i(b))$ are adjacent corners for g_i on whose joining line segment lies $(x, g_i(x))$. Further, if m denotes slope:

$$|m(\overline{AB})| = |m(\overbrace{f_i + g_{i-1}}^{\equiv g_i})| \text{ for } \overline{AB}| =$$

$$|m(f_i) + m(g_{i-1}) \text{ for } \overline{AB}| \geqslant$$

$$|m(f_i)| - |m(g_{i-1}) \text{ for } \overline{AB}| \geqslant$$

$$|m(f_i)| - \max_x |m(g_{i-1})| = \text{ (see fifth paragraph of this presentation)}$$

$$(5/2)^{i-1} - 2/3\left[(5/2)^{i-1} - 1\right] \approx 1/3(5/2)^{i-1} \to \infty$$

So we could have chosen (and do now choose) i large enough so $|m(\overline{AB})| > |L| + 1$, and by the Lemma (in $(*)$ take $x_1 \equiv b$):

$$\left|\frac{g(b) - g(x)}{b - x}\right| \geqslant |m(\overline{AB})| > |L| + 1,$$

contradicting $(**)$. A similar argument shows that g has no left-hand derivative at any x in $[0,1]$ Q.E.D.

Reference

1. B. L. van der Waerden, Ein einfaches Beispiel einer nichtdifferenzierbaren stetigen Funktion, Math. Z., 32 (1930).

MISCELLANEA

125.

(A) The most absurd science, in my opinion, and the most capable of stifling every sort of talent, is geometry. This ridiculous science deals with surfaces, lines and points, which do not exist in Nature. One can imagine inserting a hundred thousand curves between a circle and its tangent, where in reality one could not insert the thinnest straw.

—Voltaire, *Jeannot et Colin* (1764)

(B) A geometer can prove to you that you can draw infinitely many curves between a circle and a tangent, but not a single line: your eyes and your common sense say the opposite. The geometer will answer solemnly that that's an infinity of second order....

You consult a more honest geometer, who explains the mystery. "We imagine," he says, "things that are impossible in Nature, lines that have length but no width.... No actual curve or line can pass between two actual lines that touch each other: these are just intellectual games, imaginary chimeras. True geometry is the art of measuring things that exist."

—Voltaire, *L'Homme aux quarante écus* (1768)

ANSWERS TO PHOTOS ON PAGE 248

Top, J. E. Littlewood; bottom, G. H. Hardy. The Littlewood picture is from the collection of George Pólya; the Hardy photograph was taken by R. P. Boas in about 1939.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS),
H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by August 31, 1984. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3041. *Proposed by J. Martin Borden, Worcester Polytechnic Institute, and David Mason, University of Delaware.*

For positive arguments define a function R by

$$R(s_1, s_2, \ldots, s_n) = \frac{(s_1 + s_2 + \cdots + s_n)^{(s_1+s_2+\cdots+s_n)}}{s_1^{s_1} s_2^{s_2} \cdots s_n^{s_n}}.$$

(a) Show that $R(s_1, \ldots, s_n)R(t_1, \ldots, t_n) \leqslant R(s_1 + t_1, \ldots, s_n + t_n)$.

(b) Find a necessary and sufficient condition for equality to hold in the inequality of (a).

E 3042. *Proposed by A. M. Fink, Iowa State University, and Max Jodeit, Jr., University of Minnesota.*

Which real-valued functions h defined on $[0, 1]$ can be expressed as $h(x) = f(x) + g(x)$, where $f(x) \geqslant 0$ is nonincreasing, and $g(x) \geqslant 0$ is nondecreasing?

SOLUTIONS OF ELEMENTARY PROBLEMS

Interior Points and Altitudes of a Triangle

E 2885 [1981, 349]. *Proposed by T. Segiguchi, University of Arkansas.*

Let T be a triangle. Construct the set of interior points of T at which the sum of the distances to the sides of T is equal to the arithmetic mean of the lengths of the altitudes of T.

Solution by D. Hammond Smith, University of Exeter, England. The centroid G of T lies in the set since its distance from each side is one third of the corresponding altitude. Now take another

point P inside T, and let $\mathbf{p}$ be its position vector relative to G; and let $\mathbf{d}$, $\mathbf{e}$, and $\mathbf{f}$ be unit vectors in the directions of the three altitudes. Then $\mathbf{p}\cdot\mathbf{d}$, $\mathbf{p}\cdot\mathbf{e}$, and $\mathbf{p}\cdot\mathbf{f}$ measure the differences in the distances of P and G from the three sides of T. P will therefore lie in the required set if, and only if, $\mathbf{p}\cdot(\mathbf{d}+\mathbf{e}+\mathbf{f})=0$. We may thus make the following construction: Let GD, DE, EF be segments of equal length parallel to the three altitudes. Construct the line through G perpendicular to GF. The segment of this line which lies inside T is the required set.

Note that F will coincide with G if, and only if, T is an equilateral triangle, and in this case (as is well known) the required set is the whole interior of T.

Also solved by K. Bernstein, B. Cheng and D. The Hiung, L. Douarte (Spain), R. Dybvik (Norway), H. Eves, Maxim Goldberg, Michael Goldberg, J. Grane and J. Papes (Spain), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), H. Marston, W. Newcomb, I. Paasche (Germany), M. R. Railkar (India), T. Tham, J. Varilly (Costa Rica), P. Zsier, and the proposer.

An Euler ϕ-function Identity in Three Variables

E 2896 [1981, 537]. *Proposed by Stephen M. Gagola, Jr., Texas A & M University.*

Let m be a positive integer. Prove that $\Sigma\phi(abk)\phi(k)=m^2$, where the sum extends over all triples of positive integers (a, b, k) satisfying $abk|m$ and g.c.d. $(a, b)=1$.

I. *Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.* Let $f(m)=\Sigma\phi(abk)\phi(k)$. If $M=mm'$, where $(m, m')=1$ and $ABK|M$, then there is a unique decomposition $A=aa'$, $B=bb'$, $K=kk'$ with $abk|m$, $a'b'k'|m'$. Since ϕ is a multiplicative function, it follows that f is a multiplicative function. Therefore it is sufficient to prove that $f(m)=m^2$ and m is a prime power.

Let $m=p^\alpha$. Then we have

$$f(m)=\sum_{\substack{ab|p^\alpha\\(a,b)=1}}\phi(p^\alpha)+\sum_{i=1}^{\alpha}p^{i-1}(p-1)\left\{p^{i-1}(p-1)+2\sum_{j=1}^{\alpha-i}p^{i+j-1}(p-1)\right\}$$

$$=2p^\alpha-1+\sum_{i=1}^{\alpha}p^{i-1}(p-1)\{2p^\alpha-p^i-p^{i-1}\}=p^{2\alpha}.$$

This proves the assertion.

II. *Solution by K. Yocom, South Dakota State University.* Proceed by induction on the number of prime factors of m. The statement is clearly true for $m=1$. Let m be a positive integer and p be a prime and consider the sum with m replaced by mp. Express mp in the form np^r where $p\nmid n$. Group the triples (a, b, k) into the classes: (i) $p\nmid ab$, $p\nmid k$, (ii) $p|ab$, $p\nmid k$, (iii) $p|ab$, $p|k$, (iv) $p\nmid ab$, $p|k$. Note that if $r=1$, then class (iii) is empty. Group the terms of the sum accordingly, make use of the fact that ϕ is multiplicative, and use the induction hypothesis to express the sum as

$$n^2+2n^2\sum_{i=1}^{r}\phi(p^i)+2n^2\sum_{i=1}^{r-1}\sum_{j=1}^{r-1}\phi(p^{i+j})\phi(p_j)+n^2\sum_{i=1}^{r}\phi(p^i)^2.$$

Note that the double sum is zero if $r=1$. Now use $\phi(p^i)=p^i-p^{i-1}$ and simplify the expression to get n^2p^{2r} which completes the induction.

III. *Solution by M. Barr, McGill University, Montreal, Quebec, Canada.* First observe that when $\gcd(a, b)=1$, $\phi(abk)\phi(k)=\phi(ak)\phi(bk)$. This follows from

$$\phi(n)=n\prod_{p|n}\left(1-\frac{1}{p}\right)$$

by considering the possibilities of primes dividing a, b and k. Next observe that when $\gcd(a, b) = 1$, $abk = \operatorname{lcm}(ak, bk)$ so that $ak|m$ and $bk|m$ if and only if $abk|m$. Thus $(a, b, k) \mapsto (ak, bk)$ describes a correspondence between triples as described in the statement and pairs of divisors. The inverse is $(c, d) \mapsto (c/k, d/k, k)$ where $k = \gcd(c, d)$. Then

$$\sum_{(a,b,k)} \phi(abk)\phi(k) = \sum_{(a,b,k)} \phi(ak)\phi(bk) = \sum_{c|m,\, d|m} \phi(c)\phi(d)$$

$$= \sum_{c|m} \phi(c) \sum_{d|m} \phi(d) = m^2.$$

The proposer notes that a group-theoretic proof is possible, based on the following Lemma: Let $Y = G \times H$ be the direct product of two cyclic groups G and H, each of order k. Then the number of simultaneous complements for G and H in Y is $\phi(k)$.

Also solved by 29 other readers and the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be typed and mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by August 31, 1984. The solver's full post-office address should be on each sheet.

6457. *Proposed by R. L. McFarland, Yellow Springs, Ohio.*

Let p be an odd prime. Show that the $2^{(p-1)/2}$ numbers of the form $\pm 1 \pm 2 \pm \cdots \pm \frac{1}{2}(p-1)$ represent each nonzero residue class modulo p the same number of times. Determine this common number of representations and show that it differs by one from the number of representations of the zero residue class.

6458. *Proposed by Mark Haiman and David Richman, MIT.*

Determine the constant term $f(p)$ of

$$(x + y + x^{-1} + y^{-1})^p,$$

where x and y are noncommutative variables.

SOLUTIONS OF ADVANCED PROBLEMS

Series of Sines and Cosines

6299 [1980, 408]. *Proposed by Roger L. Cooke, University of Vermont.*

(a) The functions

$$f(x) = \sum_{n=1}^{\infty} n^{-1}\sin(2^{-n}x) \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} (-1)^n n^{-1}\sin(2^{-n}x)$$

are obviously uniformly continuous, real-analytic functions on the line. Is either function bounded?

(b) What can be said about $h(x) = \sum_{n=1}^{\infty}(-1)^n n^{-1}\cos(2^{-n}x)$?

Solution by David Borwein, The University of Western Ontario, Canada, and Peter Borwein, Dalhousie University, Canada. Since

$$h(x) = -\log 2 - 2\sum_{n=1}^{\infty} (-1)^n n^{-1}\sin^2(2^{-n-1}x),$$

this function is also uniformly continuous and real-analytic on the line. We show that all three functions are unbounded. Let

$$\alpha = 2\pi/15;\ x_k = 2^k\alpha \quad \text{for} \quad k = 0,1,\ldots.$$

Then

$$\sin x_k = \begin{cases} \sin\alpha & \text{when } k \equiv 0 \pmod 4, \\ \sin 2\alpha & \text{when } k \equiv 1 \pmod 4, \\ \sin 4\alpha & \text{when } k \equiv 2 \pmod 4, \\ -\sin(\alpha/2) & \text{when } k \equiv 3 \pmod 4. \end{cases}$$

Also $f(x_m) = S_m + R_m$ where

$$S_m = \sum_{n=1}^{m} n^{-1}\sin(x_{m-n}), \quad R_m = \sum_{n=m+1}^{\infty} n^{-1}\sin(2^{-n}x_m).$$

Now

$$|R_m| \leqslant x_m \sum_{n=m+1}^{\infty} n^{-1}2^{-n} \leqslant m^{-1}x_m 2^{-m} = \alpha m^{-1},$$

and, as $m \to \infty$,

$$|S_m| \sim \frac{1}{4}|\sin\alpha + \sin 2\alpha + \sin 4\alpha - \sin(\alpha/2)|\log m.$$

It follows that $|f(x_m)| \to \infty$ as $m \to \infty$, and, similarly, that $|g(x_m)| \to \infty$ and $|h(x_m)| \to \infty$ as $m \to \infty$.

Note. The continuity, analyticity and unboundedness properties persist if the factor n^{-1} in each of the sums is replaced by d_n, where $d_n \geqslant d_{n+1} > 0$ for $n = 1,2,\ldots, d_n \to 0$ and $\sum_{n=1}^{\infty} d_n = \infty$.

Part (a) was also solved by the proposer.

Linear Orderings of Finite Sets in R^n

6394 [1982, 502]. *Proposed by Jan Mycielski and Andrzej Ehrenfeucht, University of Colorado, Boulder.*

Let n and $m \geqslant n + 1$ be given. Let $X \subseteq R^n$ with $|X| = m$ be a set in general position, i.e., if $Y \subseteq X$ and $|Y| \leqslant n + 1$, then Y spans a $(|Y| - 1)$-dimensional hyperplane and no two such hyperplanes are parallel to each other if, of the corresponding two Y's, neither is a subset of the other. Let $f_n(m)$ be the number of linear orderings of X which can be obtained by a perpendicular projection of X into any directed line which is in general position relative to X.

(a) Prove that $f_2(m) = 2\binom{m}{2}$ and $f_3(m) = 6\binom{m}{4} + 4\binom{m}{3} + 2$.

(b) Is $f_n(m)$ well defined (i.e., the same for all X in general position) for $n \geqslant 4$, and, if so, can it be evaluated?

Solution by W. J. Gilbert, University of Waterloo, Canada, and A. Mandel, University of Sao Paulo, Brazil. We give a direct proof of (a) and examples to show that $f_4(6)$ is not well defined.

The projections of X onto parallel lines directed in the same way give the same ordering so, for each parallel class, we choose the representation through the origin parameterized as $l_c(\lambda) = \{\lambda c | \lambda \in R\}$, where $c \in S^{n-1}$, the unit $(n-1)$-sphere in R^n. Let $X = \{x^1,\ldots, x^m\}$. The projection of the point x^i onto the line l_c has parameter $\lambda = x^i \cdot c$; hence x^i precedes x^j in the ordering on l_c if and only if $(x^i - x^j) \cdot c < 0$. For $1 \leqslant i < j \leqslant m$, let $h(i, j)$ be the $(n-1)$-dimensional

hyperplane $(x^i - x^j) \cdot x = 0$ which passes through the origin and is perpendicular to the line joining x^i to x^j. The side of $h(i, j)$ where c lies determines whether x^i precedes or follows x^j in the ordering on l_c. Therefore the number of distinct orderings of X equals the number of full dimensional regions that the collection of hyperplanes $\{h(i, j)\}$ divides R^n, or equivalently, divides S^{n-1}.

For $n = 2$, the general position of X just implies that all the $h(i, j)$ are distinct. There are $\binom{m}{2}$ of these, each intersecting S^1 in a pair of antipodal points. Hence S^1 is divided into $2\binom{m}{2}$ arcs and $f_2(m) = 2\binom{m}{2}$.

For $n = 3$, the general position of X implies that the planes $h(i, j)$ are distinct and the only triples of $h(i, j)$ that have a one dimensional intersection are those of the form $\{h(i, j), h(i, k), h(j, k)\}$, where $1 \leqslant i < j < k \leqslant m$. Look at the complex defined by the intersection of the planes $h(i, j)$ on the sphere S^2. Let v_{2r} be the number of vertices of degree $2r$; each such vertex and its antipode lie on the one dimensional intersection of r of the planes $h(i, j)$. Hence v_6 is twice the number of triples $\{i, j, k\}$ with $1 \leqslant i < j < k \leqslant m$ and $v_6 = 2\binom{m}{3}$. Now v_4 is twice the number of pairs $\{\{i, j\}, \{k, l\}\}$ with i, j, k, l distinct and $v_4 = 6\binom{m}{4}$. The number of vertices of the complex is $v_4 + v_6$ and the number of edges, e, is half the total degree, so $e = (4v_4 + 6v_6)/2$. By Euler's formula on the sphere, the number of faces is

$$f_3(m) = e - (v_4 + v_6) + 2 = v_4 + 2v_6 + 2 = 6\binom{m}{4} + 4\binom{m}{3} + 2.$$

To show that $f_4(6)$ is not well defined consider, for example, the following matrices, identified with their column sets:

$$X_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \quad \text{and} \quad X_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 12 \\ 0 & 0 & 1 & 0 & 0 & 8 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -4 \end{pmatrix}.$$

Both X_1 and X_2 are in general position, but the above methods and Euler's formula on S^3 can be used to show that the number of orderings for X_1 is 480 while the number for X_2 is 472.

Part (a) was also solved by Pei Yuan Wu (Republic of China) and the proposers.

An Inequality

6403 [1982, 703]. *Proposed by J. L. Brenner, Palo Alto, California.*

Let $0 < r, s$. For positive reals, $A, B, A \neq B$, define

$$M_1 = \left[\frac{1}{2}(A^rB^s + A^sB^r)\right]^{1/r+s},$$

$$M_2 = \left[\frac{1}{2}(A^{rs} + B^{rs})\right]^{1/rs}.$$

Since $M_1 < M_2$ as $B \to \infty$, either $M_1 < M_2$ for all A, B or else M_1, M_2 are incomparable. Which is the case?

Solution by L. E. Mattics, University of South Alabama, Mobile, Alabama. Without loss of generality, suppose $B > A$, $r \geqslant s > 0$, and set $\theta = r/s$. We shall show that if $s \geqslant (\theta - 1)^2/(\theta(\theta + 1))$ (i.e., if $(r - s)^2 \leqslant rs(r + s)$), then $M_2 > M_1$, otherwise M_1 and M_2 are incomparable.

Set $x = B/A$ and $H(x) = \log(M_2/M_1)$. Then $H(1) = 0$ and

$$xH'(x) = \frac{r - s}{r + s}\frac{1}{1 + x^{r-s}} + \frac{s}{r + s} - \frac{1}{1 + x^{rs}},$$

so that $H'(1) = 0$ and $4H''(1) = rs - (r - s)^2/(r + s)$. Consequently, if $s < (\theta - 1)^2/(\theta(\theta + 1))$,

then $H''(1) < 0$ and so M_1 and M_2 are incomparable.

Now suppose that $s \geqslant (\theta - 1)^2/(\theta(\theta + 1))$ and set $\alpha = (\theta - 1)/(\theta + 1)$ and $u = x^{s(\theta-1)}$. Then

$$xH'(x) \geqslant G(u) = \frac{\alpha}{1+u} + \frac{1}{1+\theta} - \frac{1}{1+u^{\alpha}}.$$

To finish, it suffices to show that $H'(x) > 0$ for $x > 1$. This is certainly the case if $\theta = 1$, so we assume that $\theta > 1$. Then $G(1) = 0$ and, for $u > 1$, $G'(u)$ is a positive (nonconstant) multiple of

$$-u^{2\alpha} + (u + u^{-1})u^{\alpha} - 1 = (u - u^{\alpha})(u^{\alpha} - u^{-1}) > 0.$$

Hence $G(u) > 0$ for $u > 1$, and the desired conclusion follows.

Also solved by I. E. Leonard, William A. Newcomb.

Partially solved by Robert Breusch, C. S. Karuppan Chetty (India), Walther Janous (Austria), O. P. Lossers (The Netherlands), V. D. Mascioni (Switzerland), A. Meir (Canada), Bruce Reznick and Robert E. Shafer, and the proposer.

On Dirichlet's L-function

6404 [1982, 703]. *Proposed by Daniel Shanks, University of Maryland.*

The imaginary quadratic field $Q(\sqrt{-2 \cdot 5 \cdot 2347 \cdot 10513})$ has

$$L(1, \chi) = 0.5008000001\ldots,$$

where, in

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) \cdot n^{-s},$$

$\chi(n) = (d/n)$ is the Kronecker symbol and d is the discriminant. Show that one can find imaginary quadratic fields where the decimal value of $L(1, \chi)$ contains a string of zeros that is as long as one wishes.

Solved by the proposer. The example in the statement of the problem gives a generous hint for constructing a general solution. This quadratic field has the discriminant

$$d = -4 \cdot 2 \cdot 5 \cdot 2347 \cdot 10513 = -986960440 = -[\pi^2 \cdot 10^8],$$

and since $\sqrt{-d}$ therefore equals $\pi \cdot 10^4$ to nine decimals, while the class number h of this field is 5008, we have

$$L(1, \chi) = \frac{\pi h}{\sqrt{-d}} = 0.5008000001\ldots$$

with its string of five zeros. Note two points:

(A) Since

$$[\pi^2 \cdot 10^8] < \pi^2 \cdot 10^8,$$

we have a string of zeros whereas an equally accurate

$$-d > \pi^2 \cdot 10^8$$

would give us a string of nines instead.

(B) The construction works because Siegel's theorem:

$$h(d) = O\left(|d|^{\frac{1}{2}+\varepsilon}\right)$$

tells us that the class number h has, or nearly has, one-half of the decimal digits that $-d$ has, as $|d|$ becomes arbitrarily large.

Let

$$N_n = [\pi^2 \cdot 10^{2n}],$$

with $n = 4$ being our previous example. There are four cases:

(1) If $-N_n$ is a fundamental discriminant, as it is for $n = 4$, let our nth field be $Q(\sqrt{-N_n})$. Then its $L(1, \chi)$ has a string of n, or nearly n, zeros. This case occurs for $3/\pi^2$ or 30.4% of all n.

(2) If N_n is square-free and $\equiv 1, 2, 5$, or 6 (mod 8), $-4N_n$ is a fundamental discriminant and we again select $Q(\sqrt{-N_n})$. In these cases, h is always even and so

$$L(1, \chi) = \frac{\pi h/2}{\sqrt{[\pi^2 \cdot 10^{2n}]}}$$

as before. This occurs in $4/\pi^2$ or 40.5% of all n.

(3) In the 29.1% of the remaining n, if n is sufficiently small for convenient computation, we examine $N_n - 1, N_n - 2, \ldots$ until we find an $N_n - k$ in case (1) and (2). We then select $Q(\sqrt{-(N_n - k)})$ and the string is only slightly shortened, if at all.

(4) If n is too large in case (3) for explicit computation, we must know the maximal value that the displacement k can attain. Hooley [1] gives the rather weak result that the maximal distance between successive square-free numbers around x is

$$O(x^{\frac{2}{9}+\varepsilon}).$$

Using this, one readily finds that, in this worst possible case,

$$\frac{\pi}{\sqrt{-d}} h \approx h \cdot 10^{-n}$$

will *not* be correct to $2n$ decimal digits but only to $\frac{14}{9}n$ decimal digits. However, even this worst case gives a string of zeros that is as long as one wishes since n can be arbitrarily large.

Reference

1. C. Hooley, On the intervals between consecutive terms of sequences, Proc. Symp. Pure. Math., vol. 24, American Mathematical Society, 1973, pp. 129–140, esp. p. 138.

On Hardy's Inequality

6406 [1982, 703]. *Proposed by J. Michael Steele, Princeton University.*

Show that if $\sum_{n=0}^{\infty} a_n e^{int} = f(t)$ and $f(t) \in L^1[0, 2\pi]$, then there exist integers n_k such that

$$n_k \to \infty, n_{k+1}/n_k \to 1, \quad \text{and} \quad \sum_{k=1}^{\infty} |a_{n_k}| < \infty.$$

Solution by the proposer. By Hardy's inequality for measures of analytic type (*Math. Ann.*, 97 (1927) 159–209), we have

$$\sum_{n=0}^{\infty} |a_n|/(n+1) \leqslant \pi \int_0^{2\pi} |f(t)| dt.$$

We can thus choose a strictly increasing unbounded sequence $\{\lambda_n\}$ such that $\lambda_1 > 1$ and $\sum_{n=1}^{\infty} |a_n| \lambda_n / n < \infty$. Next, define a strictly increasing sequence of integers $\{t_k\}$ by taking $t_1 = 1$ and letting t_{k+1} be the smallest integer $t > t_k/(1 - 1/\lambda_k)$. Then $t_{k+1}/t_k \to 1$, and

$$\sum_{n=t_k+1}^{t_{k+1}} \lambda_n |a_n|/n \geqslant \frac{\lambda_{t_k}}{t_{k+1}} \sum_{n=t_k+1}^{t_{k+1}} |a_n| \geqslant \frac{\lambda_{t_k}}{t_{k+1}} (t_{k+1} - t_k) \min_{t_k \leqslant n \leqslant t_{k+1}} |a_n|.$$

By the choice of t_k, $\lambda_{t_k}(t_{k+1} - t_k)/t_{k+1} \geqslant 1$; so choosing n_k to be an integer such that

$$t_k \leqslant n_k \leqslant t_{k+1} \quad \text{and} \quad |a_{n_k}| = \min_{t_k \leqslant n \leqslant t_{k+1}} |a_n|,$$

we have $n_{k+1}/n_k \to 1$ and $\Sigma_{k=1}^{\infty}|a_{n_k}| < \infty$.

Also solved by Russell Lyons.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

The Making of Statisticians. Edited by J. Gani. Springer-Verlag, New York, 1982. vii + 263 pp.

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Statisticians do not have a good public image. I meet a stranger, we chat a little and eventually get around to our occupations. "I am a professor." My stock usually, though not always, rises. "What is your subject?" "Statistics." The drop in esteem can almost be heard in the embarrassing silence. Often my new acquaintance has been forced to take a course of statistics in college and the painful experience has remained. This happened once with a ranger in the National Parks Service, and I can only attribute his bad, and nearly disastrous, advice given after this type of conversation to a desire to get some revenge on this terrible subject.

Statisticians are human. When I was head of the department of statistics at University College London, where Egon Pearson had occupied the same post 15 years earlier, I remarked, after proving the Neyman-Pearson lemma, that the Pearson here was the gentleman the students often saw in the department. They were surprised: even lemmas have authors. "Who was Fatou?" one of them said.

In editing this collection of essays by prominent, elderly statisticians and probabilists, Joe Gani has done much to establish the humanity but has done little for the image. None of them make statistics sound the exciting subject it really is. They go around the world taking their colour slides but rarely illuminating it. An exception is Pitman, the doyen, born 1897, who writes as beautifully as ever. It is all a bit dull. Not that the authors are dull, for some of them are known to me personally as lively companions, but their text seldom sparkles. Just imagine the result if Evelyn Waugh had been asked to "sketch his life, explain how he had become interested in his subject, give an account of his major contributions, and look to the future." The tale would have been far from dull.

The point is of some importance. Let us face it: statisticians (and scientists generally) do not write as well as their literary colleagues. I do not mean that scientific writing is bad: often it is well-suited to its purpose. Rather it is their writing for the nonspecialist that is typically uninteresting. This is most unfortunate when what they have to say is so important and when they have so much information to impart; unlike their literary colleagues whose information is scanty or irrelevant to the conduct of human affairs but whose presentation is so lively. This inability of scientists to capture the public's interest has serious consequences for our society because, as a result, society fails to attach the proper importance to science. Each year the London *Observer* asks about twenty people to say which three books they have most enjoyed that year. Historians, actors, novelists, politicians, even pop stars will be asked, but never a scientist. As a result books that are informative (in the sense of providing material that enables you to do something) are

rarely displayed. An exception occurred a few years ago when Morris Kline's admirable *Mathematics: The Loss of Certainty* appeared. The person who had enjoyed that was Tom Stoppard, but then there is even probability theory in "Rosencrantz and Guildenstern are Dead." He was never asked again. "Really, Mr. Stoppard, suggesting a mathematics book. Our readers don't want that sort of material."

I do not wish to give the impression that this is not an interesting book to read. It could be brighter, but we learn a lot about the effect of the depression on their lives and of the pressure that led four of them to leave Europe for the United States where some of them initially had strange jobs. We hear too about people who influenced them. R. A. Fisher, the creator of modern statistics, emerges as a colleague who was very generous with ideas but who could not abide any criticism: strange in a man whose ideas could withstand criticism more than most. I do not know whether there is a statistical personality or whether it is I, as a statistician looking at them, but they do emerge as pleasant people, kindly disposed towards the world around them and grateful to it for the pleasures and honours it has bestowed.

What surprised me was the omission of some topics. For example: only Maurice Bartlett has the sense (and the courage?) to discuss refereeing, when he raises an old sore. The refereeing process plays such a dominant role in academic life that it deserves airing. Do others feel with me that it is not what it was? A quarter of a century ago reports on my papers were well-argued, factually correct and most important, recommended acceptance. Nowadays they are often persnickety, contain errors and demand extensive rewriting. Of course, I may have changed, but two experiences from the same, reputable journal lead me to think that the process has altered more than I. One paper, not mine, was kept six months, presumably hastily read by the referee, because his report was wrong, and then judged by the editor to be on an unimportant topic when it concerned a technique that is used (wrongly) throughout experimental science.

The second concerned a paper of mine which I had written and then, laying aside for a week to be able to look at afresh, had read and found much too long for the material. It was therefore written at half the length—a difficult task since as Churchill said "I am sorry this letter is so long, I had not time to write a short one." The referee and the editor then asked for further cuts without indicating where or how they could be made. A few months later, I had a splendid referee's report on another paper which was severely and constructively critical. The fault is perhaps not with me.

Another topic they say little about is their work styles. Many thank their wives for tolerating an unreasonable devotion to statistics (and none mentions a divorce) but they do not tell us much about the devotion. Wold is an exception, discussing serendipity. How did they obtain their original ideas? Do they work long hours? How do they write their papers, and do they enjoy doing it? Advice to young statisticians would have been welcome.

The most surprising omission, since Gani asked for it, is much discussion of the future. On reflection, I do not think old people (the youngest here was born in 1917) make good soothsayers about science. The reason is that they do not well understand the present, let alone the future. Science, especially mathematics, is a young man's game and it nowadays moves so fast that even the middle-aged have trouble in keeping up. This is illustrated in the present volume when several contributors mention how if, when they were young, they had known about some work they would have advanced more quickly. Yet none mention the same phenomenon in their later work. Indeed one, in discussing invariant estimators, asks "does anyone in practice use any others?" In fact, actuaries and psychometricians, to mention two groups, always use estimators that lack invariance and one would be critical if they adopted the standard statistical practice. Keeping up with the literature and refereeing it are two problems that we have not been able to solve. Only the adherent of the future paradigm can accurately foretell the future: and how do we recognize him? Perhaps they were wise to respond so little to the editor's request.

Statistics may not emerge as an exciting discipline, but it is clearly a varied one. Lancaster discusses medicine, Benjamin public health (and other things from this perpetual retirer), Tippett is in cotton, Finney is in agriculture, Bose enjoys the fame that came from discovering orthogonal

Latin squares with side 22, others enjoy their mathematics and de Finetti forecasts the results in Italian football. Why did the Park Ranger's course have to be so dull?

Gani has done us a service in letting us know something of the statistical profession. It is a pity that the editor of the *Observer* won't ask me: it will probably be in my best three of '83.

An Introduction to Applied Optimal Control. By Greg Knowles. Academic Press, New York, 1981. x + 180 pp.

An Introduction to Optimal Control Theory. By Jack Macki and Aaron Strauss. Springer-Verlag, New York, 1982. xiv + 165 pp.

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Although optimal control theory can be viewed as an extension of the calculus of variations, it developed in response to a variety of applied problems which were often recognized as variational in nature, but which did not quite fit into the framework of the then existing theory. One of the applied problems that stimulated the development of the theory was the following. A control surface on an aircraft must be kept at some desired position. External disturbances, such as wind gusts, occur infrequently and displace the control surface from the desired position. In the absence of any restoring torque, the surface would execute damped harmonic motion, which would be highly undesirable. A direct current electric motor, whose time constant is long relative to the length of action of the disturbance, is used to apply a torque to bring the control surface to rest in the desired position. Only the armature voltage v into the motor can be controlled. Let the desired position be the zero angle and let θ denote the angle of the control surface measured from the desired angle. Then with a suitable normalization the differential equation for θ can be written as

$$\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} + w^2\theta = u \quad \theta(0) = \theta_0 \quad \theta'(0) = \theta_0'.$$

Here u represents the restoring torque, θ_0 represents the displacement at the time the torque is first applied and θ_0' the angular velocity of the surface at this time. Since the voltage source cannot deliver a voltage that is larger in absolute value than some v_0, the magnitude of the restoring torque must be bounded in absolute value. Hence we must have $|u(t)| \leqslant A$ for some $A > 0$.

If we set $x^1 = \theta$ and $x^2 = d\theta/dt$, we can rewrite the differential equation of the motion as follows.

$$(*) \qquad \begin{aligned} dx^1/dt &= x^2 & x^2(0) &= \theta_0 \\ dx^2/dt &= -ax^2 - w^2x^1 + u & x^2(0) &= \theta_0'. \end{aligned}$$

Given that a short disturbance has resulted in a deviation $\theta = \theta_0$ and $d\theta/dt = \theta_0'$ from rest at the desired position, how should the voltage be applied over time so that the control surface is brought back to the set position ($\theta = 0$) and at rest ($d\theta/dt = 0$) in the shortest possible time? In terms of the system of differential equations (*), the problem is to choose a piecewise continuous function u, such that $|u(t)| \leqslant A$ and such that the solution $(x^1(t), x^2(t))$ of (*) corresponding to $u(t)$ reaches the origin in (x^1, x^2) space in minimum time.

The servo problem which we have just described is a special case of the following general optimal control problem. At each instant of time t the state of a system can be described by an n-vector $x(t) = (x^1(t),\ldots, x^n(t))$. The state is determined by a system of differential equations $dx/dt = f(t, x, u(t))$ and initial conditions $x(t_0) = x_0$. Here $f = (f^1,\ldots, f^n)$ is a vector valued function of (t, x, z), where t is time, x is an n-vector and z is an m-vector. The function u is called

the control or program. It is a function of time, has values in R^m and is chosen from a prescribed class of functions. The value $u(t)$ of u at time t is also required to satisfy a constraint of the form $u(t) \in \Omega(t)$, where for each t, $\Omega(t)$ is a set in R^m. In the servo problem $m = 1$ and $\Omega(t)$ is the interval $[-A, A]$. Under suitable hypotheses on f, once u is chosen and the initial condition $x(t_0) = x_0$ is specified, the state of the system at time t is determined for all $t \geqslant t_0$. The problem of optimal control is to choose the control function u so as to bring the system from the initial state x_0 at time t_0 to a preassigned terminal state at time t_1 or to one of a preassigned collection of terminal states and times $\{(t_1, x_1)\}$, called the target set, in such a way as to minimize a functional of the form

$$g(t_1, x_1) + \int_{t_0}^{t_1} f^0(t, x(t), u(t))\, dt.$$

Here g is a function defined on the target set and f^0 is a scalar function of (t, x, z) evaluated along $(x(t), u(t))$, where $x(t)$ is the solution of the differential equation corresponding to the chosen u. In the servo problem the target set in (t, x^1, x^2)-space is the set $x_1^1 = 0$, $x_1^2 = 0$ and t_1 arbitrary. The functional to be minimized can be obtained either by setting $g = t_1 - t_0, f^0 \equiv 0$ or by setting $g \equiv 0, f^0 \equiv 1$.

In addition to these problems there are problems with stochastic aspects and problems in which the state is governed by partial differential equations. Neither of the books under review treats stochastic control. Knowles' book has one chapter devoted to control by partial differential equations.

For $\Omega(t) = \Omega$, a fixed open set independent of t, for $f(t, x, z) \equiv z$ and for (t_1, x_1) fixed, the control problem reduces to the simple problem in the calculus of variations. For $\Omega(t) = \Omega$, a fixed open set independent of t, the control problem reduces to the classical problem in the calculus of variations with differential equation side conditions. The great interest and growth of optimal control theory in the past twenty-five years, however, did not result from this relationship with an important area of mathematics; it came because the mathematical formulation of the optimal control problem encompasses special problems that arise in many branches of engineering, in economics, and in resource and therapy management.

The principal questions associated with a control problem are the following. First, do there exist controls u in the set of allowable controls such that the target set will be attained? This is the controllability problem. If the answer to the first question is affirmative, does there exist a u^* that minimizes the functional? This is the existence problem. Finally, if the problem has a solution, find necessary conditions that will be useful in actually finding the solution or in characterizing the solution qualitatively in a useful way. For the person in an applied area who has a large scale problem to solve, there is the additional practical problem of getting a good numerical approximation to the solution.

The mathematical attack on the first three problems draws upon results and techniques from many areas of mathematics, especially linear algebra, ordinary differential equations, integration theory, functional analysis and convexity. Optimal control theory is an excellent argument for the proposition that basic mathematics is important for the full understanding of applications. While this is comforting to mathematicians, it poses problems for students in applied areas, whose interest in optimal control theory is primarily as a tool to solve problems. Many of them just want to understand what the theorem says and how to use it to get a number or numbers. They are willing to accept a heuristic argument or a mathematician's certification that the theorem is correct. This, of course, is not true of all workers in applied areas. Some have learned the mathematics and have made important theoretical as well as practical contributions.

The wide range of interest and background of the potential audience makes the writing of a text or monograph on optimal control theory a tricky business. At the mathematical end of the spectrum we find authors who assume—or summarize—the necessary mathematical background and proceed to use whatever tools they feel are most efficient and elegant to achieve their

development of the theory. At the applied discipline end, we find authors who emphasize the more formal aspects of the subject. They give heuristic arguments for some results, and state others without proof. The emphasis is on working out examples.

The book by Macki and Strauss is closer to the mathematical end of the spectrum. According to the preface it is addressed to "... mathematicians, graduate students and advanced undergraduates who want a concise introduction to a field of mathematics which contains nontrivial interesting applications of mathematics..." and to persons in applied areas "... who want some understanding of the mathematical foundations of optimal control." If one interprets "advanced undergraduate" as a student who has mathematical maturity and has had a first course in integration theory, a first course in functional analysis, and a very good course in linear algebra, then the authors have succeeded in their objective.

The book is a readable one. The principal results are highlighted and are clearly stated. Examples are given to illustrate and explain the statements of theorems. The authors do not strive for the greatest generality and refer the interested reader to other sources for more detailed, deeper, or alternate treatments. Readers who are not interested in plowing through all of the proofs will get a good feel for the results even if they omit the more involved and difficult proofs.

The book by Knowles, as the title suggests, is closer to the applied discipline end of the spectrum. Knowles is interested in getting to applied problems rather than in illustrating the applications of pure mathematics. He discusses more topics, but less deeply, than do Macki and Strauss. More examples are given and the "how to" is emphasized more than the mathematical foundations.

This reviewer believes that Knowles would be a good text for the purpose stated in the preface, i.e., "... to give an introduction to deterministic control theory to senior level undergraduates and first year graduate students from mathematics, engineering, and business schools,...," subject to the following reservation. The course should be taught by someone knowledgeable in the field rather than by someone who is learning the material himself from this book.

Cohomology of Groups. By Kenneth S. Brown. Graduate Texts in Mathematics No. 87, Springer-Verlag, New York, 1982. x + 306 pp.

JOSEPH ROTMAN
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Cohomology of groups sounds like a mixture of group theory and algebraic topology, and that is precisely what it is. Readers of this review are familiar with the subject matter of group theory, but let me give a brief description of algebraic topology. This branch of mathematics studies topological spaces and continuous functions by investigating algebraic systems (and their homomorphisms) associated to them. For example, the *fundamental group* is a (not necessarily abelian) group $\pi_1(X)$ defined for every topological space X; moreover, each continuous function $f\colon X \to Y$ determines a homomorphism $f_*\colon \pi_1(X) \to \pi_1(Y)$ (π_1 is a *functor*: if f is an identity function, so is f_*; if $g\colon Y \to Z$ is a continuous function, then $(g \circ f)_* = g_* \circ f_*$). As suggested by the notation, π_1 is the first of a sequence of functors, the *homotopy groups* π_n [$\pi_n(X)$ is an abelian group for every $n \geqslant 2$ and it describes the continuous functions from the n-sphere into X]. There is a second important sequence of functors, the *homology groups* H_n [$H_n(X)$ is an abelian group for every $n \geqslant 0$ that describes n-dimensional "holes" in X: what are the $(n+1)$-dimensional polyhedra whose boundaries lie in X but whose interiors do not]. Some algebraic tinkering, akin to forming dual spaces of vector spaces, transforms the homology groups $H_n(X)$ into *cohomology groups* $H^n(X)$. It is possible to equip the direct sum $H^*(X) = \sum_{n \geqslant 0} H^n(X)$ with a multiplication (cup

product) making it a ring; thus cohomology gives a ring-valued functor. Algebraic topology uses these (and similar) constructions to convert topological problems to algebraic ones.

Instead of starting with a space X and considering $\pi_n(X)$, $H_n(X)$, and $H^*(X)$, can one start with a group G, construct a space X with $\pi_1(X) \cong G$, and use X to obtain information about G? This reversal of viewpoint was made quite early. For example, Nielsen proved in 1921 that every subgroup of a finitely generated free group is itself free (in 1927, Schreier removed the finiteness hypothesis). In 1936, Baer and Levi gave a "topological proof" of the Nielsen–Schreier theorem using fundamental groups of "covering spaces."

The introduction to *Cohomology of Groups* tells what happened next. In 1936, Hurewicz considered *aspherical* spaces X, i.e., spaces whose homotopy groups $\pi_n(X)$ are trivial for all $n \geqslant 2$; he showed that two such spaces with the same fundamental group G are essentially the same (they have the same homotopy type) and so they have the same homology and cohomology groups. One is thus entitled to write $H_n(G)$ and $H^n(G)$ instead of $H_n(X)$ and $H^n(X)$ for such aspherical spaces X. Also, Hurewicz showed that $H_1(G) \cong G/G'$, where G' is the commutator subgroup of G. In 1942, H. Hopf showed $H_2(G)$ is isomorphic to the *multiplier* of G, a group introduced by Schur in 1904 in his study of representations of groups; moreover, Hopf obtained a description of this group in terms of presentations of G by generators and relations. The introduction continues the story. "Following Hopf's paper there was a rapid development of the subject by Eckmann, Eilenberg–Mac Lane, Freudenthal, and Hopf. In particular, one had by the mid-1940's a purely algebraic definition of group homology and cohomology, from which it became clear that the subject was of interest to algebraists as well as topologists. Indeed, the low-dimensional cohomology groups were seen to coincide with groups which had been introduced much earlier in connection with various algebraic problems. H^1, for instance, consists of equivalence classes of "crossed homomorphisms" or "derivations." And H^2 consists of equivalence classes of "factor sets," the study of which goes back to Schur [1904], Schreier [1926] and Brauer [1926]. Even H^3 had appeared in an algebraic context (Teichmuller [1940])." Actually, these "earlier" groups are what the contemporary student of group theory sees when he or she first encounters group extensions and the Schur-Zassenhaus theorem.

The book under review is a fine introduction to the subject whose prehistory and birth have just been recounted. It is an even-handed approach: one looks at algebra or at topology, depending on clarity. Here is the table of contents: there are ten chapters. 1. Some homological algebra. 2. The homology of a group. 3. Homology and cohomology with coefficients. 4. Low dimensional cohomology and group extensions. 5. Products. 6. Cohomology theory of finite groups. 7. Equivariant homology and spectral sequences. 8. Finiteness conditions. 9. Euler characteristics. 10. Farrell cohomology. The first six chapters contain all we mentioned above and more. For example, not only is cup product introduced in Chapter 5 so one may construct the ring $H^*(G)$ for every G, but a "Pontryagin product" is defined on $H_*(G) = \Sigma_n H_n(G)$ for abelian G making $H_*(G)$ a ring. This last ring is used to calculate (in theory) homology groups of abelian groups (I would have liked a bit more detail here: a proof of Lyndon's explicit formulas for $H_*(G)$ when G is finitely generated; calculation of the cohomological dimension of a torsion free abelian group of finite rank). In Chapter 6, Tate cohomology is discussed, as is integral duality and finite groups with periodic homology (there is also a discussion of the relation of this phenomenon with groups acting on spheres). The reader who has fully understood a course in group theory and a course in algebraic topology will find Professor Brown a knowledgeable guide to this material. But note: it is not enough to know the algebraic component of the algebraic topology course, namely, homological algebra. Topology is an essential ingredient for understanding. The last four chapters, as the author says, are "more specialized and reflect (his) own research interests." These interests form a worthwhile path for the reader. One will see links with algebraic number theory and geometry; even the Riemann zeta function enters in!

My only complaint is that some of my favorite results are not here: Evens's result that $H^*(G)$

is a finitely generated ring for every finite group G; Gaschütz's theorem (using group cohomology) that every finite p-group has an outer automorphism of order p. In fairness, the book does not claim it contains everything. For the reader with the right prerequisites, I recommend cohomology of groups, the subject and this book.

Numbers and Infinity. A Historical Account of Mathematical Concepts. By Ernst Sondheimer and Alan Rogerson. Cambridge University Press, 1981. x + 172 pp.

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Sometimes a student can best digest and assimilate a nontrivial mathematical idea by working personally through roughly the same conceptual stages that occurred in the historical development of the idea. However, textbooks often impede this beneficial process of individual evolution by presenting expositions that confound the historical order.

For example, we read in a widely used high school algebra text that a function is "a set of ordered pairs of numbers in which no two different pairs have the same first element." The student is then confronted with several finite sets of ordered pairs of numbers, and asked to determine which of these sets are functions. The notation $\{(x, f(x)):---\}$ for a function appears in a subsequent section, and it is observed that a formula (of all things) can determine precisely the sort of set of ordered pairs that constitutes a function. Finally it is revealed that our real interest is in "open sentences" (equations, apparently) associated with functions. We find that the "solution set" of an open sentence $f(x) = 0$ is the set of all first elements of those ordered pairs in f in which the second element is zero. Great emphasis is placed on the importance of the null set, it being the solution set of an open sentence that has no solutions.

Let us pass over the recent announcement that 53% of current American high school graduates cannot determine what percent 30 is of 60, and wonder how a student can attempt to overcome the handicap imposed by such a textbook. A sufficiently perceptive student might seize upon the rough but ready idea of a function as simply a formula, and then retrace in his own mind at least some of the steps toward greater understanding that Euler, Lagrange, Cauchy, and Dirichlet took. But it seems likely that at least 53% of the students will need considerable assistance along the way.

Obviously it would be too much of an antidote to follow the historical order so slavishly as to repeat its many false starts and dead ends. Concentrating on number concepts and related ideas of infinity and continuity, the book by Sondheimer and Rogerson provides a good selection of just those historical highlights that seem most likely to help elementary students, and especially their teachers and textbook writers. The authors' low-key conversational account is readily accessible (except in a few isolated spots) to readers who have yet to study calculus.

Our high school algebra textbook's lavish description of the real number system in terms of binary operations and field axioms must be studied carefully in order to be appreciated properly. But perhaps the students who read it may be pardoned for believing, as did the ancient Greeks, that the only "real" numbers are the positive integers. Indeed, the initial development of plane geometry in the Pythagorean period (the fifth and sixth centuries B.C.) was based on this belief, in the form of the assumption that any two line segments are commensurable—that is, they can be subdivided into integral multiples of some single segment. For instance, if the corresponding sides AC and $A'C'$ of two similar triangles ABC and $A'B'C'$ are commensurable, then the construction

of appropriately many parallels to the bases AB and $A'B'$ makes it clear that the sides BC and $B'C'$ are commensurable in the same fashion, so it follows that $AC/A'C' = BC/B'C'$. Thus the commensurability hypothesis provided the early Greeks with a simple proof of the proportionality of corresponding sides of similar triangles, and of the other familiar proportionality theorems of elementary geometry.

Hence the late fifth century B.C. discovery of the existence of incommensurable pairs of line segments—such as the hypotenuse and a side of an isosceles right triangle—was a devastating blow that required the foundations of geometry to be rebuilt entirely. The new keystone was the fourth century B.C. formulation by Eudoxus of the following definition of equality of ratios of (possibly) incommensurable line segments. Given four line segments $\alpha, \beta, \gamma, \delta$, he said that $\alpha/\beta = \gamma/\delta$ provided that, for any two positive integers p and q whatsoever, it is true *either* that both $q\alpha < p\beta$ (with the obvious meaning in terms of copies of α and β laid end to end) and $q\gamma < p\delta$, *or* that both $q\alpha = p\beta$ and $q\gamma = p\delta$, *or* that both $q\alpha > p\beta$ and $q\gamma > p\delta$. On the basis of this masterful definition, the foundations of geometry were reconstructed as they are presented in Euclid's *Elements* with a standard of rigor that later was lost and not regained until the nineteenth century.

We should not omit the proviso (Euclid's Definition V.3) that two segments have a ratio only if some integral multiple of each exceeds the other—the first appearance of the "axiom of Archimedes." We may interpret Eudoxus' definition of the equality $\alpha/\beta = \gamma/\delta$ as saying that every rational number p/q either is less than both α/β and γ/δ, or is equal to both, or is greater than both. Thus we have what is now called a "cut" of the rationals. Indeed, precisely this interpretation of Eudoxus was (some twenty-two centuries later) the motivation for Dedekind's famous construction of the real numbers.

Other mathematical needs motivated other extensions of the concept of number. Complex numbers appeared in the solution of quadratic equations, but the clincher was their intrusion in the Cardan solution of a cubic equation such as $x^3 - 2x^2 - x + 2 = 0$ (nicely described by Sondheimer and Rogerson in Chapter 5) despite the fact that its roots $+1$, -1, and 2 are all real. Complex numbers hovered on the fringes of respectable mathematics for about two centuries, until the time of Gauss' first attempted proof of the fundamental theorem of algebra (described in Chapter 6).

The Eudoxus–Archimedes axiom precluded infinitely small quantities but provided arbitrarily small ones, just as were needed to cinch the rigorous proof by an indirect limit process (described in Chapter 8) that the ratio of the areas of two circles equals the ratio of the areas of the squares on their two radii; of course the Greeks did not state this in terms of a certain irrational number π that appeared explicitly only much later in history. The method of exhaustion was refined by Archimedes to determine the areas and volumes of numerous geometric figures that still appear as staple examples in calculus books. His investigations provided the principal model and inspiration for the looser infinitesimal techniques that began to proliferate in the seventeenth century (as sketched in Chapter 9). And finally the completeness or continuity property of the real numbers was the key to the nineteenth century rigorization of the calculus.

Sondheimer and Rogerson weave these historical threads into an attractive informal exposition that is directed at bright high school and beginning college students, and can also be recommended to prospective school mathematics teachers. Such readers will glimpse here some of the rich lore of mathematics that can slip through the cracks in the standard undergraduate curriculum—including the prime number theorem, the insolvability of quintic equations by radicals, Fermat's last theorem and nonunique prime factorization in number fields, transcendental numbers and the Gelfond–Schneider theorem, quaternions and the only real division algebras, space-filling curves, nonstandard analysis, transfinite numbers and Cantor's uncountability arguments, and the undecidability of the continuum hypothesis. While some of these topics are mentioned without much technical detail, a useful chapter by chapter bibliography is provided as a guide to further study.

LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

Editor:

Professor Scarborough (this MONTHLY, vol. 90, p. 417) ponders the question whether courses in education are of as much value to a prospective teacher of secondary school mathematics as courses in mathematics itself.

I believe that Professor Scarborough has asked the wrong question. To me, the problem of teaching mathematics is a broad one, and it is not just a matter of whether a prospective teacher should take one or two, or possibly three, education courses to qualify as a teacher. Furthermore, it is not a question of whether education courses are as valuable in preparing the teacher as courses in the teacher's major field.

Let us postulate three *necessary conditions* for success in teaching. (The list is not complete, and the conditions might not be sufficient.)

(1) The teacher is competent in the subject on a level at least four years beyond the one at which he will be teaching.
(2) The teacher is interested in raising the level of competence of his student to that required for further courses and for applying the principles covered in the course.
(3) The teacher is willing to learn how to improve his presentation of the subject and is eager to learn how to profit from his own experiences and mistakes.

Clearly, courses in the major and related fields are necessary to successful preparation for (1) and (2). Whether extra courses in the subject are needed—beyond a reasonably well-rounded background—is debatable. Similarly, a bit of organized study of individual differences (elementary psychology) and an organized discussion of classroom management and record keeping (practice teaching) should be helpful in preparation for (3).

In either area, a minimum of preparation is clearly necessary; it is less clear that it is always sufficient. Preparation beyond the minimum is probably helpful in most cases, and lifelong learning is recommended as a standard practice.

Success in teaching requires a willingness to learn and a measure of enjoyment from helping malleable minds to grow in their knowledge and understanding of mathematics.

No amount of preparation in courses—either in the major or in the techniques of teaching—can guarantee good teaching. There must be a happy combination of knowledge of the subject, together with wisdom and understanding, in approaching the problems of teaching in order to produce good results.

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126. MISCELLANEA

The perfection of mathematical beauty is such... that whatsoever is most beautiful and regular is also found to be most useful and excellent.

—D'Arcy W. Thompson, *On Growth and Form*, vol. 2, 2nd ed., Cambridge University Press, 1942, p. 1097.

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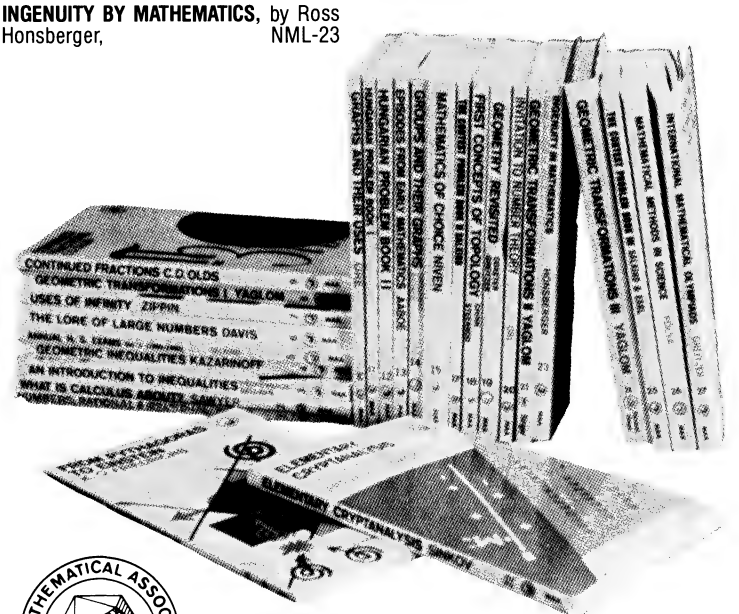
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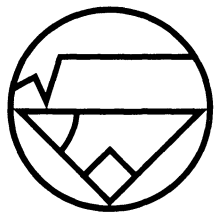
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A SUPPLEMENT TO "A MATHEMATICIAN'S GUIDE TO POPULAR SPORTS"

MICHAEL STOB
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Introduction. Tom Jech, in his MONTHLY article of April 1983 [**8**], claims to have found the "one and only one correct way of comparing teams according to their records even when their playing schedules are not identical." Although Jech's justification for his method is new, the method of comparison is old (it has been discovered independently by Zermelo [**14**], Bradley and Terry [**3**], and Ford [**7**] among others), much is known about it, and Jech's claim and his justification for it have been challenged by some MONTHLY readers. Here we intend to summarize the history of Jech's method and some of the debate over his claims. There is a vast literature on the method of paired comparisons; we don't intend this note to be exhaustive and we suggest that the reader interested in further information see David [**4**] and Bradley [**2**].

The problem is this. Teams $T_1, \ldots, T_n$ play matches according to the schedule

$$M = (m_{ij})_{i,j=1}^{n},$$

where m_{ij} is a nonnegative integer representing the number of times team T_i plays team T_j. The result of these matches is also a matrix

$$R = (r_{ij})_{i,j=1}^{n}$$

with

$$0 \leqslant r_{ij} \leqslant m_{ij}$$

and

$$r_{ij} + r_{ji} = m_{ij}.$$

For the usual examples, r_{ij} is an integer and is the number of times team T_i defeats team T_j. What is desired is a ranking of the teams on the basis of their performance. Now if all m_{ij} for $i \neq j$ are equal, the usual procedure is to rank the teams according to their scores

$$s_i = \sum_{j=1}^{n} r_{ij}.$$

However, if different teams have different schedules, ranking by scores is no longer "fair" as one team may have a lower score than another because it has played fewer games or stronger teams.

Zermelo's Solution. Zermelo [**14**] proposed the following model as the solution to the problem of determining the winner of a round-robin chess tournament which is interrupted before its completion. He first supposed that the individual contests are independent Bernoulli trials with the probability that team T_i defeats team T_j being given by π_{ij} (which is constant over the m_{ij} matches between the two teams). He then supposed that each competitor has a certain "playing

Michael Stob: Don Greenfield, Athletic Director of Timothy Christian High School, unwittingly suggested this problem to me. He ranked me very low in a class of sixteen badminton players on the basis of my record (one win—three losses) even though my opponents were four of the better players. This ego-shattering experience led me to discover Zermelo's method (as many, many others had before and after I did). I have been a connoisseur of ranking methods ever since. My usual mathematical research is in recursive function theory, and I also have a research interest in formal models of language acquisition. I have been on the staff of the Mathematics Department of Calvin College since 1979, having taken leaves of absence to teach at Massachusetts Institute of Technology (1979–81) and Wisconsin (1983–4). This research was partially supported by National Science Foundation Grant Number MCS-82-00032.

strength" which is measured by a positive real number V_i. The probabilities π_{ij} are determined by the strengths of the competitors in the following way

$$\pi_{ij} = V_i/(V_i + V_j). \tag{1}$$

Now given R, Zermelo computed estimates v_i for the strengths V_i by the method now known as maximum likelihood. For every vector $v = (v_i)_{i=1}^n$ of strengths, we can compute the probability $P(v)$ that R is the result of the schedule M. Explicitly,

$$P(v) = \prod_{i<j} \binom{m_{ij}}{r_{ij}} \cdot \left(v_i/(v_i + v_j)\right)^{r_{ij}} \cdot \left(v_j/(v_i + v_j)\right)^{r_{ji}}.$$

Then the maximum likelihood estimate of V is that vector v which maximizes $P(v)$. What Zermelo showed was that, under a certain reasonable assumption, there is a vector v which maximizes $P(v)$ and is unique up to a multiplicative constant (which clearly does not change the estimates (1) of π_{ij}). Of course it is clear that the order of the V_i determines a linear ranking of the teams with the property that team T_i is ranked at least as high as team T_j if and only if $\pi_{ij} \geqslant 1/2$. Zermelo also gave an iterative procedure which can be used to find the maximum likelihood vector v and proved that the iteration always converges provided that the matrix R satisfies the assumption mentioned above. The assumption under which this unique solution v exists and the iterative procedure converges is simply that there is no partition of the n teams into two sets S and T such that either no team in S plays any team in T (in which case there is no basis for comparison between the teams in S and those in T) or every game between a team from S and one from T results in a victory for the team in S.

Zermelo's model, proof, and algorithm for finding v were later rediscovered by Ford [7] whose motivation was to rank a number of objects on the basis of binary comparisons. The same model was also proposed by Bradley and Terry [3] in the context of paired comparison experimental design. Bradley and Terry also gave some tests of significance of various hypotheses about the experiment (for instance, that $V_1 = \cdots = V_n$) and supplied tables of the vectors v computed for small values of n and m_{ij}. They did not, however, give an existence or uniqueness proof or a procedure for finding the vector v. They also did not extend their model to the case of unequal m_{ij}; this was done by Dykstra [6]. The model (1) together with the maximum likelihood method of estimating the parameters V_i is now known in the statistical literature as the Bradley-Terry model and is the most widely used model for solving the problem.

Jech's Solution. Jech's model is different but under the same assumption on R, he arrives at the same estimates for the numbers π_{ij} as those of Zermelo. Instead of supposing the existence of strengths V_i, Jech places restrictions directly on the π_{ij}. Let

$$x_{ij} = (\pi_{ij}/\pi_{ji}).$$

Then x_{ij} represents the odds favoring T_i over T_j. Jech claims that the following relationship is plausible:

$$x_{ik} = x_{ij} \cdot x_{jk} \quad \text{for all } i, j, k. \tag{2}$$

He supplies the following argument ([8], p. 249);

> Suppose that we compare objects T_i and T_k only indirectly by comparing T_i with T_j and T_j with T_k and we do it a large number of times, say M. In $M \cdot \pi_{ij}$ cases, T_i looks better than T_j and of these $M \cdot \pi_{ij}$ cases, T_j looks better than T_k exactly $M \cdot \pi_{ij} \cdot \pi_{jk}$ times and worse $M \cdot \pi_{ij} \cdot (1 - \pi_{jk})$ times. Whenever we find T_i better than T_j and T_j better than T_k we conclude that T_i is better than T_k, but when T_i is found better than T_j and T_j worse than T_k we reserve our judgment about T_i and T_k.
>
> A similar situation arises in the $M \cdot (1 - \pi_{ij})$ cases when T_j is deemed better than T_i. Thus we have $M \cdot \pi_{ij} \cdot \pi_{jk}$ cases when T_i is declared better than T_k and $M \cdot (1 - \pi_{ij}) \cdot (1 - \pi_{jk})$ cases when T_k is considered better. It follows that $x_{ik} = x_{ij} \cdot x_{jk}$.

Instead of the method of maximum likelihood, Jech computes estimates p_{ij} for the parameters

π_{ij} by requiring that the expected scores equal the actual scores; i.e.,

$$\sum_j p_{ij} m_{ij} = s_i. \tag{3}$$

He proves that, subject to (2), there is a unique solution to (3) for every matrix R satisfying the assumption of comparability given above. (In fact, he shows that there always is a solution and it is unique unless the assumption is not satisfied.) His proof is instructive. He first shows that (2) already implies the existence of the V_i's satisfying (1). This equivalence of (1) and (2) has been noted often in the statistical literature. He then proves a theorem similar to Zermelo's but with the equalities (3) substituted for the maximum likelihood condition. However Zermelo had already shown that the maximum likelihood estimators v_i also satisfy (3). Thus Jech and Zermelo necessarily compute the very same estimates for the parameters V_i.

Differences. In order to see clearly the differences between the methods of Zermelo and Jech and to assess Jech's claims for his method, it is helpful to view the modelling process as having two steps: the specification of the model and the estimation of the parameters. In the models of Zermelo and Jech, the process of playing out the schedule M is described as a probabilistic one depending on certain (unknown) parameters π_{ij}. These parameters are assumed to satisfy certain relations. Jech's assumptions on the π_{ij}, in fact, are equivalent to those of Zermelo; the difference is in the way each argues for his assumption. Zermelo appeals to the plausibility of "playing strengths" whose ratios determine the odds; Jech appeals to (2) as capturing our intuition about the relative abilities of teams. The models also differ in the way in which the parameters V_i are estimated. Jech estimates the V_i using the criterion (3) of expected score equals actual score; Zermelo uses that of maximum likelihood. (It is this latter difference which turns out to have some interesting consequences as we shall soon see.)

Other Models. What about Jech's appeal to (2)—is (2) really forced on all reasonable people who consider the matter carefully? Apparently not. Consider the following model, due to Mosteller [**10**]. Each team T_i has a certain strength V_i as in Zermelo's model, but the result of a game between team T_i and team T_j is determined in the following way. Each team has an actual performance v_i which is a random variable that is normally distributed with mean V_i and variance σ^2, σ being a constant which does not vary from team to team. The team that wins is the team with the greater performance for that game. Under this model, then, if $\sigma = 1$,

$$\pi_{ij} = \left(1/\sqrt{2\pi}\right)\int_{-\infty}^{V_i - V_j} e^{-y^2/2} dy, \quad i \neq j. \tag{4}$$

It is easy to show by specific examples that, in this model, the π_{ij} do not satisfy (2). Nevertheless, it is an attractive model in certain situations—it was originally proposed for situations where one had to judge pairs of sensations, say, which of two objects is the heavier one. Here V_i is the actual weight and v_i is the perceived weight.

The models of Mosteller and Zermelo are both special cases of a class of models known as the linear model. We again suppose that each team has a strength V_i and we suppose that there is a symmetric distribution function H such that

$$\pi_{ij} = H(V_i - V_j). \tag{5}$$

Thus the strengths linearly order the teams by ability since $\pi_{ij} \geq 1/2$ if and only if $V_i \geq V_j$. Various candidates for H include Mosteller's model (4), Zermelo's model (2), where if we take V_i to be the natural logarithm of the V_i given there, we have

$$\pi_{ij} = 1/(1 + e^{V_j - V_i})$$

and the uniform model

$$\pi_{ij} = V_i - V_j + .5. \tag{6}$$

(Of course (6) may give probabilities > 1; it should be truncated at $|V_i - V_j| = .5$ in the obvious way.) The model (6) was used by Smith [**11**] as a model for ranking baseball teams in midseason.

Besides the class of linear models, models have been proposed based on axioms of choice behavior (Luce [**9**] but Luce's model also reduces to that of Zermelo), number and degree of upsets (Thompson [**12**]), and indirect comparisons of the teams by using such principles as transitivity (Wei [**13**]). While some of these models are similar to the linear models above in that they are models for the π_{ij}, others attack the problem of finding the desired ordering more directly. For instance, Thompson [**11**] computes the number and degree of upsets for each possible ordering of the teams and chooses the order minimizing his measure of upsets. The wealth of such models and the sometimes conflicting conclusions that may be drawn from them argue against the obviousness of (2). Still, it seems important to note that the differences among the models are not ordinarily very great. One way to see this is to notice that each linear model determines π_{ik} from π_{ij} and π_{jk}. If π_{ij} and π_{jk} are both .6 for instance, the uniform model gives $\pi_{ik} = .7$, Zermelo's model gives $\pi_{ik} = .693$, and Mosteller's model gives $\pi_{ik} = .699$. These differences are not large and often do not affect the ranking of the teams that is determined from the model.

Jech's model (2) as well as the competing versions of the linear model are hypotheses about the probability distribution functions governing the playing out of the schedule M. As such, they can be tested against the results R. Bradley [**1**] gives a test of the hypothesis that Zermelo's model is correct

$$H_0: \quad \pi_{ij} = V_i/(V_i + V_j), \quad \text{all } i \neq j$$

compared with the alternate hypothesis

$$H_a: \quad \pi_{ij} \neq V_i/(V_i + V_j) \quad \text{for some } i \neq j.$$

He used this test to test the fit of the Zermelo model to the result of experiments involving paired comparisons of taste of pork, appearance of apples, and flavor of applesauce. Mosteller and Fienberg have noted that the Zermelo model actually fits baseball data too well in the sense that collapsing home and away data introduces added stability into the data. Of course home and away differences can be accounted for by the model by treating each team as two different teams —a home team and an away team.

Estimates. The problem of estimating parameters in a probabilistic model is a statistical one. Besides the method of Jech and that of maximum likelihood there are other reasonable candidates for methods of estimation. For instance one might choose the estimates v_i of the strengths so that they minimize

$$\sum_{i,j} \left(r_{ij} - m_{ij} \cdot v_i/(v_i + v_j)\right)^2. \tag{7}$$

This is of course just the method of least squares. In fact, this is a method suggested by Mosteller for his model (4). It might be chosen by an experimenter who wishes the expected values of the r_{ij} be as near their actual values as possible. The least squares method (7) may result in orderings which are different from that of Jech, however, and the agreement of Jech's method with the method of maximum likelihood is a strong argument in its favor. For other models, such as Mosteller's model (4), the methods of maximum likelihood, expected scores equal actual scores, and least squares all may give different orderings of the teams. In fact this is the case for the baseball data of Jech using the uniform model (6). Thus even given the same model, the choice of estimator may affect the ranking and so requires justification. Jech argues for his choice on the basis of its intuitive appeal. (What sort of a claim is it that a team solely on the basis of the results should have expected to win more games than they did?) Besides arguments as to their intuitive content, there are also technical criteria which we would like an estimator to satisfy (such as unbiasedness). We will leave these issues to the statistical literature.

We might remark here that Jech did not treat this problem as a statistical one (in the sense that he considered the strengths he computed estimates for some true strengths) and some people take issue to our construal of this problem as partly a problem in statistical estimation. On this view, ranking teams by their strengths V_i determined according to (2) and (3) is no more a statistical problem than that of ranking students on the basis of their grade point averages (this illustration is due to Ford). The numbers v_i are, in this view, simply measures of performance which we compute, not estimate. We measure performance with these parameters since (2) and (3) have intutitive appeal as characteristics of a measure of performance and because of the existence and uniqueness theorems. There is no debate here, of course, as to the results—these are just theorems —but only as to the sort of arguments one is willing to entertain to justify, say, (3) and to the sort of tests one is willing to apply to the model.

There is, even in Jech's model, a distinction between maximum likelihood estimation and estimation by (3). This is the treatment of ties. On Jech's view (2) remains plausible as a model if we reinterpret the numbers π_{ij} to be the expected value of team T_i in a match with team T_j if ties are counted as half of a win. Indeed, Jech allows the result of a match to be any real number r, $0 \leqslant r \leqslant 1$. Now (2) says that the ratios of the expected values multiply. Of course the argument for (2) must be reexamined, and Jech did not supply an argument for (2) with this interpretation of the effect of ties. Given the new interpretation of (2), however, the condition on estimation of parameters that the expected scores equal the actual scores has the same meaning and the same intuitive appeal. However, to allow for ties in the case of maximum likelihood estimation, we have to estimate the probability that R is the result given the parameters V_i. In order to do that, it is necessary to include in the model a model for the probability of a tie between any pair of teams. Thus, many statisticians who favor the maximum likelihood method of estimation have proposed extensions of the basic Zermelo model to handle ties. One example is a model of Davidson [**5**]. Davidson argues for keeping (2) as a model of how the ratios of winning probabilities combine. That is, he gives π_{ij} the same meaning as in the basic Zermelo model, the probability that team T_i defeats team T_j, and he leaves (2) unchanged. He then proposes that the probability of a tie between team T_i and team T_j is proportional to $\sqrt{\pi_{ij} \cdot \pi_{ji}}$. This has the desired consequence that ties are most likely when the teams are evenly matched. The probabilities of the three outcomes in terms of the strengths V_i then becomes

$$\text{Team } T_i \text{ defeats team } T_j\colon \quad V_i/\left(V_i + V_j + \nu\sqrt{V_i \cdot V_j}\right),$$

$$\text{Team } T_i \text{ ties team } T_j\colon \quad \nu\sqrt{V_i \cdot V_j}/\left(V_i + V_j + \nu\sqrt{V_i \cdot V_j}\right),$$

$$\text{Team } T_j \text{ defeats team } T_i\colon \quad V_j/\left(V_i + V_j + \nu\sqrt{V_i \cdot V_j}\right).$$

Davidson showed that if R satisfies Zermelo's assumption, there is a unique set of positive parameters v_i and n, estimating V_i and ν, respectively, which maximizes the probability of R happening. He also gave an iterative procedure for finding the parameters and showed that, for these parameters, the expected scores equal the actual scores. Davidson's and Jech's models disagree—Davidson's model is meant to satisfy (2) only when it is interpreted as saying the winning odds multiply; Jech's model satisfies (2) for the ratios of expected values and is silent on the issue of winning odds.

A Counterexample. One way to refute a mathematical model is to use the argumentation given in its support to conclude its negation. So suppose the Zermelo conception of relative strength or the Jech idea of multiplicative odds is substantially correct.

The sport of tennis is a prime candidate for this method. Tennis tournaments are short, few matches are played relative to the number of players participating. Throughout the season, some players play against each other often while others not at all. But "objective" rankings are thought

to be desirable and are produced by the various governing bodies of the sport. Now a tennis match consists of sets, with the first player to win two (or sometimes three) sets winning the match. Now if it is reasonable to assume that the results of the match are governed by strengths according to (1), then it also might be reasonable to think that the outcome of each set could be governed by a model of the same sort (but with possibly different strengths). But it is easy to show that if strengths V_i exist so that the probabilities of winning sets are determined by (1), then it is impossible to find strengths so that the probabilities of winning a match satisfy (1). (Of course for any matrix R of results of matches M it will still be possible to find maximum likelihood estimates for the strengths. The model need not be true for these to exist.) But certainly any argument that (1) is an appropriate model for matches is as well an argument that (1) is appropriate for sets. Of course an entirely parallel argument implies that (2) cannot be a model for both matches and sets. In fact, if (2) is satisfied for sets, then what is true for matches is

$$x_{ik} > x_{ij} \cdot x_{jk} \quad \text{if } x_{ij} \cdot x_{jk} > 1.$$

The obvious rejoinder is of course that if the model applies to sets, then we can obtain the desired ranking from the data on sets. But a set is won by the first player to win 6 games and a similar argument shows that (2) cannot be true for sets if it is true for games. This argument applies not only to tennis but also to many other games which are themselves composed of minimatches.

Chess. Of course no sport chooses its winners by anything like the method described here. The chess community does however rate its players on the basis of their playing results in sanctioned events by a method, the Elo System, which has its theoretical basis in the Mosteller model (4). Integer ratings r are assigned to each player with $0 \leqslant r \leqslant 3000$. A rating difference between two players is converted to an expected value of a match between the two players (with ties counted as half of a win) by the model (4) with σ approximately equal to 280. These expected values are then used to compute the expected score of the player in a tournament. If the player does better or worse than expected, the player's rating is adjusted up or down accordingly. (Of course an added complication in the chess problem is the expected change in a player's ability over time.)

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A famous hyphenated pair. (See p. 300.)

from one of a number of triangle reconstruction problems posed by Wernick [**18**]. With respect to the triangle, it is the most symmetric of Wernick's problems; that is probably why it leads so unexpectedly to the critical circle and the acentric lacuna.

My thanks to Professors R. Blum, H. S. M. Coxeter, and L. Sauvé, for references and helpful comments.

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MISCELLANEA

127.

The mathematics that I learned...became a deeper part of my nature, as resolute consistency and something like mental courage. From a possibly very small area, which is not to be doubted, you keep on going in one and the same direction, never asking yourself where you might end up, refusing to look right or left, as though heading towards some goal without knowing which, and so long as you make no false step and maintain the connection of the steps, nothing will happen to you, you progress into the unknown—the only way to conquer the unknown *gradually*.

—Elias Canetti, *The Tongue Set Free*,
The Seabury Press, New York, 1979,
pp. 236–237

ANSWER TO PHOTO ON PAGE 283

Béla Szökefalvi-Nagy and Ciprian Foiaş, usually known as Nagy-Foiaş.

NUMBER-THEORETIC ANALYSIS AND EXTENSIONS OF "THE MOST COMPLICATED AND FANTASTIC CARD TRICK EVER INVENTED"

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1. Introduction. This paper presents a mathematical analysis, simplification, and extensions of a card trick based on number theory, and involving congruences, power residues, inverse permutations, relative primality, associated integers, and primitive roots, as well as mystery and surprise. The card trick, described by the philosopher and mathematician Charles Sanders Peirce [1], has been resurrected by Martin Gardner [2], who characterizes it as "surely the most complicated and fantastic card trick ever invented," stating that "for a teacher who wants to motivate student interest in congruence arithmetic, it is superb." A mathematical analysis shows it to be based on number-theoretic properties; it should therefore be of particular interest to mathematicians.

We shall first describe the performer's manipulations, incorporating a significant simplification, followed by mathematical validation, an extension, and generalization.

2. Initial set-up. The Ace, Jack, Queen, and King of a deck will be designated respectively as the face values 1, 11, 12 and 13. For any deck facing upward, the sequence of its cards is enumerated by starting at the bottom of the deck and proceeding towards the top. All dealings take place by holding a deck face down, turning over its top card, and placing it on the table face up.

From a deck of cards, select the spades from Ace through Queen (12 cards), to be called the "black" deck, and the hearts from Ace through King (13 cards), the "red" deck. Sequence the black deck so that the face values of successive cards represent the successive power residues modulo 13 of its primitive root 2. Cut the deck arbitrarily, yielding, let us say, the sequence $\{b_n\}$ shown in Fig. 1. Next, sequence the red deck as follows: Because the Ace of spades is in position number 9, place the 9 of hearts as the first card of the red deck. As the 2 of spades is in position number 10, place the 10 of hearts as the second card of the red deck. Proceed similarly for each successive face value of spades. At the end, append the King of hearts to the red deck. The sequence of spades thus yields a corresponding deck of hearts as shown in Fig. 1.

Black deck:	$b_n =$	3	6	12	11	9	5	10	7	1	2	4	8	
Position numbers	$n =$	1	2	3	4	5	6	7	8	9	10	11	12	
Red deck:	$r_n =$	9	10	1	11	6	2	8	12	5	7	4	3	13

FIG. 1. Initial corresponding decks.

3. Reciprocity. The two card decks have a pointer property that is mutual: For convenient reference, deal the deck of spades face down from left to right into rows of 5, 5, 2 cards, respectively. Ask a spectator to name any card of *spades*; say it is the 4. In the deck of *hearts*,

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count to the 4th card and show it to be the Jack (11). This indicates that the desired 4 of spades is the 11th card in the black deck. Show the audience that this is true, and repeat for other face values named by the audience. Similarly ask the audience to name any face value of *hearts*; say it is the 2. The second card of the deck of *spades*, which is a 6, indicates that the desired 2 of hearts is the 6th card in the red deck.

Before demonstrating the reciprocity property of the two decks, however, first apply the quasi-randomization described in the next section.

4. Red k-shuffle. Let the audience cut the red deck (into two parts, reassembled in reverse order) an arbitrary number of times, and then name an arbitrary positive integer k (< 13), say $k = 5$. From the red deck, deal k cards from left to right so as to form the bottom cards of k heaps. Continue dealing the remainder of the deck onto the k heaps by sequentially sweeping from left to right, keeping visible the upper parts of covered cards. Fig. 2 illustrates the result for the deck from Fig. 1.

8	12	5	7	4
3	13	9	10	1
11	6	2		

FIG. 2. The red deck, cut so as to start with face value 8, dealt into $k = 5$ heaps.

The heaps, arranged in vertical columns of overlapping cards, must now be carefully assembled in a particular manner: Counting heaps left to right from 1 to k (here, 5), suppose the last heart fell onto heap number z ($= 3$ in Fig. 2). Let a member of the audience point to an arbitrary heap, say number i_1. Starting at the designated heap i_1, pick up the entire heap, count heaps cyclically z positions towards the right, reaching, say, position number i_2, and place all cards from your hand on top of heap i_2. Remember that the count must NOT be i_1 positions, but z positions. Always counting z heap position numbers (NOT *remaining* heaps) towards the right, start from position i_2 and repeat the procedure, continuing until all heaps have been consolidated into a single deck. While accuracy in identifying the proper sequence of heaps is critical for success, the selection should be made nonchalantly and appear to randomize. At the end, cut once so as to bring the King to the end of the deck (to the top when the deck is held face up), ostensibly "because the King has no counterpart in the black deck." This constitutes our new, "shuffled" red deck, to be designated as the sequence $\{R_n\}$. See the illustration in Fig. 3 below.

The foregoing procedure will be termed a "(red) *k-shuffle*." Show to the audience that it radically rearranges the sequence of cards. We shall see later that $\{R_n\}$ is independent of the cuts applied to the red deck and of the choice of i_1.

When the last card falls nearer the rightmost heap, it is convenient to apply instead an equivalent, alternative method of assembly that is easier for this case: Count position numbers cyclically towards the *left*, beginning with the label 0 (NOT 1) for the rightmost heap. For Fig. 2, this alternative method yields $z = 2$ and heaps are accumulated by successive placements upon every zth heap position counted cyclically towards the *left*. The final result is the same as before.

The k-shuffle of the red deck requires a corresponding manipulation of the black deck that is surprisingly simple: Determine the new position number n of the Ace of hearts, then inconspicuously cut the black deck so that face value n occupies the first position. Alternatively, peek at the first red card R_1 ($= 6$ here); pick up the black deck and, reviewing it, inconspicuously cut it so that its Ace occupies position number R_1 ($= 6$). Denote the resulting sequence of spades as $\{B_n\}$ —see Fig. 3.

Black deck:	$B_n =$	11	9	5	10	7	1	2	4	8	3	6	12	
Red deck:	$R_n =$	6	7	10	8	3	11	5	9	2	4	1	12	13

FIG. 3. Pair of decks after a red 5-shuffle.

Point out to the audience that the red cards have been cut, rearranged into k piles, assembled,

cut again, and that the number of heaps (k) and starting point for reassembly (i_1) have been chosen *by the audience*. Yet, reciprocity is now seen to hold between the new sequence of hearts and the deck of spades, and is demonstrated, for any number of black or red cards, to the surprise of the audience.

To heighten the effect, perform two or three k-shuffles, each with a different k, before demonstrating the preservation of reciprocity. Thus with the red deck, alternate cuts by the audience and k-shuffles (various k). Apply the performer's single cut of each deck only once, at the end. No matter how often the red deck is cut and k-shuffled, reciprocity between the rearranged red and the black deck persists, seemingly by sheer miracle.

5. Mathematical validation.

a. *Reciprocity*. For a black deck sequenced arbitrarily, consider a corresponding red deck prepared as described in Section 2. Thus if, in the black deck, the face value n appears in position number i, so that $b_i = n$, the red deck correspondingly shows in position n the face value i by construction, so that $r_n = i$. Conversely, if we start with the red deck ($r_n = i$), then the black deck must correspondingly show $b_i = n$. The indicator relationship must therefore necessarily be mutual.

A somewhat deeper proof of reciprocity may be obtained by considering permutations: Let b denote the permutation of the sequence $\{b_n\}$ into the sequence of successive integers $n = 1$ to 12, i.e.,

$$\underline{b} = (b \to n) = \binom{b}{n} = \begin{pmatrix} 3 & 6 & 12 & 11 & 9 & 5 & 10 & 7 & 1 & 2 & 4 & 8 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \end{pmatrix}.$$

Ignoring r_{13}, similarly for the sequence $\{r_n\}$ define $\underline{r} = (r \to n) = \binom{r}{n}$, and $\underline{\bar{x}} = (n \to r) = \binom{n}{r}$. By construction, also $\underline{r} = (n \to b) = \binom{n}{b}$. We now find that

$$\underline{b} \cdot \underline{r} = \binom{b}{n}\binom{n}{b} = I = \binom{r}{n}\binom{n}{r} = \underline{r} \cdot \underline{\bar{x}},$$

whence firstly, $\underline{r}$ and $\underline{b}$ are mutual inverses; and secondly, $\underline{\bar{x}} = \underline{b}$, i.e., $\binom{n}{r} = \binom{b}{n}$. Thus in Fig. 1, *every* vertically positioned pair of numbers [such as $\binom{3}{1}$] found in rows 1 and 2 is also found in rows 2 and 3.

As the relationship between an arbitrary sequence $\{b_i\}$ and its pointers $\{r_n\}$ must necessarily be mutual, the surprising feature of the card trick does not consist in the mutuality of the indicator relationship, but rather in its *persistence* under shuffling! This astonishing permanence is due to the structure devised for the black deck.

b. *Basic formulae*. Preparation of the initial deck was specified by Peirce and by Gardner via a tedious procedure. Inspection of the result reveals it to represent a cyclic permutation of the successive powers modulo 13 of its primitive root 2. This observation formed the clue to all else.

Referring to Fig. 1, let

$f =$ the face value of the first card immediately following (cyclically) the Ace of spades (here, the primitive root f = 2).

$e =$ the number of cards lying at the end of the black deck beyond the Ace of spades (here $e = 3$).

$x(n) =$ the index of n mod 13.

By construction of the black deck, position number i is occupied by a spade with face value

$$b_i = f^{i'} \pmod{13} \quad \text{where } i' = i + e \pmod{12}. \tag{1}$$

Equivalently, a specified face value $n = b_i$ is found in position number $i \equiv x(n) - e \pmod{12}$.

By construction of the red deck, position number n is occupied by a heart of face value

$$r_n = i \equiv x(n) - e \pmod{12}. \tag{2}$$

c. *Red-shuffle.* We shall approach the investigation of what takes place with the deck of hearts in successive steps:

Step 1. Consider a card deck of diamonds sequenced from 1 to $p = 13$ subjected to a k-shuffle. With k heaps, the last card falls on heap z, where $p = mk + z$ (for some integer m). z will always be relatively prime to z, because any common factor would also divide p. Hence none of the cyclic counts of k position numbers will end on a vacant position until all heaps have been taken up. Heap no. i has as its bottom card the face value i, its top card $mk + i$ when $i \leqslant z$, or $(m-1)k + i$ when $i > z$. Within each heap, successive cards have the common difference k. When heap no. i is placed on top of heap no. $i + z \pmod{k}$, the difference d between the bottom card of the former and the top card of the latter is as follows:

$$\begin{aligned} &\text{When } i + z > k\text{:} \quad d = i - [mk + (i + z - k)] \quad = k - p \equiv k \bmod p; \\ &\text{When } i + z \leqslant k\text{:} \quad d = i - [(m-1)k + (i + z)] \quad = k - p \equiv k \bmod p. \end{aligned}$$

The scheme for the formation and reassembly of heaps thus assures that successive cards will have a constant difference $k \pmod{p}$ not only *within* each heap but also *between* heaps. The k-shuffle thus amounts to considering the original card deck arranged in a circle, repeatedly counting off k positions cyclically, and extracting the cards occupying the successive kth positions. Because k is relatively prime to $p = 13$, each count of k cannot reach a vacant position but lands on a spot occupied by a card, until all p cards have been selected.

Step 2. Any number of initial cuts of the deck of diamonds, and different choices of an arbitrary heap to begin the reassembly of the heaps, merely amount to cyclical permutations of the deck, which have no effect on the substance of the procedure. Cut of the final deck so that the King is in the last position thus results in the identical sequence for all the preceding variations in execution.

Step 3. What has been described above for the deck of diamonds shows what happens with the *position numbers* of the deck of hearts; namely, all variants of the intriguing procedure yield the identical result: Transformation of the sequence $\{r_n\} = r_1, r_2, \ldots, r_{13}$ into the sequence $\{R_n\} = r_k, r_{2k}, r_{3k}, \ldots, r_{13k}$, where all subscripts are modulo 13.

d. *The red and black decks.* What changes are needed in the black deck to maintain reciprocity? Applying (2) to the new deck,

$$R_n = r_{nk} \equiv x(nk) - e \pmod{12}.$$

Use of the fundamental property of indices for primes (see e.g., [3]),

$$x(nk) \equiv x(n) + x(k) \pmod{p-1},$$

yields

$$R_n \equiv r_n + r_k + e = r_n + (R_1 + e) \pmod{12}.$$

Thus the effect of the k-shuffle is to increase (mod 12) the face value of each card of the red deck by the *same* constant!

At the outset, $r_n = i$ corresponded to $b_i = n$. For the new sequence, $R_n = j$ will require the correspondence $B_j = n$ where $j \equiv i + (R_1 + e) \pmod{12}$. This means that the original black deck is updated to correspond to the shuffled red deck by merely moving $R_1 + e \pmod{12}$ cards from the bottom of the black deck to its top! This cyclic permutation is effected in the manner described in Section 4.

From (2), the number of black cards to be moved is

$$m \equiv r_k + e \equiv x(k) \pmod{12},$$

i.e., depends only on k (for fixed f) and is independent of the starting configuration of the two

decks. Moreover, if the black deck is to be updated by the movement of m cards, where m is specified, the requisite number of heaps for the corresponding k-shuffle of the red deck is given from the preceding by $k \equiv f^m \pmod{13}$.

6. Extension. An understanding of the mathematical relationships underlying Peirce's card trick allows an immediate extension: Suppose that a k-shuffle is applied not to the deck of hearts but to the deck of spades! What rearrangement will be required for the red deck to maintain reciprocity? Let us first describe the procedure, followed by mathematical validation.

Quietly note the face value of the last card of spades. Then have the audience cut the deck any number of times. Let someone pick for k one of the numbers 5, 7 or 11 (no other number will do); say $k = 5$. Apply a k-shuffle to the deck of spades. At the end, cut the deck so that the originally last card is restored to last place.

Now take a deck of diamonds with face values sequenced from 1 to 12. It is vital to remember that *the King of diamonds must be discarded at this point.* Apply a k-shuffle to the deck of diamonds, using the *same* k. Either begin reassembly with the heap that contains the last card (Queen) or cut the final deck so that the Queen occupies the last place. Now append the King of diamonds.

To illustrate, we take the decks of Fig. 3 as our new starting point (relabelling them b_n and r_n respectively). With $k = 5$, the result is shown in the first four rows of Fig. 4.

Shuffled spades:	B_n	= 7	3	5	4	11	1	6	10	8	9	2	12	
Hearts:	r_n	= 6	7	10	8	3	11	5	9	2	4	1	12	13
Position number	n	= 1	2	3	4	5	6	7	8	9	10	11	12	13
Diamonds after shuffle:	d_n	= 5	10	3	8	1	6	11	4	9	2	7	12	13
Diamonds at the end:	D_n	= 6	11	2	4	3	7	1	9	10	8	5	12	13

FIG. 4. Black 5-shuffle applied to the decks of Fig. 3.

Deal the resulting deck of diamonds into successive positions 1 to 13 from left to right, preferably face down and grouped 5 cards per row for easy reference. Now pick up the deck of *hearts* face down and deal it one card at a time. For each card, if the face value of the heart dealt is m, pick out from the table and set aside the diamond in position m face up into a single pile. The resulting deck of diamonds $\{D_n\}$, shown in Fig. 4, is our new red deck to go with the shuffled black deck. Demonstrate to the audience that despite the shuffles, reciprocity between all cards of the new black and red decks continues to hold!

The procedures of Sections 4 and 6 may be repeated any number of times, with any choices of k (for black-shuffles, from the numbers 5, 7, 11 only) without affecting the reciprocity of the black and red decks.

7. Validation of the extension. Let $f, e, x(n)$ as defined in Section 5b refer to our present starting decks of Fig. 3 (now relabelled b_n, r_n), and let $F, E, X(n)$ refer to the corresponding quantities for the new decks B_n, D_n of Fig. 4. As in Step 3 of Section 5c, the black k-shuffle amounts to rearranging the deck $b_1, b_2, \ldots, b_{12}$ to the sequence $\{B_n\} = b_k, b_{2k}, \ldots, b_{12k}$, here done with 12 cards so that the subscripts here are modulo 12. The face value of the first card immediately following (cyclically) the Ace of spades is now $F = f^k = 2^5 \equiv 6 \pmod{13}$, so that the spades now form a cyclic permutation of the successive powers mod 13 of its primitive root 6. We have $B_j = F^{j+E} = f^{k(j+E)}$ and also $B_j = b_{jk} = f^{jk+e}$, whence $kE \equiv e \pmod{12}$, and $B_j \equiv b_{12} \cdot F^j \pmod{13}$.

Let us relate the red deck R_n, required for reciprocity with B_j, to the starting deck r_n: For a fixed $n = B_j = b_{jk}$ we find from (2) and the analogous $R_n = j \equiv X(n) - E \pmod{12}$ that

$$x(n) \equiv kj + e \equiv k[X(n) - E] + e \pmod{12},$$

whence

$$r_n \equiv k \cdot R_n \pmod{12}. \tag{3}$$

Since $k^2 \equiv 1 \pmod{12}$ for $k = 1, 5, 7, 11$, the solution of (3) yields the requirement

$$R_n \equiv k \cdot r_n \pmod{12}. \tag{4}$$

The k-shuffle applied to the deck of 12 diamonds sequenced in natural order resulted in the sequence $\{d_n\} = k, 2k, \ldots, 12k \pmod{12}$ (see Fig. 4). The subsequent selection from position numbers $r_1, r_2, \ldots, r_{12}$ then furnished diamonds with face values $k \cdot r_1, k \cdot r_2, \ldots \pmod{12}$, i.e., precisely the sequence of values required by (4).

Note that the k-shuffle of the black deck is restricted to the values $k = 5, 7, 11$ because these are the only integers (other than the trivial 1) relatively prime to $p - 1 = 12$, a condition required for nondegeneracy of the procedure (see end of Step 1, Section 5c). It always leads to a cyclic permutation of successive power residues of one of the primitive roots $F = 2, 6, 7, 11$. A total of 48 different arrangements of the pairs of decks may thus be obtained.

8. Generalization. Generalization of the procedure to arbitrary primes p is now made obvious by the analysis above: For an arbitrary prime p, take a black deck of $p - 1$ cards arranged in an arbitrary cyclic permutation of successive power residues mod p of one of its primitive roots f, together with a correspondingly sequenced red deck of p cards. Apply k-shuffles of the red deck with arbitrary k, because every $k < p$ is relatively prime to p. Corresponding to it is a cyclic permutation of the black deck. Demonstrate the persistence of reciprocity.

If you wish to apply k-shuffles to the black deck, k must be chosen from the integers relatively prime to $p - 1$. In this case $p - 1 = km + z$ (for some integer m), and z will always be relatively prime to k, because any common factor would also be a common factor of $p - 1$ and k, contrary to the choice of k. Nondegeneracy of the black k-shuffle is therefore assured. However, an interesting complication now appears: If the black deck is k-shuffled, the auxiliary deck (diamonds) must be h-shuffled, where generally $h \neq k$. It is seen from (3) that its solution

$$R_n \equiv h \cdot r_n \pmod{12}$$

in reality requires h to satisfy $h \cdot k \equiv 1 \pmod{p - 1}$, i.e., $h = k^{-1} \pmod{p - 1}$. Thus h and k must be "associated integers." For practical values of p, a short table of corresponding admissible values k and h and of all relevant primitive roots is shown in Fig. 5.

Red deck: p =	11	13	17	23 (Hearts & Diamonds)
Black deck: $p - 1$ =	10	12	16	22 (Spades & Clubs)
k (rel.prime to $p - 1$):	1 3 7 9	1 5 7 11	1 3 5 7 9 11 13 15	1 3 5 7 9 13 15 17 19 21
$h = k^{-1} \pmod{p - 1}$:	1 7 3 9	1 5 7 11	1 11 13 7 9 3 5 15	1 15 9 19 5 17 3 13 7 21
f = prim.roots of p:	2 6 7 8	2 6 7 11	3 5 6 7 10 11 12 14	5 7 10 11 14 15 17 19 20 21

FIG. 5. Corresponding numerical values.

With k-shuffles of p red and $p - 1$ black cards, a total of $(p - 1)\phi(p - 1)$ different arrangements (where ϕ denotes Euler's totient function) of the pairs of decks may be obtained, all of which display reciprocity.

As is seen from Fig. 5, for a handy little deck of 10 black cards a 7-shuffle of the spades requires a 3-shuffle of the diamonds (and vice versa). This feature may be utilized to thwart the boasts of a would-be imitator: "Simplify" the above deck of 12 black cards to a deck of only 10 black cards (sequenced in successive powers mod 11 of its primitive root 2) and demonstrate by k-shuffling the red deck. Record the two final deck sequences. Then challenge him (possibly with a bet) to continue, requesting a 7-shuffle of the black deck. After he fails (because he consistently saw you use only $h = k$ with the larger deck), you can restore the two decks as recorded, do it "right" for the audience (this time with the surprise $h = 3 \neq k$), and win your bet!

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EULER LINES, TRITANGENT CENTERS, AND THEIR TRIANGLES

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1. Introduction. Notations and Early History. In the Euclidean geometry of the triangle, it has been known since ancient times that the medians concur, and so do the altitudes, the perpendicular bisectors of the sides, and the internal angle bisectors. Let us call the triangle ABC, and name these points of concurrence as centroid G, orthocenter H, circumcenter O, and incenter I_0, respectively. Let I_1, I_2, I_3 be the excenters opposite the angles A, B, C, respectively. A point which is either the incenter or one of the excenters of a triangle is called a "tritangent center," and upon occasion will be denoted by I.

The first systematic study of mutual relationships between these centers was made in the eighteenth century by Euler [**8**], who showed that O, G, H are collinear, with G dividing the segment OH in the ratio $1:2$. He also found a number of expressions for the distances between these centers. Early in the nineteenth century Brianchon and Poncelet [**7**], [**16**] discussed the nine-point circle whose center N is at the midpoint of OH. Then shortly thereafter Feuerbach [**9**], [**16**] proved that the nine-point circle touches all four tritangent circles.

In all these developments the triangle ABC was regarded as given and the properties of the centers were investigated. In the present note this process is reversed; the points O, G, H on the Euler line are regarded as given and it is investigated how the position of any one tritangent center I can serve to determine the triangle. This quickly leads to the discovery that the incenter must always lie inside the circle on GH as diameter, and that all three excenters must lie outside it. This suggests the following nomenclature:

DEFINITION 1. For nonequilateral triangles ABC the "critical circle" shall mean the circle on GH as diameter.

Furthermore, outside the critical circle there is a region, bounded by a closed bicircular quartic curve, inside which there cannot lie any tritangent center. We shall accordingly call this region the "acentric lacuna." Its precise definition must be deferred; it is given in §8.

If due allowances are made for degenerate and limiting cases, then the resulting picture is complete in the sense that all points not in the acentric lacuna are possible positions of a tritangent center I of a triangle with the given Euler line OGH. As a result the plane can be partitioned into regions corresponding to the type of tritangent center therein and the shape of the triangle ABC. The boundaries of these regions are formed by a family of bicircular quartics. These quartics can be generated as inverses of conics, but that is beyond the scope of the present note.

Since the Euler line of an equilateral triangle is indeterminate, it will be assumed initially that ABC is not equilateral. The equilateral triangle as a limiting case is discussed in §7.

2. Preliminary Results. Let R be the circumradius, r the inradius, and r_1, r_2, r_3 the radii of the excircles about the excenters I_1, I_2, I_3. Then elementary trigonometric results are [**12**]:

$$r = 4R \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C = R(\cos A + \cos B + \cos C - 1), \tag{1}$$

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$$r_1 = 4R\sin\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C = R(\cos B + \cos C - \cos A + 1), \tag{2}$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A\cos B\cos C, \tag{3}$$

$$OI_0^2 = R(R - 2r), \tag{4}$$

$$OI_1^2 = R(R + 2r_1), \tag{5}$$

$$I_0N = \frac{1}{2}R - r, \tag{6}$$

$$I_1N = \frac{1}{2}R + r_1, \tag{7}$$

$$OH^2 = R^2(1 - 8\cos A\cos B\cos C). \tag{8}$$

Of these (4), (5), and (8) are usually attributed to Euler [**5**], [**8**] though Mackay [**14**] cites several earlier forms of (4) in short-lived English journals of the mideighteenth century. Formulas (6) and (7) are equivalent to Feuerbach's theorem [**12**], [**16**].

3. The Critical Circle. By considering angles subtended at vertices it is easily shown that the circumcenter O and the incenter I_0 cannot coincide if ABC is not equilateral. Hence by (4) and (6)

$$OI_0^2 - 4 \times I_0N^2 = R(R - 2r) - (R - 2r)^2 = 2r(R - 2r) = \frac{2r}{R} \times OI_0^2 > 0,$$

since $OI_0 > 0$. Hence

$$OI_0 > 2 \times I_0N. \tag{9}$$

Similarly from (5) and (7)

$$OI_1^2 - 4 \times I_1N^2 = R(R + 2r_1) - (R + 2r_1)^2 = -2r_1(R + 2r_1) < 0,$$

whence

$$OI_1 < 2 \times I_1N. \tag{10}$$

Now the locus of points P for which $OP = 2 \times PN$ is a circle of Appollonius [**1**]. Since $OG = 2 \times GN$ and $OH = 2 \times HN$, this circle has diameter GH, and thus it is the critical circle of Definition 1. Further P lies inside or outside this circle according as $OP \gtrless 2 \times PN$, so we have:

THEOREM 1. *The incenter of a nonequilateral triangle lies inside the critical circle and all the excenters lie outside it.*

4. The Cubic Equation for Angle Cosines. Suppose that the position of the incenter I_0 relative to the Euler line is given, so that OI_0, I_0N, and OH are known quantities. Then (1), (3), (4), (6), and (8) can be used to construct a cubic whose roots are the cosines of the angles of ABC. Let $OI_0 = \rho$, $I_0N = \sigma$, $OH = \kappa$, and put

$$\alpha = 2\sigma^2/3\rho^2, \quad \beta = \kappa^2/2\rho^2.$$

Then by (4) and (6)

$$R = OI_0^2/2 \times I_0N = \rho^2/2\sigma, \tag{11}$$

and by (8)

$$\cos A\cos B\cos C = \frac{1}{8}\left(1 - \frac{\kappa^2}{R^2}\right) = \frac{1}{8}\left(1 - \frac{4\kappa^2\sigma^2}{\rho^4}\right) = \frac{1}{8}(1 - 12\alpha\beta). \tag{12}$$

Hence by (3)

$$\cos^2 A + \cos^2 B + \cos^2 C = \frac{3}{4} + 3\alpha\beta = \frac{3}{4}(1 + 4\alpha\beta). \tag{13}$$

By (6) and (11)

$$\frac{r}{R} = \frac{\frac{1}{2}R - \sigma}{R} = \frac{1}{2} - \frac{2\sigma^2}{\rho^2} = \frac{1}{2}(1 - 6\alpha). \tag{14}$$

Then by (1)

$$\cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{3}{2}(1 - 2\alpha), \tag{15}$$

and by (13) and (15)

$$\begin{aligned} &\cos B \cos C + \cos C \cos A + \cos A \cos B \\ &\quad = \frac{1}{2}(\cos A + \cos B + \cos C)^2 - \frac{1}{2}(\cos^2 A + \cos^2 B + \cos^2 C) \\ &\quad = \frac{9}{8}(1 - 2\alpha)^2 - \frac{3}{8}(1 + 4\alpha\beta) \qquad (16) \\ &\quad = \frac{3}{4}(6\alpha^2 - 2\alpha\beta - 6\alpha + 1). \end{aligned}$$

Hence by (12), (15), and (16) the cubic in c

$$c^3 + \frac{3}{2}(2\alpha - 1)c^2 + \frac{3}{4}(6\alpha^2 - 2\alpha\beta - 6\alpha + 1)c + \frac{1}{8}(12\alpha\beta - 1) = 0 \tag{17}$$

has roots $c = \cos A, \cos B, \cos C$. Expressed in terms of ρ, σ, κ it can be written

$$\rho^4(1 - 2c)^3 + 8\rho^2\sigma^2 c(3 - 2c) - 16\sigma^4 c - 4\sigma^2\kappa^2(1 - c) = 0.$$

Applying a similar argument ot the excenter I_1, we can put $OI_1 = \rho_1$, $I_1N = \sigma_1$, and

$$\alpha_1 = 2\sigma_1^2/3\rho_1^2, \quad \beta_1 = \kappa^2/2\rho_1^2. \tag{18}$$

Then, successively,

$$\begin{aligned} R &= OI_1^2/2 \times I_1N = \rho_1^2/2\sigma_1, \\ &\cos A \cos B \cos C = \tfrac{1}{8}(1 - 12\alpha_1\beta_1), \qquad (19) \\ &\cos^2 A + \cos^2 B + \cos^2 C = \tfrac{3}{4}(1 + 4\alpha_1\beta_1), \end{aligned}$$

$$\frac{r_1}{R} = \frac{\sigma_1 - \frac{1}{2}R}{R} = \frac{2\sigma_1^2}{\rho_1^2} - \frac{1}{2} = \frac{1}{2}(6\alpha_1 - 1),$$

$$\cos A - \cos B - \cos C = 1 - \frac{r_1}{R} = \frac{3}{2}(1 - 2\alpha_1), \tag{20}$$

$$\begin{aligned} &\cos B \cos C - \cos C \cos A - \cos A \cos B \\ &\quad = \frac{1}{2}(\cos A - \cos B - \cos C)^2 - \frac{1}{2}(\cos^2 A + \cos^2 B + \cos^2 C) \\ &\quad = \tfrac{9}{8}(1 - 2\alpha_1)^2 - \tfrac{3}{8}(1 + 4\alpha_1\beta_1) = \tfrac{3}{4}(6\alpha_1^2 - 2\alpha_1\beta_1 - 6\alpha_1 + 1). \qquad (21) \end{aligned}$$

Equations (19), (20), (21) are of the same form as (12), (15), (16) but with the signs of $\cos B$ and $\cos C$ reversed. Hence the cubic in c

$$c^3 + \frac{3}{2}(2\alpha_1 - 1)c^2 + \frac{3}{4}(6\alpha_1^2 - 2\alpha_1\beta_1 - 6\alpha_1 + 1)c + \frac{1}{8}(12\alpha_1\beta_1 - 1) = 0 \tag{22}$$

has roots

$$c = \cos A, -\cos B, -\cos C$$
$$= \cos A, \cos(\pi - B), \cos(\pi - C).$$

If $\theta_1, \theta_2, \theta_3$ are the inverse cosines of the roots of the cubic (22), then

$$\theta_1 + \theta_2 + \theta_3 = A + (\pi - B) + (\pi - C) = \pi + 2A.$$

Thus, if the angles $\theta_1, \theta_2, \theta_3$ are found by means of the cubic (22), then the angle opposite the given excenter can be identified as $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \pi)$, and this will be equal to one of θ_1, θ_2, or θ_3. The other angles of the triangle are then equal to the supplements of the other two thetas.

Since the cubics (17) and (22) are of the same form, the results for incenters and excenters can be combined by dropping suffixes, thus:

THEOREM 2. *If I is a tritangent center of ABC and $OI = \rho$, $IN = \sigma$, $OH = \kappa$, then the roots of the cubic in c*

$$\rho^4(1 - 2c)^3 + 8\rho^2\sigma^2c(3 - 2c) - 16\sigma^4c - 4\sigma^2\kappa^2(1 - c) = 0 \tag{23}$$

are cosines of the angles of the triangle or of their supplements. If $\cos\theta_1, \cos\theta_2, \cos\theta_3 : (0 < \theta_1, \theta_2, \theta_3 < \pi)$ are the roots of this cubic, and

(i) *$\theta_1 + \theta_2 + \theta_3 = \pi$, then I is the incenter and the angles of the triangle are $\theta_1, \theta_2, \theta_3$;*

(ii) *$\theta_1 + \theta_2 + \theta_3 \neq \pi$, then I is an excenter and the angle opposite it is $\frac{1}{2}(\theta_1 + \theta_2 + \theta_3 - \pi)$. Further, this is equal to one of $\theta_1, \theta_2, \theta_3$ and the remaining two are supplements of the remaining angles of the triangle.*

5. Loci of Tritangent Centers of Isosceles Triangles. Triage. If ABC is isosceles, with $AB = AC$, then the Euler line is the bisector of the angle at A. Hence both the incenter and the excenter opposite A lie on the Euler line.

The projections of AB and AH onto AC are identical, so

$$AB\cos A = 2R\sin C\cos A = AH\sin C, \text{ whence } AH = 2R\cos A.$$

For $0 < A < \pi/3$, as in Fig. 1, $AH = 2R\cos A > R = AO$, so

$$\kappa = OH = R(2\cos A - 1).$$

Similarly, if $\pi/3 < A < \pi$, then

$$\kappa = R(1 - 2\cos A).$$

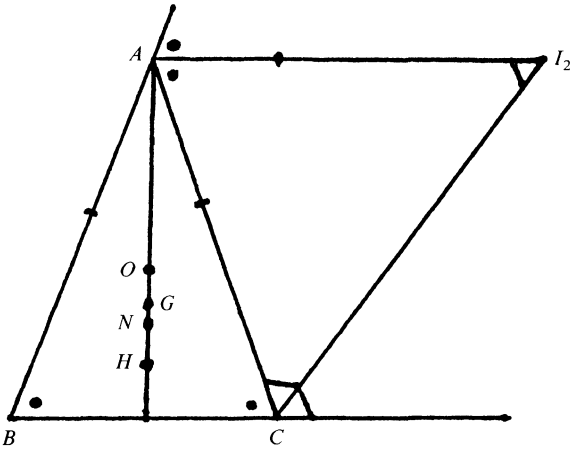


FIG. 1.

In each case AI_2 is parallel to BC so the triangle ACI_2 is isosceles, and $AI_2 = AC = 2R\sin B$. Also

$$\cos A = \cos(\pi - 2B) = 2\sin^2 B - 1.$$

Now let $\rho = OI_2$, $\sigma = I_2N$ so that (ρ, σ) can be regarded as bipolar coordinates of I_2 with respect to poles O and N [**13**]. Then

$$\rho^2 = OI_2^2 = AO^2 + AI_2^2 = R^2 + 4R^2\sin^2 B = R^2(2\cos A + 3),$$

$$AN = R \pm \frac{1}{2}\kappa = \frac{1}{2}R(2\cos A + 1) \text{ according as } A \lesseqgtr \pi/3,$$

$$\sigma^2 = I_2N^2 = AN^2 + AI_2^2 = \frac{1}{4}R^2(2\cos A + 3)^2.$$

Hence $\sigma = \frac{1}{2}R(2\cos A + 3)$ and

$$2(\sigma^2 - \rho^2) = \frac{1}{2}R^2\{(2\cos A + 3)^2 - 4(2\cos A + 3)\}$$

$$= \frac{1}{2}R^2(2\cos A - 1)(2\cos A + 3) = \pm\kappa\sigma.$$

Squaring, it follows that

$$4(\sigma^2 - \rho^2)^2 = \kappa^2\sigma^2 \tag{24}$$

is the bipolar equation of the locus of I_2 relative to the Euler line OH.

If O is taken as origin of cartesian coordinates x, y, and H as unit point $(1,0)$ on the x-axis, then

$$\rho^2 = x^2 + y^2, \quad \sigma^2 = \left(x - \frac{1}{2}\right)^2 + y^2, \quad \kappa = 1. \tag{25}$$

The cartesian equivalent of (24) then becomes

$$y^2 = 3x^2 - x.$$

This is a hyperbola of eccentricity 2 whose major axis is the Euler line. One branch touches the critical circle at G and the other branch goes through O. The excenter I_2 lies on the first branch or the second according as the apex angle A of the isosceles triangle is greater than or less than $60°$.

The hyperbola divides the plane into three regions. In the region containing the negative x-axis the point $(-1 - \sqrt{2}, 0)$ is the excenter opposite the right angle for an isosceles right-angled triangle. By continuity all other tritangent centers in this region are also excenters opposite the largest angle of their triangles. Similarly a tritangent center in the region between the two branches is an excenter opposite the intermediate angle of its triangle. The remaining region contains the critical circle, so an excenter opposite the least angle of its triangle must lie inside the region but outside the critical circle.

A classification of elements into three disjoint sets is sometimes called a "triage." Hence:

DEFINITION 2. For a given Euler line the "triage-hyperbola" shall mean the hyperbola defined by the bipolar equation

$$4(\sigma^2 - \rho^2)^2 = \kappa^2\sigma^2.$$

6. Quartic Loci of I for Triangles with One Fixed Angle. If one angle of ABC is equal to θ, then the bipolar coordinates ρ, σ of both the incenter and the excenter opposite that angle must satisfy (23) with $c = \cos\theta$. If we substitute (25) and rearrange, it follows that the cartesian coordinates of these centers satisfy

$$(2\cos\theta - 1)(2\cos\theta + 1)^2(x^2 + y^2)^2 - 8\cos\theta(2\cos\theta + 1)x(x^2 + y^2) + 2(2\cos^2\theta - \cos\theta + 2)(x^2 + y^2) + 16(\cos\theta)x^2 - 4(\cos\theta + 1)x + 1 = 0. \tag{26}$$

In general, this is the equation of a bicircular quartic symmetric about the x-axis. Let us call the curve $Q(\theta)$. Particular cases of note are:

$Q(0)$: $\{(x - \frac{2}{3})^2 + y^2 - \frac{1}{9}\}^2 = 0$,

the critical circle, repeated;

$Q(\pi/3)$: $(x - \frac{1}{2})\{(x - \frac{1}{2})^2 + y^2\} = 0$,

the line perpendicular to the Euler line at N, and a point circle at N [**12**];

$Q(\frac{1}{2}\pi)$: $(x^2 + y^2 - 2)^2 = 5 - 4x$,

the limaçon of Pascal with node at the orthocentre H [**13**];

$Q(2\pi/3)$: $2x^2 - 6y^2 + 2x - 1 = 0$,

a hyperbola of eccentricity $2/\sqrt{3}$;

$Q(\pi)$: $3(x^2 + y^2)^2 + (8x - 10)(x^2 + y^2) + 16x^2 - 1 = 0$, (27)

a closed curve touching the critical circle and the triage-hyperbola at G. It has a simple cusp at $(-1,0)$ and lies entirely between its two bitangents $x = \frac{1}{2}$ and $x = -7/4$.

DEFINITION 3. For a given Euler line the "extremal quartic" shall mean the bicircular quartic defined by equation (27).

For given values of θ and x the equation (26) is a quadratic in y^2, so it is easy to compute coordinates and plot $Q(\theta)$. For incenters the only relevant parts of the curves are those inside the critical circle.

7. Limiting and Degenerate Cases. Not all locations of I relative to the Euler line correspond to real, finite, nondegenerate triangles.

First, if I approaches the nine-point center N, then by (11) the circumradius R tends to infinity and the cubic (23) approaches the form $(1 - 2c)^3 = 0$. That is, if $\kappa = OH$ is kept constant, then N must be interpreted as the limiting position of incenters of near-equiangular triangles of increasing size.

Next, if I is on $Q(0)$, the repeated critical circle, then at least one root of (23) is $c = 1 = \cos 0$, so the triangle is degenerate, with at least one zero angle. If the zero angle is A, then B, C, I_0, I_1 all coincide with I.

If I is on the extremal quartic $Q(\pi)$, then at least one root of (23) is $c = -1 = \cos\pi$ and again the triangle is degenerate, with a zero angle; but I is now an excenter not opposite that zero angle.

If I is at G, where $Q(0)$ and $Q(\pi)$ touch, then the triangle must be regarded as doubly degenerate, with angles $0, 0, \pi$, circumradius $\kappa/3$, but all sides zero.

8. The Acentric Lacuna. A real triangle corresponds to a location of I relative to the Euler line if and only if the resulting cubic (23) has all its roots real and in the range $-1 \leqslant c \leqslant 1$. If $z = 2c + 2\alpha - 1$ is substituted in (17), then the cubic for z becomes

$$z^3 + 6\alpha(\alpha - \beta - 1)z - 2\alpha(10\alpha^2 - 6\alpha\beta - 15\alpha - 3\beta + 6) = 0.$$

The discriminant of the latter cubic is [17]

$$\Delta = 8\alpha^3(\alpha - \beta - 1)^3 + \alpha^2(10\alpha^2 - 6\alpha\beta - 15\alpha - 3\beta + 6)^2 \tag{28}$$
$$= \alpha^2\{(3\alpha - \beta + 2)^2 - 24\alpha\}\{12\alpha^2 - 8\alpha\beta - 20\alpha + 9\},$$

and the roots are all real if and only if $\Delta \leqslant 0$.

Expressed in terms of ρ, σ, κ, the second factor becomes

$$\{(4\rho^2 + 4\sigma^2 - \kappa^2)^2 - 64\rho^2\sigma^2\}/4\rho^4. \tag{29}$$

Now $\rho, \sigma, \frac{1}{2}\kappa$ are sides of the triangle ONI, so they are nonnegative and $\rho + \sigma \geqslant \frac{1}{2}\kappa$ with equality only if I lies on ON. Hence

$$\rho^2 + 2\rho\sigma + \sigma^2 \geqslant \frac{1}{4}\kappa^2,$$

and

$$(4\rho^2 + 4\sigma^2 - \kappa^2)^2 \geqslant 64\rho^2\sigma^2,$$

so by (29) the second factor of (28) vanishes if I is on ON, but is positive for I elsewhere. Also the first factor α^2 vanishes only if $\sigma = 0$. That would place I at N, the limiting equilateral case already considered.

Hence the roots are real only if the third factor of (28) is negative. Expressed in terms of ρ, σ, κ, this gives

$$27\rho^4 - 40\rho^2\sigma^2 + 16\sigma^4 - 8\sigma^2\kappa^2 \leqslant 0.$$

This inequality determines a region of the plane whose boundary has the bipolar equation

$$27\rho^4 - 40\rho^2\sigma^2 + 16\sigma^4 - 8\sigma^2\kappa^2 = 0.$$

This is precisely the form taken by (23) when $c = -1$. That is the extremal quartic $Q(\pi)$, which leads to the following:

DEFINITION 4. The acentric lacuna is the region inside the extremal quartic $Q(\pi)$.

Thus all positions of I inside the acentric lacuna except those on the Euler line lead to cubics (23) with complex roots, and are therefore impossible locations for tritangent centers of a real triangle. If I is on the Euler line and inside the acentric lacuna, then its cartesian coordinates satisfy $-1 < x < 1/3, y = 0$. For $y = 0$ the equation (23) becomes

$$(2cx + x - 1)^2(2cx^2 - x^2 - 2x + 1) = 0.$$

Its roots in c are therefore $(x^2 + 2x - 1)/2x^2$, $(1 - x)/2x$, the latter repeated. These all lie outside the permissible range $-1 \leqslant c \leqslant 1$ so no position of I inside the acentric lacuna is possible.

Positions outside the acentric lacuna and not on the critical circle or the extremal quartic are either inside the critical circle or outside both critical circle and extremal quartic. The center of the critical circle is $(2/3, 0)$ and the corresponding roots of (26) are $\frac{1}{4}, \frac{1}{4}, \frac{7}{8}$; all within the permissible range for cosines of real angles. Since these roots are continuous functions of the position of I, they cannot exceed 1 unless I moves across $Q(0)$, the critical circle, and they cannot go below -1 unless I crosses $Q(\pi)$. Hence all I within the critical circle are permissible as incenters of real triangles (provided N is regarded as a limiting case.)

Similarly if I is at $(2, 0)$, outside both critical circle and acentric lacuna, then the roots of (26) are $-\frac{1}{4}, -\frac{1}{4}, \frac{7}{8}$, also within permissible range, and a similar continuity argument applies. We thus have:

THEOREM 4. *No tritangent center can lie within the acentric lacuna, but all points outside the lacuna and outside the critical circle can be excenters of real triangles. All points inside the critical*

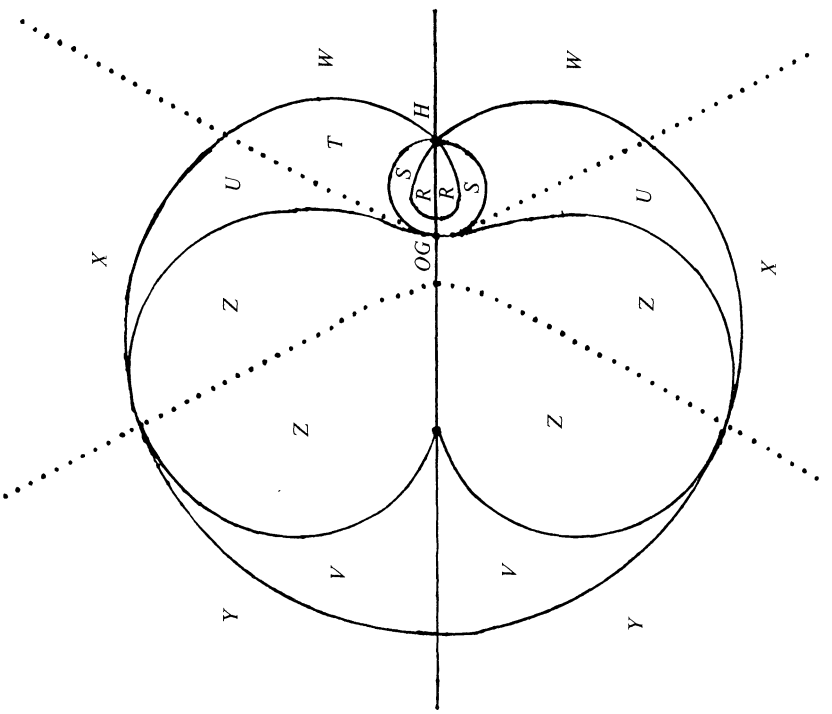

W
W
H
T
S
R
R
S
U
U
X
X
OG
Z
Z
Z
Z
V
V
Y
Y

FIG. 3.

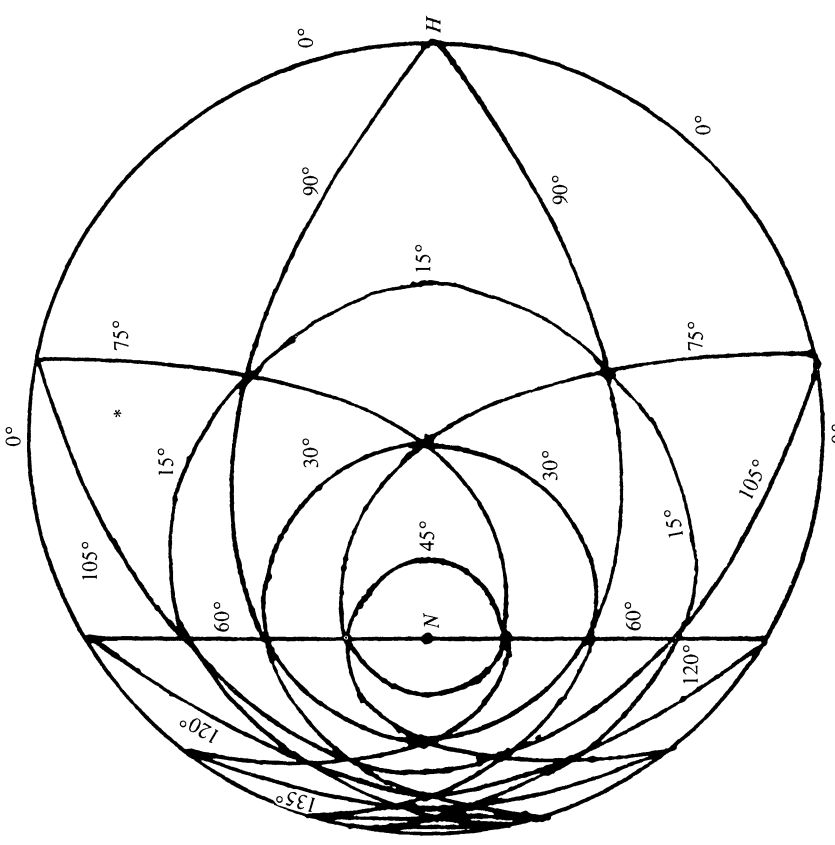

H
0°
0°
90°
90°
15°
75°
75°
0°
0°
*
15°
30°
30°
105°
105°
45°
15°
60°
60°
N
120°
120°
135°
G

FIG. 2.

circle except the nine-point center can be incenters of real, finite triangles. Points on the borders between these regions can be regarded as excenters of degenerate triangles, and the nine-point center can be regarded as corresponding to an infinite equilateral triangle.

9. Tritangent Centers and Their Regions. Fig. 2 shows the interior of the critical circle partitioned into regions by the curves $Q(\theta)$ for θ at intervals of 15°. If the position of a tritangent center is given in one of these regions, then it must be an incenter, and the angles of the corresponding triangle must lie in ranges determined by the $Q(\theta)$ surrounding that region. For example, the point marked * lies inside a curvilinear triangle. One side is part of $Q(15°)$ and the opposite vertex is on $Q(0°)$; hence one angle of the triangle with incenter * is in the range 0° to 15°. Similarly the other angles of the triangle are in the ranges 60° to 75°, and 90° to 105°. A finer network of $Q(\theta)$ would narrow these ranges.

For tritangent centers outside the critical circle, the nature of the center can be determined from the region in which it lies. Fig. 3 shows the Euler line, the critical circle, the triage-hyperbola, the extremal quartic, and the limaçon $Q(\frac{1}{2}\pi)$. These suffice to partition the plane into regions as shown in Table 1.

TABLE 1.

If I is in region:	then it is:	and the triangle is:
R	the incenter	acute-angled
S	the incenter	obtuse-angled
T	the excenter opposite the least angle	obtuse-angled
U	the excenter opposite the middle angle	obtuse-angled
V	the excenter opposite the greatest angle	obtuse-angled
W	the excenter opposite the least angle	acute-angled
X	the excenter opposite the middle angle	acute-angled
Y	the excenter opposite the greatest angle	acute-angled
Z	not a tritangent center	

10. Types of Tritangent Centers on Individual $Q(\theta)$. A tritangent center I on a particular quartic $Q(\theta)$ can be of the following three types:

(a) incenter of a triangle with an angle θ;
(b) excenter opposite an angle θ of the triangle;
(c) excenter of a triangle with an angle $\pi - \theta$, but not opposite that angle.

If I moves along $Q(\theta)$, then changes between types occur as I crosses the critical circle or touches the extremal quartic. The pattern of the changes depends on the value of θ, so it is best explained by an example. Fig. 4 shows $Q(80°)$, with various special points on it. The points and corresponding triangles are as shown in Table 2; the angles $\theta = 80°$ and $\pi - \theta = 100°$ concerned are underlined.

Changes of type of center follow the same pattern for $60° < \theta < 120°$. For other ranges of θ the pattern can be sorted out in similar fashion.

11. Remarks and Recent History. Restrictions on the location of the incenter have been noted previously in various forms. Bottema [6] attributes $I_0H\sqrt{2} \leqslant OH$ to S. G. Guba; Radford [2] shows that OI_0H is obtuse; Bankoff [2], [3] that I_0 lies in BOH if $A < B < C$; Blundon [4] that the projection of I_0 on the Euler line lies between G and H, and that GI_0H is obtuse.

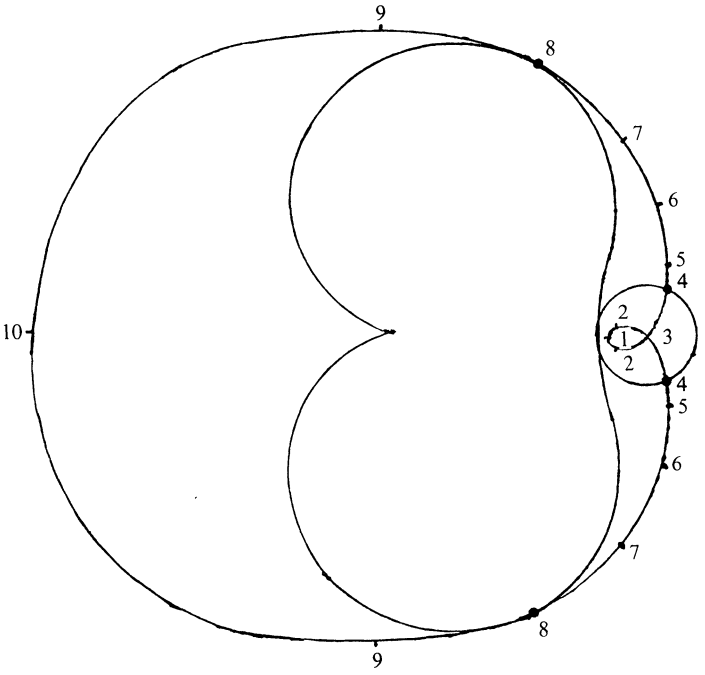


FIG. 4.

TABLE 2.

Point and location	Angles of the triangle	Type	Opposite angle
1. Euler line.	50°, 50°, 80°	(a)	—
2. $Q(60°), x = \frac{1}{2}$.	40°, 60°, 80°	(a)	—
3. node.	20°, 80°, 80°	(a)	—
4. critical circle.	0°, 100°, 80°	(a) & (c)	0°
5. bitangent contact.	8.4°, 100°, 71.6°	(c)	8.4°
6. triage-hyperbola.	40°, 100°, 40°	(c)	40°
7. $Q(60°), x = \frac{1}{2}$.	60°, 100°, 20°	(c)	60°
8. contact $Q(180°)$	80°, 100°, 0°	(c) & (b)	80°
9. triage-hyperbola.	80°, 80°, 20°	(b)	80°
10. Euler line.	80°, 50°, 50°	(b)	80°

The last three writers all commented on confusion caused by a problem in Hobson's *Trigonometry* [**12**]. In the present notation the problem is:

If OI_0H is an equilateral triangle, show that

$$\cos A + \cos B + \cos C = 3/2.$$

Both premise and conclusion imply that ABC is equilateral and hence that OI_0H degenerates to a point. It seems unlikely that Hobson had this in mind.

I have been unable to find previous mention of the rôle of the critical circle in separating incenter from excenters; still less any hint of the existence of an acentric region.

It follows readily from Theorem 2 that no general ruler-and-compass reconstruction of a triangle from Euler line and a tritangent center is possible [**11**]. The present investigation stems

from one of a number of triangle reconstruction problems posed by Wernick [**18**]. With respect to the triangle, it is the most symmetric of Wernick's problems; that is probably why it leads so unexpectedly to the critical circle and the acentric lacuna.

My thanks to Professors R. Blum, H. S. M. Coxeter, and L. Sauvé, for references and helpful comments.

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MISCELLANEA

127.

The mathematics that I learned...became a deeper part of my nature, as resolute consistency and something like mental courage. From a possibly very small area, which is not to be doubted, you keep on going in one and the same direction, never asking yourself where you might end up, refusing to look right or left, as though heading towards some goal without knowing which, and so long as you make no false step and maintain the connection of the steps, nothing will happen to you, you progress into the unknown—the only way to conquer the unknown *gradually*.

—Elias Canetti, *The Tongue Set Free*,
The Seabury Press, New York, 1979,
pp. 236–237

ANSWER TO PHOTO ON PAGE 283

Béla Szökefalvi-Nagy and Ciprian Foiaş, usually known as Nagy-Foiaş.

PROGRESS REPORTS

EDITED BY THOMAS BANCHOFF AND RICHARD MILLMAN

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

THE INNER FUNCTION PROBLEM

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Do there exist nonconstant inner functions in the unit ball B of $\mathbb{C}^n$ when $n > 1$?

This is a question about several complex variables. To define our terms, B is the set of all points $z = (z_1,\ldots,z_n)$ in $\mathbb{C}^n$ (the n-dimensional vector space over the complex field $\mathbb{C}$) whose coordinates satisfy $\Sigma|z_i|^2 < 1$. A complex-valued function f, defined in B, is "holomorphic" (which is another word for "analytic") if f is holomorphic in each variable separately. If such an f satisfies $|f(z)| \leqslant 1$ for all z in B, and if its radial limits $f^*(z) = \lim_{r\to 1} f(rz)$ have absolute value 1 at almost every point z in the sphere S that bounds B, then f is said to be "inner."

The existence of these radial limits follows easily, for every bounded holomorphic f, from the analogous one-variable situation, where B and S are replaced by the more familiar open unit disc U and the unit circle T, and Fatou's classical theorem is available: if g is a bounded holomorphic function defined in U, then $\lim g(re^{i\theta})$, as $r \to 1$, exists for almost every $e^{i\theta}$ in T. What makes inner functions special is that $|f^*(z)|$ is to be 1 a.e. on S.

For brevity, the class of all *nonconstant* inner functions in B will be called $I(B)$.

The analogous class $I(U)$ has been intensively studied for decades and plays an all-pervasive role in the function theory of the disc. A glance at Garnett's 1981 book "Bounded Analytic Functions" should convince anyone of this. Functional analysts became interested in $I(U)$ when Beurling showed, in 1949, how $I(U)$ describes the closed invariant subspaces of the unilateral shift operator on a Hilbert space. This involved the so-called inner-outer factorization, and was only the first of a large number of theorems in that area.

When the function theory of B began to be investigated, it was thus perfectly natural that one wanted to know all about $I(B)$. A difficulty arose immediately: No one knew any member of $I(B)$! Hence the question with which this article begins. It was first asked, as far as I know, in 1966.

The following facts were quickly discovered [3]: No f in $I(B)$ can have even one point of continuity on S. Worse than that: To every f in $I(B)$ corresponds a dense subset D_f of S (a G_δ-set,

in fact) such that f maps every radius of B that ends at a point of D_f onto a dense subset of U. And even worse things are true.

In spite of these wild oscillations that *every* f in $I(B)$ must have at *every* boundary point of B, it was also known that the radial limits f^* of every f in $I(B)$ must behave so well that they induce measure-preserving maps from S to T. More precisely, the equality

$$\int_S (h \circ f^*)\, d\sigma = \int_T h\, dm$$

must hold for every continuous function h on T if f is in $I(B)$ and $f(0) = 0$. Here σ and m are the positive rotation-invariant measures on S and T that have total mass 1.

In view of all this, it is no wonder that "$I(B)$ is empty" was the prevailing conjecture. (A "heuristic proof" of it appears on p. 503 of the May 1983 issue of Bull. Amer. Math. Soc.) In fact, many stronger conjectures looked plausible [3].

But in November, 1981, A. B. Aleksandrov [1] came up with a surprise (for which he was awarded the 1982 Salem prize):

$$I(B) \textit{ is not empty}!$$

Eric Løw [2] proved the same about six weeks later, without knowing of Aleksandrov's work, by pushing an earlier construction of Hakim and Sibony a bit further than they had done.

A remarkable feature of both proofs is that they use essentially nothing from the theory of several complex variables. Aleksandrov made good use of his previous experience with H^p when $p < 1$; the Hakim-Sibony-Løw proof is more geometric. Both are complicated, and both are basically real-variable proofs. To give some of the flavor, here is a simple case of Aleksandrov's main theorem ("Re" denotes "real part"):

If ϕ is a positive lower semicontinuous integrable function on S, and u is the Poisson integral of ϕ, then there is a holomorphic function h in B, with $h(0) = 0$, so that $\operatorname{Re} h \leqslant u$ *in* B *and* $(\operatorname{Re} h)^* = \phi$ a.e. *on* S.

Taking $\phi = 1$ gives $f = \exp(h - 1)$ in $I(B)$.

Not only is $I(B)$ nonempty, it is huge. For example:

(a) *$I(B)$ is dense in the unit ball of $H^\infty(B)$, in the topology of uniform convergence on compact subsets.*

(b) *If g is holomorphic in the unit ball of $\mathbb{C}^{n-1}$ and $|g| < 1$, then some f in $I(B)$ satisfies*

$$f(z_1, \ldots, z_{n-1}, 0) = g(z_1, \ldots, z_{n-1}).$$

(c) *If $F \not\equiv 0$ is a uniformly continuous holomorphic function in B, then some f in $I(B)$ has the same zeros in B as F, including multiplicities.*

But—and this kills all hopes of using $I(B)$ in a decent theory of factorization as it exists in U—the quotient F/f *cannot* be bounded in B; it cannot even lie in any H^p-space, for any $p > 0$.

The significance of the Aleksandrov-Løw breakthrough is not so much that we now have lots of inner functions to play with. For the above-mentioned reasons, it is most unlikely that they will ever be as important in B as they are in U. The important thing is that this work has developed techniques for constructing holomorphic functions (bounded, or in H^p) of several complex variables that solve many other previously untouchable problems concerning, for example, prescribed boundary values of $|f|$, prescribed zero-varieties, and interpolation on analytic subvarieties. These techniques are the real benefit that has come from the solution of the inner function problem.

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NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

TAYLOR EXPANSION OF $\exp(\sum_{k=0}^{\infty} a_k z^k)$ AND SOME APPLICATIONS

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1. Introduction. The application of Taylor series expansion of functions, in particular that of $\exp(z)$, is well known in pure and applied mathematics. Functions such as $h(z) = \exp(\sum_{k=0}^{\infty} a_k z^k)$ appear in several areas of mathematics such as complex and harmonic analysis, probability theory and statistics, and numerical analysis. In all these areas the Taylor (coefficients) expansion of this function plays extremely important roles. Therefore, for computational reasons, it is necessary to have explicit formulae expressing the Taylor coefficients of $h(z)$ in terms of the (known) constants a_k, $k = 0,1,2,\ldots$.

It is recommended by several authors including Wiener and Masani [**5**, p. 103] that the Taylor coefficients of $h(z)$ can be obtained by writing the function in the form $\sum_{k=0}^{\infty} \frac{1}{k!}(\sum_{j=0}^{\infty} a_j z^j)^k$. Although this can be used to find the first few coefficients of $h(z)$, for k moderately large the task of finding the coefficient of z^k becomes extremely difficult as there is no simple method to expand $(\sum_{j=0}^{\infty} a_j z^j)^m$, $m = 2,3,\ldots$.

Another method used by Kendall and Stuart [**2**, p. 68] amounts to writing the function $h(z)$ as

$$\exp(a_0)\prod_{k=1}^{\infty}\exp(a_k z^k) = \exp(a_0)\prod_{k=1}^{\infty}\left(\sum_{j=0}^{\infty}\frac{a_k^j}{j!}z^{kj}\right).$$

Then the nth Taylor coefficient of $h(z)$ can be written as $\exp(a_0)\Sigma_n'\prod_{k=1}^{n} a_k^{P_k}/P_k!$, where the summation Σ_n' extends over all n-tuples $(P_1,\ldots,P_n)$ of nonnegative integers with $\sum_{k=1}^{n} kP_k = n$. Although this method provides an explicit formula for the nth Taylor coefficient of $h(z)$, it is not computationally feasible for n large.

It should be noted that in the previous two methods $\exp(z) = \sum_{k=0}^{\infty} z^k/k!$ plays an important role, but the fact that the Taylor coefficients of $\exp(z)$, that is $b_k = 1/k!$, can be generated recursively from $b_{k+1} = \frac{1}{k+1} b_k$ with $b_0 = 1$ does not play any role at all.

In this note, motivated by this simple observation on the recursion of the Taylor coefficients of $\exp(z)$, we provide a very simple computationally feasible recursive formula for finding the Taylor coefficients of $\exp(\sum_{k=0}^{\infty} a_k z^k)$. In fact, it is shown that if $\exp(\sum_{k=0}^{\infty} a_k z^k) = \sum_{k=0}^{\infty} c_k z^k$, then the c_k's can be obtained recursively via the relation

$$c_{k+1} = \sum_{j=0}^{k}\left(1 - \frac{j}{k+1}\right)a_{k+1-j}c_j, \quad k = 0,1,2,\ldots \tag{1}$$

with $c_0 = \exp(a_0)$.

It is interesting to note that c_{k+1} is obtained by means of a "weighted Césaro average" of the previous c_k's with weights a_k, $k = 1,2,\ldots$. In particular, when $a_k = 1$ for $k = 0,1,2,\ldots$, we have

$$c_{k+1} = \sum_{j=0}^{k}\left(1 - \frac{j}{k+1}\right)c_j$$

with $c_0 = e$, which is exactly the Césaro average of the previous c_k's. Thus the Taylor coefficients

of $\exp(\sum_{k=0}^{\infty} z^k)$ can be obtained recursively by computing the Césaro average of the previous Taylor coefficients. Of course, it is easy to check that the relation (1) is the natural extension of the recursion $b_{k+1} = \frac{1}{k+1} b_k$ which holds for the Taylor coefficients of $\exp(z)$.

As byproducts of our approach, we find recursive formulae for the Taylor coefficients of functions like $\log(\sum_{k=0}^{\infty} a_k z^k)$ and $(\sum_{k=0}^{\infty} a_k z^k)^m$, even when m is not an integer. In section 3, we discuss some of the applications of such recursive formulae in several computational problems arising in harmonic analysis, probability theory, and numerical analysis.

2. Taylor Coefficients of $\exp(\sum_{k=0}^{\infty} a_k z^k)$. Let $\mathcal{N}$ be a neighborhood of zero in the complex plane. In most applications we may take $\mathcal{N}$ to be the open unit disc $D = \{z \in \mathbb{C}: |z| < 1\}$. The following proposition provides recursive formulae for the Taylor coefficients of $\exp(\sum_{k=0}^{\infty} a_k z^k)$ and some other functions.

PROPOSITION. (a) *Let* $h(z) = \exp(\sum_{k=0}^{\infty} a_k z^k)$, *where* $\sum_{k=0}^{\infty} a_k z^k$ *is analytic in* $\mathcal{N}$, *and let* c_k *be the kth Taylor coefficient of* $h(z)$. *Then we have*

$$c_{k+1} = \sum_{j=0}^{k} \left(1 - \frac{j}{k+1}\right) a_{k+1-j} c_j, \quad k = 0,1,2,\ldots \text{ with } c_0 = \exp(a_0).$$

(b) *Let* $a_0 = 1, a_1, a_2, \ldots$ *be such that* $h(z) = \log(\sum_{k=0}^{\infty} a_k z^k)$ *is analytic in* $\mathcal{N}$. *Let* c_k *denote the kth Taylor coefficient of* $h(z)$, *then*

$$c_{k+1} = a_{k+1} - \sum_{j=1}^{k} \left(1 - \frac{j}{k+1}\right) a_j c_{k+1-j} \tag{2}$$

$$= a_{k+1} - \frac{1}{k+1} \sum_{j=1}^{k} j a_{k+1-j} c_j, \quad k = 1,2,\ldots,$$

with $c_1 = a_1$.

(c) *Let* $a_0 = 1, a_1, a_2, \ldots$ *be such that the function* $(\sum_{k=0}^{\infty} a_k z^k)^m$ *is analytic in* $\mathcal{N}$ *and let* c_k *denote its kth Taylor coefficient, then*

$$c_{k+1} = \sum_{j=0}^{k} [m - (m+1)j/(k+1)] a_{k+1-j} c_j, \quad k = 0,1,2,\ldots, \tag{3}$$

with $c_0 = 1$.

Proof. (a) Let $\sum_{k=0}^{\infty} c_k z^k = \exp(\sum_{k=0}^{\infty} a_k z^k)$, $z \in \mathcal{N}$. By differentiating both sides of this identity with respect to z and replacing $\exp(\sum_{k=0}^{\infty} a_k z^k)$ by $\sum_{k=0}^{\infty} c_k z^k$, we obtain

$$\sum_{k=1}^{\infty} k c_k z^{k-1} = \left(\sum_{k=1}^{\infty} k a_k z^{k-1}\right)\left(\sum_{k=0}^{\infty} c_k z^k\right),$$

or equivalently,

$$\sum_{k=0}^{\infty} (k+1) c_{k+1} z^k = \left(\sum_{k=0}^{\infty} c_k z^k\right) \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k = \sum_{k=0}^{\infty} \left[\sum_{j=0}^{k} (k+1-j) a_{k+1-j} c_j\right] z^k.$$

Next, by comparing the coefficients of like powers of z in the identity above we get,

$$c_{k+1} = \sum_{j=0}^{k} \left(1 - \frac{j}{k+1}\right) a_{k+1-j} c_j, \quad k = 0,1,2,\ldots.$$

(b) follows immediately from (a) by exchanging the roles of a_k's and c_k's.

(c) Let $(\sum_{k=0}^{\infty} a_k z^k)^m = \sum_{k=0}^{\infty} c_k z^k$. By differentiating both sides of this identity we get

$$m\left(\sum_{k=0}^{\infty} a_k z^k\right)^{m-1}\left(\sum_{k=1}^{\infty} k a_k z^{k-1}\right) = \sum_{k=1}^{\infty} k c_k z^{k-1}.$$

Multiplying both sides of this relation by $\Sigma_{k=0}^{\infty} a_k z^k$ and replacing $(\Sigma_{k=0}^{\infty} a_k z^k)^m$ by $\Sigma_{k=0}^{\infty} c_k z^k$ gives

$$m\sum_{k=0}^{\infty} c_k z^k \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k = \sum_{k=0}^{\infty} a_k z^k \sum_{k=0}^{\infty} (k+1) c_{k+1} z^k,$$

or equivalently,

$$m\sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} (k+1-j) a_{k+1-j} c_j\right] z^k = \sum_{k=0}^{\infty}\left[\sum_{j=0}^{k} (k+1-j) c_{k+1-j} a_j\right] z^k.$$

By comparing the coefficients of like powers of z in the identity above, we get

$$(k+1)c_{k+1} + \sum_{j=1}^{k} (k+1-j) c_{k+1-j} a_j = m\sum_{j=0}^{k} (k+1-j) a_{k+1-j} c_j.$$

Thus

$$\begin{aligned} c_{k+1} &= m\sum_{j=0}^{k} (1 - j/(k+1)) a_{k+1-j} c_j - \sum_{j=1}^{k} (1 - j/(k+1)) c_{k+1-j} a_j \\ &= \sum_{j=0}^{k} [m - (m+1) j/(k+1)] a_{k+1-j} c_j, \quad k = 0,1,2,\ldots \text{ with } c_0 = 1. \end{aligned}$$

Q.E.D.

Generally the Taylor coefficients of $(\Sigma_{k=0}^{\infty} a_k z^k)^m$, when m is an integer, are found by repeated convolution of the coefficient sequence $\{a_k\}_{k=0}^{\infty}$. However, for large m the number of convolutions will increase rapidly and it becomes extremely difficult to write down explicit or computationally feasible formulae for these coefficients. Proposition (c) provides recursive and computationally feasible formulae for these coefficients even when m is not an integer.

Remembering that recursive formulae are particularly important for computation on digital computers, it seems to be a good idea to acquaint students with this alternative method of generating (finding) the Taylor coefficients of some special functions which are encountered often in applications.

In the next section we discuss some of the applications of the recursive formula (1) in problems arising in harmonic analysis, time series analysis and prediction theory, probability theory, etc.

3. Applications.

3.1 *Factorization of nonnegative polynomials* (*Féjer-Riesz Theorem*). Let $f(\lambda)$ be a nonnegative trigonometric polynomial of degree n, that is, $f(\lambda) = \Sigma_{k=-n}^{n} \gamma_k e^{ik\lambda} \geq 0$, for all $\lambda \in [-\pi, \pi]$, where γ_k's are complex-valued constants. A well-known theorem of L. Féjer and F. Riesz [**4**, p. 4] states that for any such nonnegative trigonometric polynomial which is not identically zero, there exists a unique analytic polynomial $P(z) = \Sigma_{k=0}^{n} c_k z^k$ with no zeros in the unit disc D and $c_0 > 0$ such that

$$f(\lambda) = |P(\lambda)|^2 = \left|\sum_{k=0}^{n} c_k e^{ik\lambda}\right|^2.$$

The problem of finding c_k's in terms of γ_k's is an extremely important problem in time series analysis, control and network theory [**6**], and the theory of orthogonal polynomials [**4**]. One method of finding c_k's is based on locating the zeros of the equation $\Sigma_{k=-n}^{n} \gamma_k z^k = 0$. However,

this is very inefficient when n is large. An iterative Newton-Raphson type algorithm for finding c_k's is given by Wilson [**6**]. Here, we show that the c_k's can be obtained recursively from the relations

$$c_{k+1} = \sum_{j=0}^{k} (1 - j/(k+1)) a_{k+1-j} c_j, \quad k = 0,1,\ldots,n-1,$$

with $c_0 = \exp(a_0/2)$, where $a_k = \int_{-\pi}^{\pi} e^{-ik\lambda} \log f(\lambda)\, d\lambda/2\pi$ is the kth Fourier coefficient of the logarithm of the given polynomial. This recursive formula is the result of applying Proposition (a) to the identity $\sum_{k=0}^{n} c_k z^k = \exp(a_0/2 + \sum_{k=1}^{\infty} a_k z^k)$, which is obtained as a result of the following standard arguments in complex analysis.

Since $\sum_{k=0}^{n} c_k z^k$ has no zeros in D, then

$$\log\left(\sum_{k=0}^{n} c_k z^k \right) = \log\left| \sum_{k=0}^{n} c_k z^k \right| + i \arg\left(\sum_{k=0}^{n} c_k z^k \right)$$

is analytic in D and thus $\arg(\sum_{k=0}^{n} c_k z^k)$ is the harmonic conjugate of $\log|\sum_{k=0}^{n} c_k z^k|$. Furthermore, the integrability of $\log|\sum_{k=0}^{n} c_k e^{ik\lambda}|$ on $[-\pi, \pi)$ implies that

$$\log\left| \sum_{k=0}^{n} c_k z^k \right| = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} a_k z^k,$$

with a_k's as defined before. Also, from the definition of the harmonic conjugate of an integrable function we get

$$\arg\left(\sum_{k=0}^{n} c_k z^k \right) = -\frac{i}{2} \sum_{k=-\infty}^{\infty} \operatorname{Sgn}(k) a_k z^k,$$

where

$$\operatorname{Sgn}(k) = \begin{cases} 1 & \text{if } k > 0, \\ 0 & \text{if } k = 0, \\ -1 & \text{if } k < 0. \end{cases}$$

Hence,

$$\log\left(\sum_{k=0}^{n} c_k z^k \right) = \tfrac{1}{2} \sum_{k=-\infty}^{\infty} a_k z^k + \tfrac{1}{2} \sum_{k=-\infty}^{\infty} \operatorname{Sgn}(k) a_k z^k = a_0/2 + \sum_{k=1}^{\infty} a_k z^k.$$

3.2 *Taylor coefficients of* $\exp[(1+z)/(1-z)]$ *and* $[(1+z)/(1-z)]^m$. Since

$$\left(\frac{1+z}{1-z} \right) = 1 + 2 \sum_{k=1}^{\infty} z^k,$$

it follows from Proposition (a) that the kth Taylor coefficients c_k of $\exp[(1+z)/(1-z)]$ can be obtained from

$$c_{k+1} = 2 \sum_{j=0}^{k} (1 - j/(k+1)) c_j, \quad k = 0,1,2,\ldots$$

with $c_0 = e$. Thus c_{k+1} is twice the Césaro average of the previous c_k's. Similarly, from Proposition (c) it follows that the Taylor coefficients of $[(1+z)/(1-z)]^m$ satisfy

$$c_{k+1} = 2 \sum_{j=0}^{k} [m - (m+1) j/(k+1)] c_j = \frac{2}{k+1} \sum_{j=0}^{k} [m(k+1) - (m+1) j] c_j,$$

$k = 0,1,2,\ldots$ with $c_0 = 1$. In this case c_{k+1} is twice the "weighted" arithmetic average of the previous c_k's.

3.3 *Hermite polynomials*. These are polynomials $H_k(x)$, $k = 0,1,2,\ldots$ on $(-\infty, +\infty)$ defined by

$$\sum_{k=0}^{\infty} H_k(x)\frac{z^k}{k!} = \exp\left(xz - \tfrac{1}{2}z^2\right).$$

The function on the right hand side is called the generating function of these polynomials [4]. From Proposition (a), it follows that

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x),$$

with $H_0(x) = 1$ and $H_1(x) = x$.

This list of applications can be amended to include important problems such as the factorization of nonnegative functions (Szegő's Theorem) which is a natural extension of the Fejer-Riesz Theorem, finding the Taylor coefficients of inner and outer functions in harmonic analysis [1, chap. 5], finding relationships between moments and cumulants of random variables [2, p. 68] and random walks [3]. Areas such as orthogonal polynomials and numerical analysis offer several problems of interest. In particular, the generating functions of many important orthogonal polynomials [4] and numerical sequences, for example Bernoulli's numbers, contain expressions of the form $\exp(\sum_{k=0}^{\infty} a_k z^k)$ and $(\sum_{k=0}^{\infty} a_k z^k)^m$.

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ON VAN DER WAERDEN'S NOWHERE DIFFERENTIABLE FUNCTION

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Van der Waerden's function is $f(x) = \sum_{k=0}^{\infty} a_k(x)$, where $a_0(x)$ is the distance from x to the nearest integer and $a_k(x) = 2^{-k}a_0(2^k x)$. In a recent note [1], Patrick Billingsley gave a simple proof that f is a continuous function with no two-sided derivative at any point. Here we show that a generalization of his argument proves that f has no one-sided derivative at any point.

Assume, to the contrary, that $f'_+(x) = L$ for some $x \in R$. For each integer $n > 0$, let $u_n = i2^{-n}$ and $v_n = (i+1)2^{-n}$ where $(i-1)2^{-n} \leqslant x < i2^{-n}$. Define r_n and s_n such that

$$f(u_n) - f(x) = (L + r_n)(u_n - x)$$

and

$$f(v_n) - f(x) = (L + s_n)(v_n - x).$$

Then

$$\begin{aligned} f(v_n) - f(u_n) &= \big(f(v_n) - f(x)\big) - \big(f(u_n) - f(x)\big) = (L + s_n)(v_n - x) - (L + r_n)(u_n - x) \\ &= L(v_n - u_n) + s_n(v_n - x) - r_n(u_n - x) = 2^{-n}L + s_n(v_n - x) - r_n(u_n - x), \end{aligned}$$

CENTER SECTION
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Telegraphic Reviews

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books submitted for review should be sent to Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

Telegraphic reviews are designed to give prompt notice of all new books in the mathematical sciences. Certain of these books will be selected for more extensive review in the Reviews section of the Monthly.

Special Codes:

T:	Textbook	13-18:	Grade Level
S:	Supplementary Reading	1-4 :	Time in Semesters
P:	Professional Reading	** :	Special Emphasis
L:	Undergraduate Library	?? :	Questionable

General, S, P, L. Superior Beings: If They Exist, How Would We Know. Steven J. Brams. Springer-Verlag, 1983, xix + 202 pp, $21.95; $11.95 (P). [ISBN: 0-387-91223-1; 0-387-90877-3] Continuing in the tradition of Blaise Pascal (Pensées, 1670), Steven Brams brings mathematics and formal deductive methods to bear on classical problems in metaphysics. The modern theory of two-person game theory provides the context for tightening up such definitions as omniscience, omnipotence, immortality, incomprehensibility. The result is a refreshing and stimulating look at some of the age-old religious-theological-philosophical questions. LCL

General, T(14), S. Logic, Proof, and Sets, Second Edition. Marvin L. Bittinger. Addison-Wesley, 1982, 131 pp, $7.95 (P). [ISBN: 0-201-10384-2] A supplement for students making the transition from routine calculus exercises to theoretical proofs. Content includes discussions on quantifiers, how to negate a sentence, mathematical induction, proof by contradiction, proof by cases, equivalence relations. LCL

Education, T, S, P, L. The Art of Problem Posing. Stephen I. Brown, Marion I. Walter. Franklin Inst Pr, 1983, 147 pp, $11.95 (P). [ISBN: 0-89168-052-7] The authors discuss the importance of problem posing at all levels of mathematics, and the strategies that exist for incorporating it in the classroom. Several examples of the "What-If-Not" problem-posing strategy. LCL

Education, P*, L*. Computing and Mathematics: The Impact on Secondary School Curricula. Ed: James T. Fey. NCTM, 1984, ix + 100 pp, $7.50 (P). [ISBN: 0-87353-212-0] An "unequivocal dissent" from "efforts to improve school mathematics that seek broad agreement on conservative curricula." This report, edited from ideas presented at an October 1982 conference at the University of Maryland, offers a wide variety of suggestions for adapting algebra, geometry, discrete mathematics, and calculus to a computer-based classroom environment. LAS

History, S(18), P*, L. Emmy Noether: Gesammelte Abhandlungen, Collected Papers. N. Jacobson. Springer-Verlag, 1983, viii + 777 pp, $62.10. [ISBN: 0-387-11504-8] The collected papers (in German) of one of the most influential mathematicians of our time. Includes a memorial address by Paul Alexandroff and an introduction by Nathan Jacobson (both in English). The legacy of her life in thought and style was to ignore the unimportant, and to strive for the simple and general; only in this way could one reach a deep understanding which computation alone could never provide. LCL

Combinatorics, T*(14-16: 1, 2), S*. Introductory Combinatorics. Kenneth P. Bogart. Pitman Pub, 1983, xii + 388 pp, $24.95. [ISBN: 0-273-01923-6] A broad and comprehensive survey of modern combinatorics: enumeration (including a very nice presentation of inclusion-exclusion, generating functions, recurrence relations), graph theory and trees (including combinatorial algorithms for matching and searching), combinatorial designs, algebraic structures (partially ordered sets, Boolean algebras). Well motivated examples and exercises; conversational style. LCL

Combinatorics, S(18), P, L. Surveys in Combinatorics. Ed: E. Keith Lloyd. London Math. Soc. Lect. Note Ser., No. 82. Cambridge U Pr, 1983, xi + 256 pp, $27.50 (P). [ISBN: 0-521-27552-0] The invited papers for the Ninth British Combinatorial Conference held at the University of Southhampton, July 11-15, 1983. CEC

Discrete Mathematics, T*(13-14: 1, 2), S. Discrete Mathematics for Computer Scientists. Joe L. Mott, Abraham Kandel, Theodore P. Baker. Reston Pub, 1983, xi + 562 pp, $27.95. [ISBN: 0-8359-1372-4] Designed as a sophomore-level introduction to logical and algebraic structures and to combinatorial mathematics including enumeration methods and graph theory. No specific mathematical background is prerequisite, yet the level of mathematical sophistication is roughly parallel to that of first-year calculus. Large selection of exercises allow the instructor to tailor the course to the class. LCL

Discrete Mathematics, T(13: 1), S. Discrete Mathematics: A Computational Approach Using Basic. Marvin Marcus. Computer Sci Pr, 1983, xvii + 329 pp, $19.95 (P). [ISBN: 0-914894-38-2] Narrative sections (logic, sets, functions, induction and combinatorics, probability, matrices, linear equations, linear programming) followed by computer exercises designed to illustrate, reinforce, and extend narrative material. Workbook format, with space for write-in answers. Also includes problems for traditional pencil and paper work. LCL

Number Theory, P. Foundations of Semiological Theory of Numbers, V. 1. H.A. Pogorzelski, W.J. Ryan. U of Maine at Orono Pr, 1982, (P). [ISBN: 0-89101-053-X] As "nearness" is characterized by various topological spaces, "nextness" is characterized by presemiological spaces. An introduction to presemiological spaces which is particularly concerned with the almost semisemiological structures endowed on the primes. The aim is to deal with the Goldbach Conjecture to the extent that these methods permit. Heavy reading. CEC

Algebra, S(16-18), P, L. Additive Groups of Rings. S. Feigelstock. Research Notes in Math., No. 83. Pitman Pub, 1983, 113 pp, $15.95 (P). [ISBN: 0-273-08591-3] A self-contained, easily accessible account, with special emphasis on results not contained in L. Fuchs' classic book. LCL

Algebra, P. Lecture Notes in Mathematics-1023: Asymptotic Prime Divisors. Stephen McAdam. Springer-Verlag, 1983, ix + 118 pp, $8 (P). [ISBN: 0-387-12722-4] This monograph attempts to present the bulk of the present knowledge of asymptotic prime divisors. Includes a list of references. CEC

Algebra, S(17-18), P. Factor Categories with Applications to Direct Decomposition of Modules. Manabu Harada. Lect. Notes in Pure & Appl. Math., V. 88. Dekker, 1983, ix + 328 pp, $59.75 (P). [ISBN: 0-8247-1897-6] In addition to providing a survey of the author's results on direct decomposition of modules, the first half of the book provides a thorough, but swift, treatment of groups and rings in a very categorical setting. The reasonably complete index may make this first part independently useful to a graduate student concerned with certain aspects of categorical algebra. JAS

Calculus, S(14), P, L. Fundamentals of Mathematical Analysis. V.A. Ilyin, E.G. Poznyak. Transl: Irene Aleksanova. MIR Pub, 1982, $18.95 set. Part 1, 637 pp; Part 2, 438 pp. Includes differential and integral calculus, real analysis and multivariable calculus along with geometrical applications. This massive two-volume work would be difficult to use as a text because it has no problems, but as a reference work for all of undergraduate real analysis it seems like a steal for the price. CEC

Complex Analysis, S(18), P. Eisenstein Series on the Metaplectic Group: An Algebraic Approach. G.F. Helminck. Math. Centre Tracts, No. 161. Math Centrum, 1983, vii + 73 pp, Dfl. 9,90 (P). [ISBN: 90-6196-257-9] This monograph investigates to what extent results similar to Siegel's hold for Eisenstein. Series defined on the mataplectic group. Includes a brief list of references. CEC

Complex Analysis, T*(18: 2), S, P. Holomorphic Functions of Several Variables: An Introduction to the Fundamental Theory. Ludger Kaup, Burchard Kaup. Transl: Michael Bridgland. Walter de Gruyter, 1983, xv + 349 pp, DM 112. [ISBN: 3-11-004150-2] Introduces function theory of several complex variables, first on domains in $\mathbb{C}^n$, then on analytic sets, finally on Stein spaces. Appropriate for self-study--includes many examples, exercises in very detailed presentation. A little knowledge of sheaves and cohomology is assumed, but references are given for what is needed. Nicely organized, with attractive graphics. PZ

Complex Analysis, T(16-17: 1, 2), S. The Elements of Complex Analysis. B. Choudhary. Halsted Pr, 1983, xi + 262 pp, $17.95. Brief, readable introduction to standard elementary theory through Runge's theorem and analytic continuation. Depth of coverage somewhere between standard graduate and undergraduate texts. Many diagrams, varied exercises--quite a few solved. Emphasizes differential equations and applications. PZ

Differential Equations, T(14-15: 1). Linear and Nonlinear Differential Equations. Ian Huntley, R.M. Johnson. Math. & Its Applic. Halsted Pr, 1983, 190 pp, $49.95; $22.50 (P). [ISBN: 0-85312-441-8] Intended to fill gap between books on linear ordinary differential equations and advanced nonlinear equations. Aimed mostly to non-specialists in mathematics. The theory is reduced and there are many concrete examples from physics and engineering. Concentrates mainly on qualitative methods; does not discuss numerical methods. CU

Differential Equations, P. Free Boundary Problems: Theory and Applications. Ed: A. Fasano, M. Primicerio. Pitman Pub, 1983, $27.50 (P) each. Volume I, 320 pp, [ISBN: 0-273-08589-1]; Volume II, 390 pp. [ISBN: 0-273-08590-5] Proceedings of an interdisciplinary symposium held in Montecatini, Italy in June 1981. Topics: fluid mechanics, the dam problem, the porous media equation and percolation, soil freezing, phase-change problems, Stefan-like problems, inverse problems and control, electrochemical machining and flame fronts, etc. CU

Differential Equations, T(17: 1), S. Similarity Solutions of Nonlinear Partial Differential Equations. Lawrence Dresner. Research Notes in Math., No. 88. Pitman Pub, 1983, 124 pp, $17.95 (P). [ISBN: 0-273-08621-9] The goal of the book is to popularize the method of similarity solutions as a practical technique for technologists. Mathematical rigor is restrained (on purpose). There is an abundance of brief physical motivations. Topics: linear and nonlinear diffusion, boundary-layer

problems, wave propagation problems, etc. CU

Differential Equations, P. Lecture Notes in Mathematics-1017: Equadiff 82. Ed: H.W. Knobloch, K. Schmitt. Springer-Verlag, 1983, xxiii + 666 pp, $26.50 (P). [ISBN: 0-387-12686-4] Proceedings of the international conference held in Würzburg, West Germany, August 23-28, 1982. JAS

Differential Equations, P. Systems of Nonlinear Partial Differential Equations. Ed: J.M. Ball. NATO Adv. Sci. Institutes Ser. C., No. 111. D Reidel Pub, 1983, ix + 481 pp, $65. [ISBN: 90-277-1629-3] Proceedings of an advanced study institute held in Oxford, July 25-August 7, 1982. Includes 13 expository lectures and papers presented at special sessions. LAS

Numerical Analysis, T(14-16: 1). Numerical Calculations and Algorithms. Royce Beckett, James Hurt. Krieger Pub, 1983, xi + 298 pp, $19.50. [ISBN: 0-89874-415-6] A reprint of the 1967 edition, published by McGraw-Hill at the time of the better texts. Expresses algorithms in flowcharts. Topics: algebraic equations, linear systems, interpolation, integration, initial value problems, boundary value problems and approximation. An appropriate level of error analysis for engineering and science students. RWN

Numerical Analysis, S(16). Fundamental Concepts in the Numerical Solution of Differential Equations. J.F. Botha, G.F. Pinder. Wiley, 1983, xii + 202 pp, $24.95. [ISBN: 0-471-87546-5] Written as a "reference on the methodology available for the solution of ordinary and partial differential equations in science and engineering." Discusses finite difference, finite element, collocation, and boundary elements using the theory of interpolating polynomials. A few problems at the end of each chapter. BH

Functional Analysis, P. Lecture Notes in Mathematics-1003: Spaces of Vector-Valued Continuous Functions. Jean Schmets. Springer-Verlag, 1983, vi + 117 pp, $8 (P). [ISBN: 0-387-12327-X] Characterizes the locally convex properties of C(X;E), the space of continuous functions from a Hausdorff completely regular space X to a Hausdorff locally convex topological vector space E. Complements the first four chapters of the author's previous book, which treats the case E = R or C. BH

Analysis, P. Approximation Theory (Selected Papers). George Alexits. Akademiai Kiado, 1983, 298 pp, $24. [ISBN: 963-05-3204-2] 30 of Alexits' 88 mathematics papers (those on approximation theory, multiplicative function systems, and function series), preceded by a brief biography and a complete list of Alexits' publications. An appendix contains errata to the original papers. LAS

Algebraic Geometry, S(18), P. Algebraic Varieties and Analytic Varieties. Ed: S. Iitaka. Adv. Stud. in Pure Math., No. 1. North-Holland Pub, 1983, 384 pp, $74.50. [ISBN: 0-444-86612-4] The proceedings of a symposium on algebraic varieties and analytic varieties which was held at the University of Tokyo from July 13-24, 1981. Consists of 15 research papers. CEC

Differential Geometry, P. Lecture Notes in Mathematics-1022: Geometry Seminar "Luigi Bianchi". Graziano Gentili, Simon Salamon, Jean-Pierre Vigué. Springer-Verlag, 1983, vi + 177 pp, $9.50 (P). [ISBN: 0-387-12719-4] Amplified versions of the lecture cycles, "Distances in convex cones" by Graziano Gentili, "Topics in four-dimensional Riemannian geometry" by Simon Salamon, and "Domaines bornés symétriques" by Jean-Pierre Vigúe, given at the Scuola Normale Superiore, Pisa, Italy, in the winter and spring of 1982. JAS

Differential Geometry, P. Lecture Notes in Mathematics-1007: Geometric Dynamics. Ed: J. Palis, Jr. Springer-Verlag, 1983, ix + 827 pp, $32.50 (P). [ISBN: 0-387-12336-9] Proceedings of a 1981 summer conference held at the Instituto de Mathemática Pura e Aplicada (IMPA) in Rio de Janeiro in celebration of the opening of IMPA's new building. LAS

Geometry, S(13), P. Geometry (A Textbook for Technical Schools). Ed: G.N. Yakovlev. Transl: Leonid Levant. MIR Pub, 1982, 288 pp, $7.45. A terse but comprehensive presentation of vectors in R^2 and R^3, lines, planes, conic sections, polyhedra, volumes of solids and areas of surfaces. Numerous problems. An inexpensive resource. JNC

Geometry, S(17-18), P. Relativity and Geometry. Roberto Torretti. Found. & Philo. of Sci. & Tech. Series. Pergamon Pr, 1983, xi + 395 pp, $45. [ISBN: 0-08-026773-4] A "historico-critical" exposition of Einstein's theorems of special and general relativity with an emphasis on geometrical ideas; intended to "elucidate the motivation and significance of the [resultant] changes in physical geometry;" extensive use of primary sources. JNC

Topology, T(17: 1), S, P, L. Knot Theory. K. Reidemeister BCS Assoc, 1983, xv + 143 pp, $14.95 (P). [ISBN: 0-914351-00-1] A translation of a 1932 German edition, using modern terminology. A highly readable introduction with a wealth of material. CEC

Topology, T(16-18: 1, 2), S. General Topology. Mangesh G. Murdeshwar. Halsted Pr, 1983, xi + 369 pp, $19.95. [ISBN: 0-470-26916-2] Covers standard topics of elementary general topology through separation and countability, but with more than standard attention paid to foundations (sets, filters, nets). "Continuous function" is defined on page 135. For use in Indian universities where instructor contact may be limited, text attempts to partly fill instructor's role in explaining, commenting informally, discussing heuristics. Many exercises. PZ

Dynamical Systems, P. Advances in Hamiltonian Systems. Ed: J.P. Aubin, A. Bensoussan, I. Ekeland. Birkhauser Boston, 1983, viii + 195 pp, $19.95. [ISBN: 3-7643-3130-5] Proceedings of a February 1981 conference at the University of Rome. Main topic: periodic solutions of Hamiltonian systems. All results local. CU

Probability, P. Lecture Notes in Mathematics-990: Probability in Banach Spaces IV. Ed: A. Beck, K. Jacobs. Springer-Verlag, 1983, 234 pp, $11 (P). [ISBN: 0-387-12295-8] Proceedings of the seminar held in Oberwolfach, July 1982. JAS

Probability, S(16-17), P, L. Introductory Applied Probability. G.P. Beaumont. Ser. in Math. & Its Applic. Halsted Pr, 1983, 235 pp, $54.95. [ISBN: 0-470-27481-6] Applications include birth and death processes, queueing theory, renewal theory, reliability, and inventory control. The approach is intermediate in nature, building on the probability normally covered in a first course in calculus-based statistics. A nice collection of good problems together with brief solutions and comments. Note price! LCL

Computer Programming, T(13), S. BASIC/APPLE II: A Programming Guide. Allen B. Tucker, Jr. Holt, Rinehart & Winston, 1983, xiv + 253 pp, $16.95 (P). [ISBN: 0-03-061769-3] An introduction to programming and BASIC from the viewpoint of the Apple BASIC system. Well written with lots of exercises, including lab exercises from business, mathematics, and science. Also has an introduction to file manipulation, multidimensional arrays and graphics, as well as the usual topics. CEC

Computer Programming, T?, S(13-16), L. Programming the Z80, Third Revised Edition. Rodnay Zaks. Sybex, 1982, 624 pp, $16.95 (P). [ISBN: 0-89588-069-5] A how-to-book from binary data representation through a number of sample programs. Very chip specific. Follows the same formula as the author's Programming the 6502, though with fewer and less extensive examples. JAS

Computer Programming, S(13). Graphics for the IBMpc. B.J. Korites. Kern Pub, 1983, 274 pp, $28.50 (P); disk $21.50. Back to BASICS! The first program lights up a single point in the middle of the screen. The book then moves at a reasonable pace through the core material of drawing, translation, scaling, rotations and the like for both two- and three-dimensional graphics. Matrices are used mechanically and discussion of homogeneous coordinates is completely avoided. The result is more a book for the hacker in BASIC than a text for computer graphics. The program disks use atomized BASICA but could be translated fairly easily to such machines as the Heath/Zenith H/Z-100. JAS

Computer Programming, T(13: 1, 2), S(13), P, L. Problem Solving Using PL/I and PL/C. Keith Harrow. Prentice-Hall, 1984, xvi + 447 pp, $19.95 (P). [ISBN: 0-13-711796-5] An introduction to PL/I or PL/C through a series of problems. Each of the sixteen chapters start with the statement of a problem and is then devoted to solving the problem with a PL/I program. Well organized, with lots of exercises. CEC

Computer Programming, T, S. PASCAL for Fortran Programmers. Ronald H. Perrott, Donald C.S. Allison. Computer Sci Pr, 1984, xi + 335 pp, $18.95 (P). [ISBN: 0-914894-09-9] This book is written for those who already know some FORTRAN and who wish to learn about PASCAL. All the features of PASCAL are examined and explained by identifying and comparing them with their corresponding features in FORTRAN. Several case studies are given to illustrate the advantages of structured programming, and the rich data and control structures offered in PASCAL. LCL

Software Systems, T(15-17). Foundations of Software Design. J.F. Leathrum. Reston Pub, 1983, x + 182 pp. [ISBN: 0-8359-2094-1] A brief introduction to many of the important topics in software design: life cycles, productivity, requirements, analysis, software tools, the design process, packaging, software design languages, programming methodology, verification, correctness and testing. Includes some case studies. As a text, this book needs supplementation and extension. RWN

Software Systems, S(13-16), L. The Programmer's CP/M Handbook. Andy Johnson-Laird. Osborne/McGraw-Hill, 1983, ix + 501 pp, $21.95 (P). [ISBN: 0-88134-119-3] Written for experienced programmers who are reasonably familiar as users with CP/M and who want to "do their own thing." Contains a 55-page BIOS with bells and whistles and devotes a whole chapter to discussing this BIOS. Discusses, with technical detail, track buffering, RAM-disks, and system error messages. A reference of considerable substance on the subject. JAS

Software Systems, S(13-14), L? The CP/M Handbook with MP/M. Rodnay Zaks. Sybex, 1980, xii + 320 pp, $14.95 (P). [ISBN: 0-89588-048-2] A start-from-nothing user's guide to standard issues of CP/M and MP/M, including 35 pages on ED. No editorializing, no bells and whistles, no details of diskette logical structure, but a handful of appendices for novices presenting such things as checklists for computer room organization. JAS

Computer Science, L. The Microprocessor Handbook. Elmer Poe. Howard W Sams & Co, 1983, 236 pp, $14.95 (P). [ISBN: 0-672-22013-X] Presents pinouts, signal definitions, timing diagrams, architecture, addressing modes, and instruction sets for 10 popular 8- and 16-bit processors as well as for a few I/O chips. Dry, technical, useful. JAS

Control Theory, S(16-18), P, L. Polynomials and Linear Control Systems. Stephen Barnett. Pure & Appl. Math., V. 77. Dekker, 1983, xi + 452 pp, $35. [ISBN: 0-8247-1898-4] A unique collection of topics not readily available in book form: a matrix approach to properties of polynomials with applications to control systems. Large bibliography; many worked examples. Strong undergraduates

could use this book to approach the frontiers of current research in this area. LCL

Applications (Engineering), S(14-15), P. Digital Image Processing of Remotely Sensed Data. R. Michael Hord. Comp. Sci. & Appl. Math., No. 5. Academic Pr, 1982, xiii + 256 pp, $27.50. [ISBN: 0-12-355620-1] A general review of several physical, mathematical and computational techniques, with applications, especially to LANDSAT. RWN

Applications (Physics), P. Lecture Notes in Physics-181: Gauge Theories of the Eighties. Ed: R. Raitio, J. Lindfors. Springer-Verlag, 1983, 644 pp, $35.50 (P). [ISBN: 0-387-12301-6] Proceedings of the Arctic School of Physics held in Äkäslompolo, Finland, August 1-13, 1982. Twelve days (and no nights) of current physics. JAS

Reviewers

RJA: Richard J. Allen, St. Olaf; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JD-B: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; PH: Paul Humke, St. Olaf; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MT: Michael Tveite, St. Olaf; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

The Mathematical Association of America
The Sixty Seventh Annual Meeting of the Association
Louisville, Kentucky

The Sixty Seventh Annual Meeting of the Association was held from January 26 through January 28, 1984 in conjunction with meetings of the American Mathematical Society and the Association for Women in Mathematics. The Retiring Presidential Address was given by Richard D. Anderson. The abstracts for this address and the Invited Addresses are as follows:

Reflections on the Mystique of R.L. Moore, by Richard D. Anderson, Louisiana State University.

As a former student of R.L. Moore, and after consulting with many other former students, the author discusses the teaching procedures and personal characteristics of R.L. Moore that made him, over the years, such a successful teacher. He was, indeed, an inspiration to his students both in his devotion to research and in his commitment to his profession. He was especially proud of the research accomplishments of his students. It is a touch of irony that while five of his students became Presidents of the Association, he himself had little time or inclination for involvement in the MAA (although he reportedly did join it three times).

The address was preceded by a showing of the film "Challenge in the Classroom," a film by the late R.L. Moore.

Beta and Gama Functions from Euler to Selberg and Beyond, by Richard Askey, University of Wisconsin-Madison.

Euler defined the gamma function as an infinite product, obtained an integral representation for it, and used it to evaluate a couple of beta integrals. Cauchy evaluated another beta integral. Dirichlet used these integrals to evaluate some multidimensional beta integrals. Stieltjes solved a problem in two dimensional physics which started the deeper multidimensional study of gamma and beta integrals. A few statisticians and number theorists evaluated beta, gamma and normal integrals on groups, but the deepest result was found by Selberg. MacDonald found a geometric setting for Selberg's integral. There are applications and open problems as well.

Turning Good Mathematics Into Good TV, by Ross L. Finney and Donald Berman, COMAP.

Non-Expansive Maps, by Andrew M. Gleason, Harvard University.

Let M be a metric space whose distance function is denoted by ρ. Assume that M is complete; i.e., that every Cauchy sequence is convergent. A map f: M -> M is called non-expansive if $\rho(f(x),f(y)) \leq \rho(x,y)$ for all x,y in M. f is called a strict contraction if it satisfies $\rho(f(x,)f(y)) \leq K\,\rho(x,y)$, where K is a fixed number less than 1. Banach's theorem asserts that a strict contraction always has a unique fixed point p which can be found by interaction from any start; i.e., if x_0 is any point of M and we define $x_{n+1} = f(x_n)$, then the sequence $x_0,x_1,x_2,\ldots$, converges to p. Variations of Banach's theorem will be discussed, and a general structure theorem will be developed for non-expansive maps of a compact M.

The Computer as a Grader, by Melvin J. Maron, University of Louisville.

The problem addressed is the use of the computer to grade problems whose answers are numbers which are obtained using a sequence of mathematical calculations, rather than by inspection. Such problems, typical of science and engineering as well as mathematics courses, do not lend themselves to a multiple-choice format. Alternative approaches will be described and compared, and one approach will be demonstrated.

Mathematics in Industry--How Do Problems Arise? by Henry O. Pollak, Central Staff Organization for the Regional Bell Operating Companies.

It is not customary for the written exposition of mathematical research, even applied mathematics, to go into the history of the particular problem. How did the particular formulation come about, and who was interested in the results? In this talk, several such problems from the speaker's experience in mathematical and statistical research at Bell Laboratories will be described from their original conception to their application. This is perhaps particularly timely in view of the recent breakup of the telephone system.

Some Examples of Combinatorial Averaging, by Herbert S. Wilf, University of Pennsylvania.

We can often gain insight into combinatorial problems by averaging over suitable sets of objects, and of course, by finding further moments of the probability distribution that are involved. This talk is intended to give a number of examples of such situations, drawn from the theory of the Hadamard determinant problem, the enumeration of cliques in a graph, the computational complexity of graph coloring, the size of the permanent function, etc.

Computational Geometry: Paradigms and Applications, by F. Frances Yao, Xerox Palo Alto Research Center.

Computational geometry is an active area of research in theoretical computer science. In this talk, a number of important problems in computational geometry will be considered, and the bearing that classical geometry has on this modern subject will be highlighted. We will also discuss the current status of the field and speculate on its future trends.

Special Sessions

Panel Discussions.

Joan R. Leitzel moderated a panel discussion titled "Issues of Remediation." The panel members were Dean Patterson, Ronald Wenger and Geoffrey Akst. Clarence Stephens was scheduled to react to the presentations.

The Committee on Corporate Members sponsored a panel discussion on "Mathematics Publishing, Copyright, and Software." Jerry Lyons was the moderator.

A panel discussion, titled "Beginning Integration: Calculus and Discrete Mathematics in the First Two Years," was moderated by Martha Siegel.

A "Progress Report on the Clarkson Institute for Retraining in Computer Science under the Auspices of the Joint ACM/MAA Committee" was moderated by Donald L. Kreider.

Mini-Courses

The Association sponsored eight mini-courses. Their titles and organizers were: "Linear Programming" by Charles E. Huff; "Discrete Algorithmic Mathematics" by Stephen B. Maurer; "Teaching Problem Solving" by Alan H. Schoenfeld; "Applications of Discrete Mathematics" by Fred S. Roberts; "Problems from Industry" by Jeanne L. Agnew and Marvin S. Keener; "Applications of Computer Graphics" by Joan Wyzkoski; "CONDUIT Microcomputer Software" by David A. Smith; and "NonCONDUIT Microcomputer Software" by David A. Smith.

Board and Business Meeting

The Board of Governors met at 9:00 A.M. on January 26, 1984. The major items of business will be announced to the membership in FOCUS.

The business meeting of the Association was held at 12:00 Noon on January 27, 1984. It was announced that Lynn A. Steen of St. Olaf College has been elected President-Elect and that Gerald L. Alexanderson of the University of Santa Clara was elected First Vice President.

The Chauvenet Prize was presented to R. Arthur Knoebel of New Mexico State University for his Monthly article "Exponentials Reiterated." Professor Knoebel made some remarks concerning recent developments in the area.

The Lester R. Ford Awards for Monthly expository articles were awarded to:

Robert F. Brown, UCLA
"Fixed Point Property and Cartesian Products"

Tony Rothman, University of Texas
"Genius and Biographers: Galois"

Robert S. Strichartz, Cornell University
"Radon Inversion--Variations on a Theme"

President Niven read a Citation honoring those who have furthered the progress of mathematics by enhancing significantly the status of women in mathematics. The Citation was received by Julia B. Robinson, President of the American Mathematical Society, and by Linda P. Rothschild, President of the Association for Women in Mathematics. The Citation will be published in FOCUS.

Election of Members.

At its meeting on January 26, 1984, the Board elected 1056 applicants for individual membership and 10 institutional members, namely: Texas A & I University; Southern Oregon State College; Normandale Community College; Triton College; University of Northern Iowa; Washington and Lee University; West Georgia College; Columbia Christian College; Franklin College of Indiana; Lewis-Clark State College.

Respectfully submitted,

Kenneth A. Ross
Secretary

Citation Honoring Women in Mathematics

At the annual meeting in Louisville in January, 1984, the Mathematical Association of America announced a special Citation honoring those who have furthered the progress of mathematics by enhancing significantly the status of women in mathematics. The full text of the Citation appears in the March-April issue of FOCUS, the MAA newsletter.

Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

Louisiana-Mississippi Section

The sixty-first annual joint meeting of the Louisiana-Mississippi Section and the Louisiana-Mississippi Branch of NCTM met at Southeastern Louisiana University, Hammond, Louisiana on February 17-18, 1984. There were 175 persons in attendance.

Invited Lectures:

"Geometrical Gems," by Donald Albers, Menlo College.
"Mathematicians Observed," by Donald Albers, Menlo College.

Short Course:

"Mathematical Problems in Computer Graphics," by Willis Bourque, University of Southwestern Louisiana.

Mini-Conference on Developmental Mathematics:

Coordinator: John Tilley, Mississippi State University; keynote address was given by Thomas Meredith, Board of Trustees of State Institutions of Higher Learning, Mississippi. Papers were presented by: Lester W. Jones, Xavier University; Diane Daniels, Mississippi State University; Lunelle Miller, Northeast Mississippi Junior College; Gerhart Moore and Lawrence Davis, Southeastern Louisiana University; Keith Alford, Alcorn State University; Gail Fulton, Copiah-Lincoln Junior College; Robert Lax and Gerry Vidrine, Louisiana State University; Judy Pate, Mississippi University for Women; Allean Ursery, Hinds Junior College. (Publication of the Proceedings of the Mini-Conference was funded by a grant from The Mathematical Association of America.)

Contributed Papers:

"Structured Proof to Determine a Complex and A Map," by Margaret M. LaSalle, University of Southwestern Louisiana.
"Calculating the Fundamental Group of a Knot," by Mark Dugopolski, Southeastern Louisiana University.
"Decompositions of Uncountable Sets and S-Lindelof Spaces," by Charles Dorsett, Louisiana Tech University.
"Some Remarks on Equations of Non-negative Matrices," by Cecil E. Robinson, University of Southern Mississippi.
"An Inductive Proof that No Permutation is Both Even and Odd," by David M. Berman, University of New Orleans.
"The Changing Face of Placement in Developmental Mathematics," by Roberta Yellott, McNeese State University.
"Automorphism Groups," by Gary L. Walls, University of Southern Mississippi.
"The Effect of Adjacent Presentation on Ranking," by Margaret Maxfield and John Maxfield, Louisiana Tech University.
"The Algebra of Operators for the Quantized Harmonic Oscillator," by Henry Heatherly and Fred Rolfes, University of Southwestern Louisiana.
"Convergence to Zero of the Non-oscillatory Solution of a Second Order Differential Equation with Integrable Coefficients," by John R. Graef and Paul W. Spikes, Mississippi State University.
"Fixed Point Theorems for Multi-valued Quasi-contractions," by Hideaki Kaneko, Mississippi State University.
"Some Attitudes that Affect Student Attitudes Toward Science and Technology Courses," by Bill Attebery, Margaret Maxfield, and Mary Livingston, Louisiana Tech University.
"The Effect of Errors in the National Mortality Rate on Stock Size Estimates Using Cohort Analysis," by S. Ugene Sims, Louisiana Tech University.
"The MVUE of the P.D.F. of the LaPlace Distribution," by Badiollah R. Asrabadi, Nichols State University.
"Monotonic Properties of the Zeros of Orthogonal Polynomials," by Shafique Ahmed, University of Southern Mississippi.
"G-Quasi-Interior Functions," by W. Travis Thompson, Northeast Louisiana University.
"Topological Obstructions to a Totally Geodesic Riemannian Foliation," by Vincent E. Coll, Loyola University.

Panel Discussion:

"Concerns of Mathematics Departments," by Duane Blumberg (Moderator), University of Southwestern Louisiana; Stephen Doblin, University of Southern Mississippi; Kathleen A. Drude, Northeast Louisiana University; George Mead, McNeese State University; James Porter, University of Mississippi.

Student Papers:

"Series Summation Through Integral Calculus," by Dominic Milioto, Southeastern Louisiana University.
"Modelling the Mind," by Michael Boudreaux, Loyola University.
"Implementing Finite Automata," by Erik Humel, Loyola University.

The following presentors were awarded a $100 U.S. Savings Bond:

"Completions of the Integers," by John M. Graves, University of Southern Mississippi.
"Generating Sets for the Alternating Group on n Objects," by Karen Fawcett, University of Southern Louisiana.

The following presentors were awarded a student membership in the MAA:

"Haiku Poetry Processing," by Karen Collins, Loyola University.
"McNaughton-Yamada Algorithm," by Robert B. Davis, Loyola University.
"An Upper Bound for the Error of the Cubic Interpolating Polynomial," by John C. Beasley, Louisiana Tech University.

Louisiana-Mississippi Branch NCTM

Short Presentations:

"Some Techniques of Composing Challenging Exercises Involving Rational Expressions and Equations," by John Hornsby, University of New Orleans.
"Applications of Geometric Series," by Mark Dugopolski, Southeastern Louisiana University.

Student Paper:

"Rubic's Cube," by Michael Ledford, Southeastern Louisiana University.

3.3 *Hermite polynomials.* These are polynomials $H_k(x)$, $k = 0, 1, 2, \ldots$ on $(-\infty, +\infty)$ defined by

$$\sum_{k=0}^{\infty} H_k(x)\frac{z^k}{k!} = \exp\left(xz - \tfrac{1}{2}z^2\right).$$

The function on the right hand side is called the generating function of these polynomials [4]. From Proposition (a), it follows that

$$H_{k+1}(x) = xH_k(x) - kH_{k-1}(x),$$

with $H_0(x) = 1$ and $H_1(x) = x$.

This list of applications can be amended to include important problems such as the factorization of nonnegative functions (Szegő's Theorem) which is a natural extension of the Fejer-Riesz Theorem, finding the Taylor coefficients of inner and outer functions in harmonic analysis [1, chap. 5], finding relationships between moments and cumulants of random variables [2, p. 68] and random walks [3]. Areas such as orthogonal polynomials and numerical analysis offer several problems of interest. In particular, the generating functions of many important orthogonal polynomials [4] and numerical sequences, for example Bernoulli's numbers, contain expressions of the form $\exp(\sum_{k=0}^{\infty} a_k z^k)$ and $(\sum_{k=0}^{\infty} a_k z^k)^m$.

References

1. K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, New Jersey, 1962.
2. M. G. Kendall and A. Stuart, The Advanced Theory of Statistics, London, 1958.
3. F. Spitzer, A combinatorial lemma and its applications to probability theory, Trans. Amer. Math. Soc., 82 (1956) 323–339.
4. G. Szegő, Orthogonal Polynomials, Amer. Math. Soc. Colloq. Publ., vol. 23, 1939.
5. N. Wiener and P. Masani, The prediction theory of multivariate stochastic processes: II, Acta Math., 99 (1958) 93–137.
6. G. Wilson, Factorization of the covariance generating function of a pure moving average process, SIAM J. Numer. Anal., 6 (1969) 1–7.

ON VAN DER WAERDEN'S NOWHERE DIFFERENTIABLE FUNCTION

F. S. CATER

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Van der Waerden's function is $f(x) = \sum_{k=0}^{\infty} a_k(x)$, where $a_0(x)$ is the distance from x to the nearest integer and $a_k(x) = 2^{-k}a_0(2^k x)$. In a recent note [1], Patrick Billingsley gave a simple proof that f is a continuous function with no two-sided derivative at any point. Here we show that a generalization of his argument proves that f has no one-sided derivative at any point.

Assume, to the contrary, that $f'_+(x) = L$ for some $x \in R$. For each integer $n > 0$, let $u_n = i2^{-n}$ and $v_n = (i+1)2^{-n}$ where $(i-1)2^{-n} \leqslant x < i2^{-n}$. Define r_n and s_n such that

$$f(u_n) - f(x) = (L + r_n)(u_n - x)$$

and

$$f(v_n) - f(x) = (L + s_n)(v_n - x).$$

Then

$$\begin{aligned} f(v_n) - f(u_n) &= \big(f(v_n) - f(x)\big) - \big(f(u_n) - f(x)\big) = (L + s_n)(v_n - x) - (L + r_n)(u_n - x) \\ &= L(v_n - u_n) + s_n(v_n - x) - r_n(u_n - x) = 2^{-n}L + s_n(v_n - x) - r_n(u_n - x), \end{aligned}$$

and

$$2^n(f(v_n) - f(u_n)) = L + 2^n s_n(v_n - x) - 2^n r_n(u_n - x).$$

But $0 < u_n - x < 2^{-n}$, $0 < v_n - x < 2^{1-n}$, so

$$|2^n(f(v_n) - f(u_n)) - L| \leqslant |r_n| + 2|s_n|.$$

It follows that $\lim r_n = \lim s_n = 0$ and hence $\lim 2^n(f(v_n) - f(u_n)) = L$. But

$$2^n(f(v_n) - f(u_n)) = \sum_{k=1}^{\infty} 2^n(a_k(v_n) - a_k(u_n)).$$

Now a_k is a linear on the interval $[u_n, v_n]$ for $1 \leqslant k < n$ and $a_k(u_n) = a_k(v_n) = 0$ for $k \geqslant n$. Thus

$$2^n(f(v_n) - f(u_n)) = \sum_{k=1}^{n-1} a_k^+(u_n)$$

where a_k^+ is the right derivative of a_k. But $a_k^+ = \pm 1$, so $2^n(f(v_n) - f(u_n))$ is an odd or even integer depending on whether n is an even or odd integer. But this is impossible. The argument for the left derivative is analogous.

Thus we have a simple example (and proof) of a continuous function with no left or right derivative at any point. It is not difficult to prove in a similar way that $\Sigma_{k=1}^{\infty} b_k(x)$ where $b_k(x) = 2^{-k}a_0(2^{3k}x)$ is a continuous function with an infinite left and an infinite right derived number at each point.

We could have shortened our proof by referring to the general principle that if $f'_+(x)$ is finite and if $x < u_n < v_n$, $v_n \to x$ and $|(u_n - v_n)/(u_n - x)| \geqslant 1$ for all n, then $(f(v_n) - f(u_n))/(v_n - u_n)$ approaches $f'_+(x)$.

Reference

1. Patrick Billingsley: Van der Waerden, Continuous nowhere differentiable function, this MONTHLY, 89 (1982) 691.

THE TEACHING OF MATHEMATICS

EDITED BY MARY R. WARDROP AND ROBERT F. WARDROP

Material for this department should be sent to Professor Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.

MATRICES WITH "CUSTOM-BUILT" EIGENSPACES

W. P. GALVIN
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In a recent note [1], J.-C. Renaud poses the question of constructing an $n \times n$ matrix with integer entries and integer eigenvalues. The motivating problem is the need to provide suitable matrices for classroom exercises in linear algebra. He proves an interesting theorem which characterises such matrices and can be used in their construction.

The motivating problem can be solved pragmatically in a way which may not impress the purist, but will (if thoughtfully applied) provide a supply of suitable classroom exercises and

answer the more general motivating question:

How can one produce an $n \times n$ matrix possessing each of the following: (a) integer entries; (b) a set of n integer eigenvalues in a specified proportion to each other; (c) a corresponding set of n specified independent eigenvectors having integer entries?

Such a matrix, A, is diagonalisable and so a nonsingular matrix P exists such that

$$P^{-1}AP = D,$$

where D is a diagonal matrix with entries which are the eigenvalues of A. The matrix P, of course, has the specified eigenvectors as columns. It follows that

$$A = PDP^{-1}.$$

If we begin by constructing matrix P, using the specified n independent eigenvectors as columns, and the matrix D, using a corresponding set of integers in the proportion specified in (b), then the matrix A' produced by the product

$$A' = PDP^{-1}$$

will possess properties (b) and (c) but not, in general, property (a).

This problem can be overcome by a variety of ruses. For example, A' will have integer entries if the determinant of P, det P, is a factor common to each eigenvalue.

A less restrictive approach is to begin with a set of relatively prime eigenvalues and then multiply each entry of the matrix obtained, A', by det P. This will ensure that the resulting matrix, A'', has the desired properties. However, each entry of A'' and each of its eigenvalues may have a factor, f, in common with det P. In that case a more manageable and still suitable matrix A''' can be obtained from A'' by dividing each of its elements by f. If $\{\lambda_i\}_{i=1}^n$ is the set of eigenvalues chosen for matrix A', then $\{((\det P)/f)\lambda_i\}_{i=1}^n$ will be the set of eigenvalues of matrix A'''. The original set of eigenvectors selected for A' remain the eigenvectors of A'''.

The calculations involved are extensive and require the writing of a suitable computer program. This would invite the input of the order of the matrix, its eigenvalues and the corresponding eigenvectors. Subroutines for transposing a matrix, inverting a matrix and multiplying matrices would quickly produce the matrix A'. A further subroutine could find det P and the factor f. These values could then be used to display matrix A''' and its eigenvalues.

Earlier, it was remarked parenthetically, that in using this method to generate suitable classroom exercises, a thoughtful application is needed. Clearly the eigenvectors need to be chosen so that the value of det P is small, otherwise the magnitude of the elements of A''' may be too large to make pencil and paper calculation convenient. Of interest in this context is an article by Robert Hanson [**2**], in which an algorithm for constructing matrices which have determinant $+1$ or -1 is established. This can be used in the present situation to construct a convenient matrix P and hence a suitable set of eigenvectors with integer entries. If it is desired, the algorithm could be coded as a further subroutine.

The program will be useful in a variety of teaching and learning situations, including the production of matrices (with integer entries) possessing a set of specified orthogonal eigenvectors. This problem has been investigated by Konrad J. Heuvers [**3**]. He provides several results which can be used to construct a real $n \times n$ symmetric matrix having prescribed orthogonal eigenvectors and prescribed eigenvalues. The program may also be useful in the search for possible further characterisations of matrices with integer entries and integer eigenvalues.

References

1. J.-C. Renaud, Matrices with integer entries and integer eigenvalues, this MONTHLY, 90 (1983) 202–203.

2. Robert Hanson, Integer matrices whose inverses contain only integers, Two-Year Coll. Math. J., 13 (1982) 18–21.

3. Konrad J. Heuvers, Symmetric matrices with prescribed eigenvalues and eigenvectors, Math. Mag., 55 (1982) 106–111.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS),
H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

EDITOR EMERITUS: EMORY P. STARKE. COLLABORATING EDITORS: VINCENT BRUNO, FRANK S. CATER, GULBANK D. CHAKERIAN, A. M. DAWES, MICHAEL J. DIXON, UNDERWOOD DUDLEY, RICHARD A. GIBBS, CLARK GIVENS, RICHARD M. GRASSL, DOUGLAS A. HENSLEY, ISRAEL N. HERSTEIN, ROBERT H. JOHNSON, ELGIN H. JOHNSTON, MURRAY S. KLAMKIN, DANIEL J. KLEITMAN, JOSEPH D. E. KONHAUSER, FREDERICK W. LUTTMANN, MARVIN MARCUS, LOUISE E. MOSER, M. J. PELLING, C. M. REIS, J. O. SHALLIT, B. L. R. SHAWYER, EDWARD T. H. WANG, AND ALBERT WILANSKY.

*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara CA 95053. Please include solutions, relevant references, etc.*

An asterisk () indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by September 30, 1984. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3043. *Proposed by Gunnar Blom, University of Lund, Sweden.*

A has a house with one front door and one back door.

(a) A places n pairs of walking shoes at each door. For each walk, he chooses one door at random, puts on a pair of shoes, returns after the walk to a randomly chosen door and takes off the shoes at the door. Find the average number of finished walks until A discovers that no shoes are available at the door he has chosen for a further walk.

(b) A walks bare-footed if no shoes are available. Find the long run probability that A performs a given walk bare-footed, i.e., find the limit as $k \to \infty$ of the probability that A's kth walk is bare-footed.

E 3044. *Proposed by Jordi Dou, Barcelona, Spain.*

Construct a triangle ABC given: r, the inradius, $\overline{AI}$ where I is the incenter, and $\overline{AH}$ where H is the orthocenter.

E 3045. *Proposed by Călin P. Popescu, Bucharest, Romania.*

Let H be a hexagon inscribed in a circle. Show that H can be circumscribed about a conic if and only if the product of three alternate sides equals the product of the other three.

SOLUTIONS OF ELEMENTARY PROBLEMS

Difference Between Consecutive Primes

E 2844 [1980, 577; 1983, 286]. *Proposed by Barry Powell, Kirkland, Washington.*

Give as elementary a proof as possible of the following. Let p_n be the sequence of primes. For infinitely many n, the difference $p_{n+1} - p_n$ exceeds $2 \log p_n$. For infinitely many n, the difference

$p_{n+1} - p_n$ is less than $\log p_n$. (More precise results can be obtained by advanced methods.)

Solution to part (a) by the proposer. Let p_n denote the nth prime. We show that for infinitely many n

$$\frac{p_{n+1} - p_n}{\log p_n} < \lambda = 1.2, \tag{1}$$

using Chebyshev's elementary bounds for $\pi(x)$, the number of primes $\leq x$. Chebyshev's arguments (cf. [**1**] pp. 182–184) suffice to show that

$$.9\frac{x}{\ln x} < \pi(x) < 1.1\frac{x}{\ln x}$$

for all sufficiently large x, say $x \geqslant x_0$. We show the stronger result that for all $x \geqslant \max(x_0, e^{10})$ the interval $[x, 5x]$ contains at least one prime p_n for which (1) holds. We argue by contradiction. Let $\{p_m, \ldots, p_{m+n}\}$ denote all primes in the interval $[x, 5x]$ and suppose

$$p_{n+1} - p_n > \lambda \log p_n$$

for $m \leqslant n \leqslant m + k - 1$. Then

$$\begin{aligned} 4x &\geqslant p_{m+k} - p_m \\ &\geqslant \lambda \sum_{i=0}^{k-1} \ln p_{m+1} \\ &\geqslant \lambda k \ln x \end{aligned} \tag{2}$$

where

$$\begin{aligned} k &\geqslant \pi(2x) - \pi(x) - 1 \geqslant .9\frac{5x}{\ln(5x)} - 1.1\frac{x}{\ln x} - 1 \\ &\geqslant 3.3\frac{x}{\ln x} \end{aligned} \tag{3}$$

for $x \geqslant e^{10}$. Then (2) and (3) give

$$4x \geqslant 3.3\lambda x > 4x,$$

a contradiction.

Using the prime number theorem that $\pi(x) \sim x/\ln x$, this argument can be sharpened to improve (1) to

$$\liminf_{n \to \infty} \frac{p_{n+1} - p_n}{\ln p_n} \leqslant 1.$$

Solution to part (b) *by Robert E. Shafer, Berkeley, California.* Let p_n denote the nth prime. We show that for infinitely many n

$$\frac{p_{n+1} - p_n}{\log p_n} \geqslant 1.7. \tag{4}$$

This proves (b) since $1.7 \log p_n \geqslant \log p_{n+1}$ for all sufficiently large n, since $\log p_n / \log p_{n+1} \to 1$ as $k \to \infty$.

For any n, there exists a number $J < p_{n+1}p_{n+2}/2$ such that the number

$$M = p_1 \cdots p_n \cdot J$$

has all $M + k$ composite for $-(p_{n+1} - 1) \leqslant k \leqslant p_{n+1} - 1$. We pick J using the Chinese remainder theorem so that

$$\begin{aligned} p_1 \cdots p_n \cdot J &\equiv -1 \pmod{p_{n+1}}, \\ p_1 \cdots p_n \cdot J &\equiv 1 \pmod{p_{n+2}}. \end{aligned}$$

This ensures that $M + 1$ and $M - 1$ are composite, while for all other $2 < |k| \leqslant p_{n+1} - 1$, $M + k$ is divisible by all primes k. Now let p_m and p_{m+1} be two consecutive primes such that $p_m < M < p_{m+1}$, so that

$$p_{m+1} - p_m \geqslant 2p_{n+1}. \tag{5}$$

But

$$\ln p_m \leqslant \ln M \leqslant \sum_{i=1}^{n+2} \ln p_i. \tag{6}$$

Using elementary arguments of Chebyshev, it can be shown that (cf. [1], p. 186)

$$p_{n+1} > .9n \ln n \tag{7}$$

and that

$$\sum_{i=1}^{n+2} \ln p_i < 1.1n \ln n \tag{8}$$

for all sufficiently large n. Then (5)–(8) give

$$p_{m+1} - p_m \geqslant 1.8n \ln n \geqslant \frac{1.8}{1.1} \ln p_m > 1.7 \ln p_m.$$

Reference

1. I. Niven and H. S. Zuckerman, An Introduction to the Theory of Numbers, 2nd ed., Wiley, New York, 1966.

Comment by the editors. The distribution of gaps between consecutive primes is an extensively studied problem. Concerning small gaps between primes, M. Huxley [1] has shown that

$$\frac{p_{n+1} - p_n}{\log p_n} < 0.4426,$$

for infinitely many n. Concerning large gaps between primes, R. A. Rankin [2] showed that

$$\frac{p_{n+1} - p_n}{\log p_n} > e^{\gamma}(\log_2 p_n)(\log_4 p_n)(\log_3 p_n)^{-2}$$

for infinitely many n, where $\log_2 p_n = \log\log p_n$, $\log_3 p_n = \log\log\log p_n$, etc., and logarithms are to the base e.

References

1. M. Huxley, Small differences between consecutive primes II, Mathematika, 24 (1977) 142–152.
2. R. A. Rankin, The difference between consecutive prime numbers V, Proc. Edinburgh Math. Soc., 13 (1962/3) 331–332.

The Number of Tangent-Normals of an nth Degree Polynomial

E 2886 [1981, 349]. *Proposed by C. O. Oakley, Haverford College.*

Let $n > 2$. How many tangent-normals can the graph of an nth-degree polynomial have? (A tangent-normal is a line that is tangent to the graph at one point and normal at another.)

Partial solution by L. Kuipers, Sierre, Switzerland. Let $N(n)$ be the least integer such that each polynomial of nth degree has at most $N(n)$ tangent-normals. If a line is tangent to the graph of $y = f(x)$ at $(\xi, f(\xi))$ and normal to the graph at $(\eta, f(\eta))$, then $F(\xi, \eta) = 0$ and $G(\xi, \eta) = 0$, where $F(x, y) = f'(x)f'(y) + 1$ and

$$G(x, y) = \sum_{k=2}^{n} \frac{f^{(k)}(x)(y - x)^{k-2}}{k!}.$$

Thus (ξ, η) is one of the points of intersection of the two algebraic curves given by $F(x, y) = 0$ and $G(x, y) = 0$. Now F has degree $2n - 2$ while G has degree $n - 2$. Hence by Bézout's theorem the total number of points of intersection can be at most (deg F) · (deg G) = $(n - 2)(2n - 2)$ and so $N(n) \leqslant (n - 2)(2n - 2)$.

Partial solution by the proposer. A detailed analysis of the nth = $(2m + 1)$th degree polynomial $f(x) = kx(x - 1) \cdots (x - 2m)$ leads to the lower bound $N(n) \geqslant \frac{1}{2}(n - 1)(3n - 5)$ in case n is odd. Similarly $N(n) \geqslant \frac{1}{2}(n - 2)(3n - 2)$ in case n is even. The analysis proceeds by enumeration of Type I and Type II tangent-normals. A Type I tangent-normal is a tangent line near a maximum or minimum point that is normal to a "vertical" part of the graph. A Type II tangent-normal is a tangent-normal that has both endpoints near maximum or minimum points.

Editor's note: The proposer's lower bounds probably represent the exact value of $N(n)$.
Also partially solved by R. Cheng and Dinh Thế Hùng.

Strongly Closed Finite Sets "Often" Have Cardinality 4

E 2909 [1981, 705]. *Proposed by I. M. Isaacs and Mark Manasse, University of Wisconsin.*

Let us say that a set of nonnegative real numbers is "closed under $\pm$" if for every $x, y \in S$, either $x + y \in S$ or $|x - y| \in S$. For instance, if $\alpha > 0$ and $n \geqslant 0$ is an integer, then the set $S(n, \alpha) = \{0, \alpha, 2\alpha, \ldots, n\alpha\}$ has this property. Show that, if a finite set is closed under $\pm$ and it is not of the form $S(n, \alpha)$, then the set contains exactly four elements.

I. *Solution by M. Doob, University of Manitoba, Winnipeg, Manitoba, Canada.* We are given the closure property "CLO": x and y in S implies that either $x + y$ or $x - y$ is in S. Since S is finite, by using $x = y$ we see that 0 is in S. If S satisfies CLO, then rS also satisfies CLO for any real number r; thus we may assume that any nonempty S other than $\{0\}$ has 1 as its least positive element. Let M be the maximum element of S. Using $x = M$ and CLO, we get the property "SYM": y in S implies $M - y$ in S. Thus the elements of S in $[M - 1, M]$ are precisely M and $M - 1$ themselves. Now suppose z is in S with $1 < z < M - 1$. Then using CLO with $x = M - 1$ and $y = z$ we get $M - 1 - z$ in S, and by SYM we get $z + 1$ in S. Applying this argument repeatedly, we get an integer t such that $z + t = M - 1$. Using the same z and SYM, we get $y = M - z$ is in S, and using CLO with this y and $x = M - 1$, we get $z - 1$ in S. Thus we get an integer u such that $z - u = 1$. Hence if a z exists in S satisfying $1 < z < M - 1$, then $M = z + t + 1 = u + t + 2$ and $S = S(M, 1)$. Hence, in general, if S is finite and satisfies CLO, then S is empty, $S = \{0\} = S(n, 0)$, $S = \{0, a\} = S(1, a)$, $S = \{0, a, b, a + b\}$, or $S = S(n, a)$.

II. *Solution by H. P. von Ohlweiler, Technische Hochschule, Darmstadt, West Germany.* If S is finite, '$\pm$'-closed, then let $a_0 < a_1 < \cdots < a_n = s$ be the elements of S. Clearly $s - s = 0 = a_0$ and for $i \neq 0$, $s + a_i \notin S$, so $s - a_i \in S$, which makes the map ϕ_s with $\phi_s(x) := s - x$ bijective, order reversing, distance preserving from S to S.

Claim: *If* $|x - y| \in S$ *for all* $x, y \in S$, *then* $S = \{0, \alpha, \ldots, k\alpha\} = S(k, \alpha)$. Indeed, for the induction step the elements of S are $a_0 < a_1 < \cdots < a_k < a_{k+1}$. But $\{a_0, \ldots, a_k\}$ is closed under $|x - y|$ again, so the inductive hypothesis gives: $S = \{0, \alpha, \ldots, k\alpha, s\}$. Again with ϕ_s as above we find $\phi_s(k\alpha) = \alpha$, so $s - k\alpha = \alpha$ whence $s = (k + 1)\alpha$.

Now let S be '$\pm$'-closed again, then with the help of ϕ_s we find that

$$\phi_s(s) = 0 < a_1 < \cdots < a_k \leqslant s - a_k < \cdots < s - a_1 < s$$

are the elements of S. For $0 < i < j \leqslant k$ we find $(s - a_i) + (s - a_j) \notin S$, so

$$(s - a_i) - (s - a_j) = a_j - a_i \in S.$$

So $\{0, a_1, \ldots, a_k\}$ is closed under $|x - y|$, so with the "claim" the elements of S are

$$0 < \alpha < \cdots < k\alpha \leqslant s - k\alpha < \cdots < s - \alpha < s.$$

If $|S|$ is odd, then $s - k\alpha = k\alpha$ and $S = S(2k, \alpha)$. So we assume $s - k\alpha > k\alpha$ and, from now on, $k > 1$. Then $(s - \alpha) + k\alpha \notin S$, so

$$(s - \alpha) - k\alpha = (s - k\alpha) - \alpha \leqslant k\alpha,$$

hence $s \leqslant (2k + 1)\alpha$. But on the other hand $(s - (k - 1)\alpha) + k\alpha \notin S$, so

$$s - (k - 1)\alpha - k\alpha \geqslant \alpha.$$

Equality would imply $s - k\alpha = k\alpha$, so we must in fact have

$$s - (k - 1)\alpha - k\alpha \geqslant 2\alpha$$

and therefore $s \geqslant (2k + 1)\alpha$ and with the above $s = (2k + 1)\alpha$, so $S = S(2k + 1, \alpha)$. Finally the case of $k = 1$ and $|S|$ even yields $|S| = 4$ and indeed here we find that $\{0, \alpha, \beta, \alpha + \beta\}$ is a solution for every $0 < \alpha < \beta$.

Also solved by 28 other readers.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by September 30, 1984. The solver's full post-office address should be on each sheet.

6459. *Proposed by Mark Bowron (student), University of Washington.*

Let C be Cantor's ternary set, and let $g : [0,1] \to [0,1]$ be the Cantor-Lebesgue function. Define a measure μ on C by setting $\mu(S) = mg(S)$ for $S \subseteq C$, where m is Lebesgue measure. We know that $g'(x) = 0$ a.e. (m) in $[0,1]$. Show that $g'(x) = \infty$ a.e. (μ) in C.

SOLUTIONS OF ADVANCED PROBLEMS

Newton's Iteration

6293 [1980, 308]. *Proposed by W. M. Kahan, University of California, Berkeley.*

Assume that $f(x)$ never takes negative values but that it does vanish somewhere inside an interval throughout which the graph of $f(x)$ is convex. To find where $f(x)$ vanishes one might use Newton's iteration

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \quad \text{for } n = 0, 1, 2, 3, \ldots .$$

This iteration will converge to the desired zero of $f(x)$ from any starting iterate x_0 in the interval, though convergence may be arbitrarily slow; so much is already well known. The problem is to prove that, no matter how slowly the iterates $\{x_n\}$ converge, $\Sigma_1^\infty (2^n f(x_n))^2$ is finite.

Solution by the proposer. Abbreviate $f(x_n) = f_n$ for $n = 0, 1, 2, 3, \ldots$. The desired result is true trivially if any $f_n = 0$ since that terminates the iteration, so suppose instead that every $f_n > 0$. Since the graph of f is convex, it lies above its tangent; $f(x) \geqslant f_n + (x - x_n)f'(x_n)$. Hence

$$f_{n-1} \geqslant f_n + (x_{n-1} - x_n)f'(x_n) = (1 + d_{n-1}/d_n)f_n,$$

where $d_n = x_{n+1} - x_n$. After observing that all $d_{n-1}/d_n > 0$ we infer

$$f_n/f_{n-1} \leqslant 1/(1 + d_{n-1}/d_n) \leqslant \sqrt{(d_n/d_{n-1})}/2,$$

and therefore

$$f_n/f_0 \leqslant \sqrt{(d_n/d_0)}\,/2^n.$$

From this follows

$$\sum_0^\infty (2^n f_n)^2 \leqslant (f_0^2/d_0)\sum_0^\infty d_n = (f_0^2/d_0)\Big(\lim_{n\to\infty} x_n - x_0\Big) < \infty.$$

Also solved by Chen Dao-qi.

An Identity Involving Gaussian Polynomials

6407 [1982, 703–704]. *Proposed by L. Van Hamme, Free University of Brussels, Belgium.*

Define $\begin{bmatrix} n \\ k \end{bmatrix}$ by means of the relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = F_{n,k}/F_{k,k},\ F_{n,k} = (q^n - 1)(q^{n-1} - 1)\cdots(q^{n-k+1} - 1),$$

so that $\begin{bmatrix} n \\ k \end{bmatrix}$ is the so-called Gaussian polynomial. Prove the identity

$$\sum_{k=1}^n \frac{q^k}{1-q^k} = \sum_{k=1}^n \frac{(-1)^{k-1}}{1-q^k} q^{k(k+1)/2} \begin{bmatrix} n \\ k \end{bmatrix}.$$

Solution by Michael Hoffman, Memorial University of Newfoundland, St. John's, Newfoundland, Canada. The recursion relation

$$q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix} \tag{1}$$

is easily proved directly from the definition. We shall use the following lemma:

LEMMA.

$$\sum_{k=1}^n (-1)^{k-1} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = 1.$$

Proof. The result holds for $n = 1$: suppose inductively that

$$\sum_{k=1}^{n-1} (-1)^{k-1} q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} = 1. \tag{2}$$

Then

$$q^{n-1}\sum_{k=0}^{n-1} (-1)^k q^{k(k-1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix} = 0,$$

which can be rewritten as

$$\sum_{k=1}^n (-1)^{k-1} q^{k(k-1)/2} q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = 0.$$

Add this to (2) to get

$$\sum_{k=1}^n (-1)^{k-1} q^{k(k-1)/2}\left(q^{n-k}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}\right) = 1,$$

from which the result follows by (1).

Now we prove the desired identity by induction on n. The identity holds for $n = 1$: suppose

that

$$\sum_{k=1}^{n-1} \frac{q^k}{1-q^k} = \sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{1-q^k} q^{k(k+1)/2} \begin{bmatrix} n-1 \\ k \end{bmatrix}.$$

By the use of (1), to prove the identity from this it suffices to show that

$$\sum_{k=1}^{n} \frac{(-1)^{k-1}}{1-q^k} q^{k(k+1)/2} q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \frac{q^n}{1-q^n}. \tag{3}$$

But

$$\frac{1-q^n}{1-q^k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix},$$

so (3) is equivalent to

$$\sum_{k=1}^{n} (-1)^{k-1} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} = 1,$$

which is the lemma.

Also solved by George E. Andrews, Paul S. Bruckman, C. S. Karuppan Chetty (India), O. P. Lossers (The Netherlands), Pei Yuan Wu (Republic of China), Hang Fai Yeung (Australia), and the proposer.

Rotating a Set into a Set

6408 [1982, 704]. *Proposed by Alexander Kovaćec, University of Vienna, Austria.*

Let $T := \{z \in \mathbb{C} : |z| = 1\}$. Let λ denote Lebesgue measure on T, normalized so that $\lambda(T) = 1$. For a given positive integer n let A be a subset of T with $\lambda(A) > 1 - 1/n$. Now suppose B is a set in T containing exactly n points. Show that there exists a rotation of B which carries it into A, i.e., there exists $c \in T$ such that $cB \subset A$.

Solution by J. G. Mauldon, Amherst College, Amherst, Massachusetts. Let $B = \{b_i : i = 1,2,\ldots,n\}$, $R_i = T \setminus b_i^{-1}A$ and $R = \cup_{i=1}^{n} R_i$ so that $R = T \setminus \cap_{i=1}^{n} b_i^{-1}A$. Then, since λ is rotation invariant,

$$\lambda(R_i) = 1 - \lambda(A) < 1/n$$

and so $\lambda(R) < 1$. Hence,

$$T \setminus R = \cap_{i=1}^{n} b_i^{-1}A \neq \varnothing$$

and any $c \in \cap_{i=1}^{n} b_i^{-1}A$ has the required property that $cB \subset A$.

Also solved by K. F. Andersen (Canada), Antonio Sá Barreto (Brazil), F. S. Cater, Roy O. Davies (England), Adam Fieldsteel, Hans Goller (Germany), Jay Hook, Yeung Sai Kee (Hong Kong), Ivo Klemes, Matti Lehtinen (Finland), N. J. Lord (England), O. P. Lossers (The Netherlands), Russell Lyons, Curt McMullen, Mark D. Meyerson, Nicholas Miller, Victor Pambuccian (Romania), James Propp (England), John Rainwater, Bruno Remillard (Canada), C. Ray Rosentrater, Jean-Marc Roy (Canada), Chin-Chi Shan, Aristomenis Siskakis, David M. Wells, Pei Yuan Wu (Republic of China), and the proposer.

A Fixed Point Theorem

6409 [1982, 787]. *Proposed by Dilip Kumar Bayan and S. K. Chatterjea, Calcutta University, India.*

Let $f: K^n \to E^n$ be a continuous function (where E^n is Euclidean n-space and $K^n = \{x | x \in E^n, \|x\| \leqslant 1\}$) such that for every $y \in S^{n-1} = \{x | x \in E^n, \|x\| = 1\}$ there is an $m > 1$ with $f(y) = my$. Show that f has a fixed point. Compare this MONTHLY 78 (1971) 310, #5721.

Solution by G. A. Heuer, Concordia College, Moorhead, Minnesota, and Karl W. Heuer, Washington State University, Pullman, Washington. Assume f has no fixed point. Then the function $g: K^n \to S^{n-1}$ given by $g(x) = (x - f(x))/\|f(x) - x\|$ is continuous. By the Brouwer Fixed Point Theorem, there exists a point y such that

$$g(y) = (y - f(y))/\|f(y) - y\| = y,$$

implying that $y \in S^{n-1}$ and $f(y) = y\{1 - \|f(y) - y\|\} = my$, where $m < 1$, a contradiction. [Note that the hypothesis may be weakened to read: For every $y \in S^{n-1}$ there is no $m < 1$ with $f(y) = my$.]

Also solved by 24 others including the proposers.

Composition of Polynomials

6410 [1982, 787–788]. *Proposed by Gary Gundersen, University of New Orleans, and Steve Osborn, Dallas, Texas.*

Let $f(z)$ and $g(z)$ be two nonconstant rational functions such that if $g(z_0)$ is an integer for some z_0, then $f(z_0)$ is also an integer. Show that $f(z) = P(g(z))$, where P is a polynomial with rational coefficients.

Solution by I. N. Baker, Imperial College, London. Suppose that g has a pole at α. Then for some sequence z_n which tends to α, $g(z_n) = n$, $n = 1, 2, 3, \ldots$, and so $f(z_n)$ is an integer $m(n)$. Since f takes each value only finitely often, the sequence $m(n)$ is unbounded and α is a pole of f. By making a Moebius transformation of the z variable, if necessary, we can arrange that $\alpha = \infty$.

Then for some positive integers n and m there is a neighbourhood of ∞ in which we have

$$w = g(z) = a_m z^m + \sum_{k=-\infty}^{m-1} a_k z^k, \quad a_m \neq 0, \qquad f(z) = b_n z^n + \sum_{k=-\infty}^{n-1} b_k z^k, \quad b_n \neq 0,$$

and for certain branches of the inverse of g near $w = \infty$

$$z = g^{-1}(w) = a_m^{-1/m} w^{1/m} + \lambda + \mu w^{-1/m} + \cdots,$$

$$\psi(w) = f\big(g^{-1}(w)\big) = A w^{n/m} + \text{lower powers of } w^{1/m}, \quad A \neq 0.$$

Since if w is an integer, then $\psi(w)$ is an integer and so real, the argument of the solution to Advanced Problem 6208 in this MONTHLY [vol. 87, 1980, p. 228] shows that the expansion of ψ reduces to

$$\psi(w) = \sum_{i=-\infty}^{k} c_i w^i, \quad c_i \in \mathbb{R}, \quad w \text{ near } \infty, \tag{1}$$

$$= P(w) + h(w),$$

where k is a positive integer, $P(w) = \sum_{i=0}^{k} c_i w^i$, $h(w) = \sum_{i=-\infty}^{-1} c_i w^i$.

Putting $\Delta F(w) \equiv F(w+1) - F(w)$, we see that, for each $j \in \mathbb{N}$, $\Delta^j \psi$ is analytic for large w and, in particular, this is true for

$$\Delta^k \psi(w) = k! c_k + \Delta^k h(w).$$

Since this last expression is an integer for all large integral w, we see that $k! c_k$ is an integer and $\Delta^k h(w) = 0$ for such w and hence identically for all w. If p is the largest integer such that $H(w) = \Delta^p h(w) = \sum_{i=-\infty}^{-1} c_i' w^i \not\equiv 0$ and if r is the largest i such that $c_i' \neq 0$, then $|\Delta H(w)| \sim |c_r w^{r-1}| \neq 0$ for large r—which contradicts $\Delta^{p+1} h(w) \equiv 0$. Hence in (1), $h(w) \equiv 0$ and $\psi(w) \equiv P(w)$, $f(z) = P(g(z))$.

Now $\Delta^i \psi(w)$ is an integer if w is any integer, in particular, $w = 0$ shows that $i! c_i$ is an integer. The proof is complete.

Also solved by Peter Ungar.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

WHAT'S A GOOD BOOK REVIEW? AN EDITORIAL COMMENT

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What's a good book review? We won't tell you (we're not sure there's only one answer anyway), but we will say what we (the editors) think a good MONTHLY book review is.

First, what's a bad book review? That's easy—a bad book review is dull. It faithfully reports the contents of a book in concise terms so that the experts who are interested in reading the book can determine exactly which topics the book covers. It may tell you that on page 351 there are several typos, the notation in Chapter 7 is atrocious, and the author was really quite negligent in omitting the references to Professor Baker's 1968 paper in the bibliography. It may end (and probably does end) with the illuminating statement that in the reviewer's opinion this is a "useful" addition to the literature of pseudo-convex algebroids. That's dull!

A review for the MONTHLY should be a chatty, informal essay based on the book rather than a stodgy, formal report; it ought to be a commentary on the subject rather than a list of contents. The essay should be informative and, most importantly, interesting—interesting to a *broad* audience of mathematicians. Such an essay should address an audience of thousands rather than a dozen or so experts.

We receive hundreds of books to review each year and we can only review about 60. How do we choose? Part of the answer is practical: we can only review books for which we can find suitable reviewers. But our main selection criterion is this: we try to review books that can be used to learn from—at the undergraduate, the graduate, or even the research level. Proceedings and lecture notes don't often fall in this category; undergraduate and graduate texts usually do, and so do many research level books. Of course, this means that the books being reviewed cover a wide range of areas and levels.

Reviews of research level books may say very little about the book itself; the book should serve as a framework about which the review is written. (A house can't be built without support, but in most cases the support doesn't show when it's completed.) Most readers find a description of the contents of such a book uninformative as well as dull. A typical reader might be interested in the contents of only one or two research level books reviewed during the course of a year. On the other hand, even though I may not be interested in the contents of a book on pseudo-differential operators, I would like to know what the subject is about. When did people get interested in these things? (Or *did* they?) Is it algebra? Analysis? Topology? What are the major problems? That kind of review may interest 25 percent of all readers; a detailed accounting of what's in the book may interest only one percent.

Of course, it's quite nice when a reviewer can both write an essay based on the book and give the reader some rough idea of how this particular book fits into the literature. That can often be done without giving a table of contents and without boring the majority of readers who never intend to pick up the book (no matter *what* the reviewer says). But given the choice between finding out about the book or finding out about mathematics, we'll take mathematics every time.

Reviews of undergraduate and beginning graduate texts may often focus on pedagogical and curricular matters. It is often still appropriate, however, to convey some real mathematics. (A reader who took a graduate course in measure theory 20 years ago, but has worked in finite group theory ever since, needs some gentle reminders about just what Egorov's theorem says.) Once

again, the review should be based on the book rather than focused on it. How does this subject fit into the curriculum? (Or does it?) What are the essential topics and major results? Are some topics more important than others? What are the "standard" texts?

By their nature, reviews of textbooks are likely to be opinionated. That's fine. How do you teach such-and-such? You'll never find out from someone who gives you a long list of alternatives. But reading someone's strong, unequivocal opinion—even if you disagree with it—will usually give you more insight than you had before.

Are there good books and bad books? Of course there are. We orally communicate our feelings about good and bad books to our colleagues all the time—at lunch, at tea, in the hall. We ought to do no less in writing a review, and we ought to use the same clear, unmistakable language even if it raises the ire of publishers and authors. Mathematicians do painfully little thinking about the way we teach our subject; strong, responsible opinions force us to think—they stir the waters.

A "responsible" opinion, by the way, is *not* necessarily an opinion with which the editors agree. (While we quite like our own opinions, we like to hear other opinions even if we disagree with them—and we often do.) The editorial pages of newspapers would be pretty dull if they only published columns and letters with which the editors agreed; our book reviews would be equally dull.

The "perfect" book review is a mixture of all the above. It contains history, mathematics, opinions; it tells about the subject and it tells about the book (without being obvious); it informs, it entertains, and, above all, it is interesting. Remarkably, it's supposed to do all this in four to six typed pages (our recommended length: we tell authors to write at least three pages and never more than ten).

Is this kind of review unfair to authors? Is it conceivable that a reader will interpret the lack of detailed information about a book as a subtle comment on the book's merit? We hope not; we want to be fair to authors (and publishers) but we also want to be fair to our readers. And publishing prim reviews that almost no one reads is not fair to anyone.

What's a good MONTHLY book review? It's interesting reading based on a book.

Introduction to Number Theory. By Hua Loo Keng. Translated from the Chinese by Peter Shiu. Springer-Verlag, New York, 1982. xviii + 572 pp.

A Classical Introduction to Modern Number Theory. By Kenneth Ireland and Michael Rosen. Springer-Verlag (Graduate Texts in Mathematics, vol. 84), New York, 1982. xiv + 341 pp.

JOSEPH B. ROBERTS
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The theory of numbers is one of the oldest branches of mathematics and is a subject that has been worked and reworked by a full range of intellects from the most amateur to the most exalted mathematical practitioners. Some of the unsolved problems of the subject are well known to people who know nothing else about number theory. Does there exist an odd integer which is equal to half the sum of all its divisors? (The odd perfect number problem.) Is there an integer n, greater than 2, for which an integral nth power is the sum of two integral nth powers? (The Fermat conjecture.) Indeed the very attractiveness of the subject lies partly in this fact. One of the greatest mathematicians of all time, Carl Friedrich Gauss, has had the following to say about this:

> A great part of its theories derives an additional charm from the peculiarity that important propositions, with the imprint of simplicity upon them, are often easily discoverable by induction, and yet are of so profound a character that we cannot find their demonstration till after many vain attempts....

In the preface to the first edition of one of the recent classics in number theory, *An Introduction to the Theory of Numbers*, by G. H. Hardy and E. M. Wright, the authors say "... we can hardly have failed completely, the subject-matter being so attractive that only extravagant incompetence could make it dull."

As with any large and well-worked field, the subject is no longer a single subject but a loose conglomerate of many subjects. The points of contact between, for instance, those branches of number theory called elementary number theory, algebraic number theory, and analytic number theory may seem to the nonexpert to be tenuous and not to justify their being considered part of the same subject. In fact, among mathematicians themselves, even among those calling themselves number theorists, the name "number theory" does not always evoke similar mental responses. Many will think of the basic contents of a first course in number theory which embraces such topics as the elements of prime number theory, the theory of congruences, quadratic reciprocity, quadratic forms, and continued fractions. Others will think of the Riemann zeta function and the distribution of prime numbers. Still other may think of one or more of: transcendence theory, the theory of elliptic curves, modular forms, representation of adele groups, the continuing investigation of Fermat's conjecture, the Riemann hypothesis, or arithmetical algebraic geometry.

Not only might one be in the dark about what a mathematician, unknown to us, means by number theory, but one would not even dare to guess the contents of contemporary books on the subject from their titles—witness, for example, the books *Elementary Theory of Numbers* by W. Sierpinski in which the author says that the subject of elementary number theory, as distinguished from other parts of number theory, is a part of the subject "which does not use the notion of limit," and *Basic Number Theory* by A. Weil in which (near the beginning of a four-page detail on prerequisites and notations) one finds the phrase "Already in Chapter I,..., essential use is made of the basic properties of locally compact commutative groups, including the existence and unicity of the Haar measure,...."

The books prompting this essay are also examples of this phenomenon—though less extreme in nature. The titles are very similar, yet their contents are quite different.

More so than with many subjects, once one goes beyond the material of a first course, the subject breaks down into areas of research, each associated with some famous problem or conjecture.

For example, during much of the 19th century one of the most famous assertions of the analytic theory of numbers—that the number of prime numbers not exceeding x is asymptotically equal to $x/\log x$ (the prime number theorem)—was extensively studied and was proved, after much effort, only in 1896. In that year two proofs were independently given, one by Jacques Hadamard, a French mathematician, and the other by Ch.-J. de la Vallée Poussin, a Belgian mathematician. Both proofs made heavy use of the theory of functions of a complex variable and during the first half of the century, theorems in analytic number theory were sometimes judged as to their "depth" by how much complex variable theory was used in their proofs. Even in recent times, this theorem has played an important role in research, and only in 1948 did Paul Erdős and Atle Selberg succeed in giving an "elementary" proof of the theorem (i.e., one not using complex variable theory).

On the algebraic side, the entire subject now referred to as algebraic number theory may be considered to be an outgrowth of work prompted by the Fermat conjecture. As other examples of special conjectures or theorems, around which much work has coalesced, one might cite: the search for reciprocity laws and the work in class field theory stretching all the way from Gauss through the work of Artin to the recent work of Langlands; Roth's theorem, proved in 1955 and culminating a half century of effort, which states that for $\delta > 0$ and ξ any irrational algebraic number, there are only finitely many rational numbers p/q such that $|\xi - p/q| < 1/q^{2+\delta}$; the Gelfond-Schneider theorem, proved in 1934 and included in the 1900 list of 23 problems posed by David Hilbert, tells us that if α and β are algebraic ($\alpha \neq 0$ or 1, β irrational) then α^{β} is transcendental. Each of these has played a significant role in 20th century number theory.

Continuing in this vein, Goldbach's conjecture, made in 1742, states that all even integers greater than 2 can be written as the sum of two primes. This conjecture has been under intensive investigation for 200 years and only in recent times have Schnirelmann, Vinogradoff, and Chen made real progress on the problem.

Finally we mention one of the most famous unproved conjectures of all—the Riemann hypothesis. In the 19th century Riemann, while engaged in work on the distribution of prime numbers, investigated the properties of the so-called Riemann zeta function

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for s complex. Riemann conjectured that all of the zeros of this function which satisfy $0 \leqslant \operatorname{Re} s \leqslant 1$ lay on the line $s = \frac{1}{2}$. So many deep and fascinating consequences follow from this hypothesis that at one time it was rumored that a very well-known mathematician had written a book based on its truth and was waiting anxiously for a proof to appear so that it could be included. In any event, the original hypothesis has been extended and modified, and various forms (though not the original) have been proved by A. Weil, P. Deligne, and others. From these a variety of results have been deduced. The so-called "extended Riemann hypothesis" for algebraic number fields would, for example, imply the still unproved conjecture of E. Artin which asserts for any nonsquare integer a, $a \neq -1$, there are infinitely many primes p for which a is a primitive root modulo p. Making use of the truth of another variant of the Riemann hypothesis, an analogue of the Artin conjecture for $k[x]$ has been proved. These questions continue to be the focus of much research.

All of the above examples furnish us with landmarks of the subject of number theory even though they are not all part of a nicely unified theory.

Both books under review deal with a number of aspects of the subject as exemplified by our examples. By proper selection, both books could be used as texts for a first course. At the same time, each of them goes far beyond this. Almost all of the above topics are treated in Hua's book which exhibits in its broad scope something of the rather disparate nature of the history of the subject. The exposition places an emphasis on analytic number theory and there is relatively little use of modern algebra in the presentation. One can dip into Hua's book at many points: it is not necessary to read it from beginning to end.

The book by Ireland and Rosen, on the other hand, attempts to give a unified approach to a somewhat narrower set of topics drawn for the most part from algebraic number theory. Their presentation makes fairly heavy use of modern algebra—especially in the last 2/3 of the book in which they deal with some of the interconnections of the Riemann zeta function and various questions on elliptic curves; reciprocity laws; and the solution of equations in finite fields. Unlike the book by Hua, this book is written with an eye towards a reader wanting a unified treatment of the topics with which it deals; it is more difficult to dip into at random.

Both books make an attempt, through chapter notes, to bring the reader abreast with recent developments, and both books are modern in that the material is in a form not foreign to current research articles. Perhaps the book by Ireland and Rosen is more fashionable at the present time since there are some hopes that extensive unification of seemingly separate branches of number theory will come from such considerations as treated therein. Certainly the book requires a more algebraically sophisticated reader than does Hua's book, and it does take one closer to the research frontiers of the subject. On the other hand, Hua's book offers a more comprehensive look at a wider range of material than does that of Ireland and Rosen and is perhaps more suited to a reader interested in gaining insight into the full range of topics treated in modern number theory.

Not many years ago there were embarrassingly few books in number theory suitable for a text. That has not been true for some time. Nevertheless, both of these books are texts and are a welcome addition to the literature since they each offer a rich body of material not readily available elsewhere.

Introduction to Algebra. By A. I. Kostrikin. Springer-Verlag, New York, 1982. xiii + 575 pp.

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Recently, P. J. Hilton (this MONTHLY, October, 1982) reviewed a rather comprehensive mathematics text used by first-year students in the Soviet Union. Hilton noted the absence of certain topics from that text, including abstract algebra. That particular gap is filled by Kostrikin's text, which is derived from first and third semester courses of the Mechanics-Mathematics Faculty of Moscow University. (The book is in two parts; it is assumed that the student takes a course in abstract linear algebra in between.) As with other texts, the level of sophistication is higher than that customarily used for students of the same age in the United States. This presumably reflects the European tendency toward early specialization. Unfortunately, it is not clear that the average American mathematics major ever reaches the level of this text, as far as his acquaintance with algebra is concerned.

I assume that the most selective universities continue to require a strong curriculum in basic mathematics, including a thorough introduction to modern algebra. However, I feel that other universities may be slipping in this regard, both because of the decline in the quality of students and because of distracting factors that have intruded on the curriculum. My own institution, which is required to accept almost all in-state applicants, formerly required a two-semester algebra sequence at the level of Birkhoff-Mac Lane. That requirement has now been cut back to only one semester (in which not much material is covered). Still, many of our majors need several tries to pass the requirement; some find it the motivation to transfer to other departments. These students simply have no genuine mathematical talent. On the other hand, the recent rush to embrace applicability has also caused a dilution of the mathematics curriculum. Seeking to prepare our students for jobs in business and government, we teach them computer science, number-crunching numerical analysis, and applied statistics. Given the constraints of time, this means that something has to go; too often, the something is algebra, topology, or rigorous analysis. The result is a generation of mathematics graduates who can program a chi-squared test in any of several computer languages, but are unfamiliar with solvable groups or Hausdorff spaces; who can numerically approximate a definite integral to a hundred decimal places, but who never heard of Lebesgue measure. Some may feel that no great loss is occurring here, especially for those students who do not go on to graduate school in mathematics. I suspect that they are wrong. Even if we adopt a vocational view of higher education, we must realize that the applied mathematician of the present and of the future needs a strong general background in mathematics.

It was not so long ago that "applied mathematics" referred mainly to the explicit or numerical solution of differential equations that arise in physical or engineering problems. Of course, the laws of physics have not been suspended, so interest in these problems continues. However, the electronic revolution, among other things, has broadened the spectrum of applied mathematics considerably. The computer itself, and the new needs of data transmission (satellite communications, direct computer linkage) have given rise to many new problems about discrete, algebraic, and topological structures. The theoretical background for computing and computer design is highly algebraic, and the information processed in a digital computer is inherently discrete. Data transmission requires the use of error-correcting codes, which are constructed by studying modules over finite rings. Recent developments in linear system theory, which lies at the heart of the theory of automatic control, are based on Lie algebras and the properties of algebraic varieties. Pólya's enumeration theory helps to sort out organic chemical compounds. All this is with us now; the future may hold more unlikely applications.

It would seem that a book like Kostrikin's could be the basis for a fine course that gives the undergraduate a thorough introduction to abstract algebra, with indications of some of the applications included. A first chapter called "Sources of Algebra" starts with motivating problems from physics and coding theory, and then covers basic topics such as systems of linear equations

and integer arithmetic. Later chapters cover the usual vector spaces, determinants, groups, rings, and so forth. Interesting special topics include the theory of equations (including symmetric functions, the fundamental theorem of algebra and stability of real polynomials), group actions on sets, group representations and an introduction to algebras (including the nonassociative case). The discussion is rigorous, but sufficiently informal to be quite readable. There are plenty of exercises, and the translation (by Neal Koblitz) captures the idiosyncratic charm very well.

Probability and Statistics with Reliability, Queuing, and Computer Science Applications. By K. S. Trivedi. Prentice-Hall, 1982.

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I knew I would never raise peafowl, or even want to think about raising peafowl. And so it was, in my third year as an undergraduate student of mathematics, that the charm of Probability and Statistics I and II began to wane, taking an unfortunate direction from which it was never to recover.

The course had had such a vibrant beginning! When we read of how, in 1675, long before any formal notion of probability, the Dutch mathematician Christiaan Huygens had expressed the expected gain from a game of chance by "$[na + (N - n)b]/N$," it touched an important chord in each of us. It was not that we were gamblers, or ever would be gamblers. It was that this wee bit of mathematics had a certain quality that transcended its "use." Like all mathematics that I call "good" mathematics, it called out to us to explore the consequences, to ask questions, to extend, to generalize. We did. We drew a shiny coin out of a velvet sack, selected at random from three such, one of which had two gold coins, one a gold and a silver, and the third two silvers. We saw that it was gold, and were asked to determine the probability that the other was silver. That the answer was not $1/2$ surprised some more than others, but the resulting explanation, and Bayes' Theorem, delighted us all.

And yet, by the end of the first semester, we found ourselves raising peafowl and measuring salmon and watching Johnny take an IQ test, and all the while feeling much like bits of flotsam caught in a tidepool, motionless and somehow cut off from that main body which had earlier propelled us with such excitement. The difficulty was a pedagogical one, to be sure, but just where did it all go wrong?

To provide an answer we need only take cursory glance at the standard course outline then and now. I speak here of the calculus-based introduction to probability and statistics found in most universities as a third or fourth year undergraduate course for mathematics majors or as a first year graduate course in a related discipline such as computer science, econometrics, or operations research. It goes (and went):

1. relative frequency
2. axioms for a probability space
3. direct consequences of the axioms
4. the distributions of Bernoulli, Poisson, Erlang and other assorted curios
5. the Gaussian distribution
6. the central limit theorem
7. applications of the central limit theorem
8. applications of the central limit theorem

⋮

N. applications of the central limit theorem

So my opinion of the problem is evident.

Of course it would be ludicrous to contend that points 5–N represent "bad" mathematics, for nothing could be further from the truth. The issue is rather one of emphasis. We need to examine just why points 5–N have become a pedagogical sink and what is to be done, where the emphasis should instead be placed.

First, there is a certain delightful comfort in the central limit theorem. It allows us, as mathematicians, to take unassailable positions on the nature and worth of our predictions, all the while discussing a multitude of events from a world over which we have embarrassingly little control. Thus we predict the collective performance of our students with high certainty, acknowledging the existence of a contribution from great-grandmother's uncanny ability to remember recipes, but otherwise ignoring the explicit nature of such contributions, for there are far too many to consider. I contend that it is just this comfort that has led to some pedagogical complacency. "Here, in one neat package, is all you need to know."

As a second factor which has forced us in the current direction, but might also provide our salvation, I nominate a surprise candidate: the foundational work of Norbert Wiener on Brownian motion. Keep in mind that, upon first glance, two aspects of Wiener's original work stand out:

(1) the heavy involvement of the Gaussian distribution,
(2) the formalization of the notion of stochastic process.

Now it is not surprising that this cornerstone of modern theory impacts pedagogy. Nor is it surprising that aspect (1) has received the major pedagogical emphasis (we can teach this well established stuff to undergraduates!), while aspect (2) has not (this new stuff we can't!). The surprise is that we continue to ignore a lesson which apparently has been easily learned in other branches of mathematics: one of the choicest fruits of research is the heightened accessibility of previous work! We *can* teach stochastic processes to undergraduates and first-year graduates, and we can do it in a first course. Further, doing so will offer the course a much-needed breath of new life.

We should begin gently, with the exponential distribution, $F_X(t) = 1 - e^{-\lambda t}$. This is the basic building block, whose full consideration and exploitation will return us from the role of observer to that of participant. A fundamental theorem due to D. R. Cox tells us that any continuous random variable having rational Laplace transform can be decomposed into a series-parallel cascade of exponentially distributed random variables. When we combine this result with the "memoryless property" of the exponential, $P(X - s \leqslant t | X > s) = P(X \leqslant t)$, which has obvious implications for state-space classification, and the plethora of results on approximation by rationals, we find ourselves in the midst of a wonderful wrangle with the properties of the system under study. We are no longer "observing" a random variable with a yawn, but rather "grabbing it by the throat."

Consideration of an independent sequence of such random variables gives us the general renewal process, and, regarding successive values as inter-event times, we obtain the general counting process, the simplest of which is, of course, the all-important Poisson.

I am not hereby claiming that the exponential distribution and the theorem of Cox should replace the Gaussian distribution and the central limit theorem on the tips of undergraduate tongues. We have, rather, a new player in a different role, and it is time to share the spotlight.

Trivedi's text offers precisely the pedagogical course correction of which I speak. After the delightful, but now standard, preliminaries of my points 1–3, it launches into two full chapters on expectation (in the spirit of Huygens!), followed by the much-needed introduction to stochastic processes. Other vital concepts are then developed within this context. It is loaded with examples and problems which touch us all, from why the telephones never work to what performance we might expect from the computer program we just wrote.

I have taught Probability and Statistics I and II from this text and from many others, and, in each case, student enthusiasm waxes strong in the initial phases of the course. With the Trivedi text, it never wanes.

The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors. By Colin Sparrow. Applied Mathematical Sciences, Number 41. Springer-Verlag, New York, 1982. xii + 269 pp.

JOHN GUCKENHEIMER
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Colin Sparrow has written an entire book about the system of ordinary differential equations

$$\frac{dx}{dt} = \sigma(y - x)$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

which are called the *Lorenz equations*. Edward Lorenz is a distinguished meterologist who studied this system of equations in the early 1960's and discovered some of their remarkable properties. This book provides a comprehensive collection of most of the known theory that has been developed since that time for the Lorenz system. It is a study that is of interest not only for its own sake but also stands as an example of a style of mathematics which is likely to become much more common as widespread computational abilities permeate mathematics.

The basis for Sparrow's book is data obtained from numerical solutions of the Lorenz system. Finding numerical solutions is routine: the equations are not stiff for moderate values of the parameters (σ, r, b) and integration for modest periods of time using standard algorithms gives accurate answers. Still, there is little to be found that is rigorously proved for the Lorenz system. Instead, the powerful geometric intuition developed by watching solutions plotted on a computer graphics terminal is translated into theory about geometrically defined systems that *appear* to have the properties of the Lorenz system. The theory of the Lorenz system has developed with little attention to issues of numerical analysis that would give rise to rigorous results about the Lorenz system. Some historical background is in order before we discuss these issues of rigor further.

The Lorenz system is not arbitrary. It was developed from the Boussinesq equations which describe fluid flow in a convecting layer. The Rayleigh-Bénard experiment of convection between two horizontal plates held at (different) fixed temperatures has a long history that provides a physical backdrop to the mathematics. To obtain the Lorenz system, one expands the Boussinesq equations (with appropriate boundary conditions) in terms of the trigonometric normal modes of the conducting solution (no motion, linear temperature gradient). The Lorenz system is a minimal truncation of the evolution equations for the normal modes which contains the information necessary to analyze the qualitative behavior which occurs as the conducting solution loses stability. The parameter r in the equations represents Rayleigh number: $r = 1$ is the critical Rayleigh number at which the origin loses its stability for the Lorenz system.

In the late 1950's as computers began to appear in universities, it was soon noticed that the solutions of the Lorenz system can behave erratically for values of r much larger than the range in which the system is a good model for the Boussinesq equations. Lorenz pursued the matter systematically, motivated by the desire to explore the inherent limitations of numerical weather prediction. Simultaneously with the work of Lorenz, Smale embarked on the enterprise of extending the theory of structural stability for systems of differential equations to dimensions larger than two. His starting point was the theory of geometric "phase-plane" methods (originating with Poincaré) which culminated around 1960 with Peixoto's theorem proving that structurally stable vector fields are dense on compact, orientable two-dimensional manifolds. An early product

Partially supported by the National Science Foundation.

of Smale's efforts was the *horseshoe*, a geometrically defined example which gives a structurally stable three dimensional flow having the same type of aperiodic trajectories observed numerically by Lorenz.

The recent history of dynamical systems theory might have been substantially different had Smale and Lorenz been aware of each other's work. The "Smale school" of dynamical systems theory was born in the mid 60's and developed an extensive theory of structural stability. The examples studied were usually defined algebraically in terms of Lie groups or geometrically in terms of pictures. Only by the mid 70's was there any effort to turn to the "real" world of systems defined in terms of formulas, although work of Cartwright, Littlewood and Levinson on the van der Pol equation provided inspiration to Smale in his initial efforts that led to the horseshoe. At this point Lorenz's work was "rediscovered." (I believe Jim Yorke was responsible for calling it to the attention of Smale and his students.) It was quickly discovered that the Lorenz attractor did not fit neatly into the class of objects dealt with by the Smale school. The Lorenz attractor pointed to new phenomena distinguishing discrete and continuous flows related to the presence of equilibrium points in an attractor of a continuous flow.

The outcome of these events has been some interesting mathematics. The first steps toward understanding the Lorenz attractor was the geometric abstraction of properties from which one could deduce its *apparent* structure. As we noted above, the numerical estimates required to verify that the hypotheses of the geometric models are satisfied by the Lorenz system have not been carried out. This remains an interesting area for further mathematical work in which the issues appear to be somewhat different from those of classical numerical analysis. Let us pose the following fundamental question: with numerical computations having a specified precision and with an algorithm of your choice, how accurately can one integrate a given system of differential equations for one unit of time? The problem involves a balance between the compounding of round-off errors with the need for small step sizes in making the algorithm accurate.

One of the features of the Lorenz attractor is the presence of an infinite number of periodic orbits inside the attractor. Sparrow places heavy emphasis upon efforts to numerically find periodic orbits and classify them. A surprising recent discovery (not discussed by Sparrow) by Williams is that these periodic orbits are generally knotted and linked with each other. Work of Birman, Franks and Williams is developing these topological aspects of the theory and enriching knot theory in the process.

This history of the Lorenz system illustrates, in my opinion, three features of contemporary mathematics. The first is the importance of a systematic and thorough study of special examples which can motivate the development of more general theory. Ten years ago, I don't think anyone could have imagined how much there was to be learned from a study of this system of differential equations or from a detailed study of the iteration of quadratic functions (a related problem). The ramifications for "pure" mathematics have been astonishing. The second feature illustrated by this work is the importance of the computer as a mathematical tool. The Lorenz system cannot be solved accurately with hand calculation. The results of numerical calculations are compelling even if they do not yet form the direct basis of a rigorous theory. The third feature is the *necessity* for having general mathematical theory that can guide numerical investigations and the interpretation of its results. The interaction between computation and theory is a dialogue; without theory the computations are capable of producing reams of indigestible information that is virtually useless. Sparrow's book is a very successful case study that illustrates this dialogue between mathematics and machine.

LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

Editor:

I was surprised to see the note [**5**] by Liu Wen, less because it is merely a less elegant version of Schoenberg's construction [**6**] than because neither the referee nor the editor noticed this. I can easily understand that the author was unfamiliar with recent literature, but referees are supposed to be more knowledgeable. That the author's only reference was to a book [**4**] published in 1927 should have been a warning. It is true enough that many particular proofs are rediscovered every few decades, but usually only when they have failed to find a home in textbooks.

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Editor:

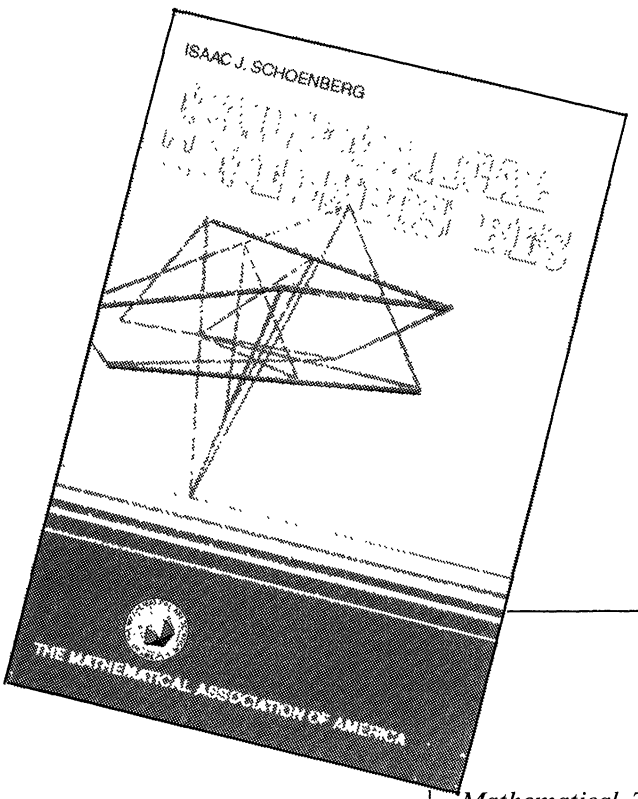
Many educated laymen believe that mathematics is a field in which no research is done because everything is already known, or that if research is done, no creativity is involved, or even that mathematicians hate creativity and spend their time on rote memorization of formulas. In the hope of clearing up such misconceptions, I am collecting a rare species of popular articles on mathematics: articles about mathematical research that was recent at the time of publication, written in the popular press by journalists for an audience of laymen. I hope that reading such articles could clear up misconceptions, and could be useful in teaching "mathematics appreciation." An anthology of such articles might also influence the quantity and quality of mathematical journalism in the future.

Here are some examples of the kind of article I mean: the article in *Newsweek* on 1st August 1983 about Gerd Faltings' proof of Mordell's conjecture; the article in the *New York Times* on 22nd June 1980, 4th section, p. 11, about the classification of finite simple groups; the article in *Time* on 30th April 1945, about Hans Rademacher's erroneous refutation of the Riemann hypothesis.

I would appreciate hearing of others from readers of the MONTHLY.

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The author manages to bring together concepts from geometry, number theory, algebra, and analysis, frequently mixing them together in the same chapter. The arts are not neglected. Discussions on the tuning of keyboard instruments, the guitar, and the vibrations of strings are discussed, as well as the suggestion of rectilinear models for outdoor sculpture.

Many of the chapters in this book may be read independently, and at least half of them with only a knowledge of precalculus mathematics.

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P. R. HALMOS, *Editor*

FACTORIZATION AND PRIMALITY TESTS

JOHN D. DIXON
Department of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada

1. Introduction. Although unique factorization for integers is sometimes called the "fundamental theorem of arithmetic," it is only occasionally that a student learns anything about the constructive aspects of this theorem beyond the most elementary facts. Yet there are interesting unsolved mathematical and computational problems involved in factorization of integers and tests of primality, and many of the ideas involved are accessible to undergraduate students. As L. E. Dickson's "History of the Theory of Numbers" [**12**] shows, these problems have attracted the interest of some of the great mathematicians of the past such as Fermat, Euler, Legendre and Gauss, as well as numerous less well-known names; and much of what has been done recently has its roots in this early work. The general availability of computers has had a positive influence on this field, leading to the development of algorithms which previously would not have been feasible, and raising new questions of a theoretical nature concerning the complexity of the underlying problems. The study of new methods of factorization also has taken on an "applied" flavor since the proposal of [**50**] of a form of public-key cryptosystem whose security lies in the assumption that some large integers are hard to factor; an efficient factoring method would enable an opponent to break the system.

John Dixon completed his bachelor's degree (1958) and master's degree (1959) in number theory at the University of Melbourne, Australia. He received his Ph.D. (1961) in group theory under the supervision of Hans Schwerdtfeger from McGill University. Since then he has taught at California Institute of Technology, University of New South Wales, and Carleton University. He is the author of three books: *Problems in Group Theory* (1967), *The Structure of Linear Groups* (1971), and *Modular Representations of Finite Groups* (with B. M. Puttaswamaiah, 1976). During the past 15 years he has had an increasing interest in algorithmic problems in algebra and number theory. His mathematical heroes are the generalists such as Gauss and Hilbert.

In the survey which follows my aim is to describe the problems and the progress which has been made. Most of the survey will refer to work published in the last 10 years, but the reader should be aware that the history of the area goes back much further and that, frequently, methods have been rediscovered in slightly variant forms. In some cases it is difficult to be sure to what extent an idea is new or how much it is implicit in earlier (sometimes much earlier) work.

2. The Ring $\mathbf{Z}_n$. Let n be the integer in whose factorization we are interested. To avoid uninteresting special cases we shall always assume that n is odd and greater than 1, and suppose that its canonical prime factorization is

$$n = \prod_{i=1}^{s} r_i^{k_i} \qquad (r_i \text{ distinct primes}, k_i \geqslant 1). \tag{1}$$

(In what follows, p, q and r, with or without subscripts, will always designate primes.) We now recall some basic number theoretic facts.

Let $\mathbf{Z}_n$ denote the ring of integers modulo n, and let $x \mapsto x \bmod n$ be the canonical homomorphism of $\mathbf{Z}$ onto $\mathbf{Z}_n$. If d divides n (notation $d|n$), then there is a natural ring homomorphism of $\mathbf{Z}_n$ onto $\mathbf{Z}_d$ which (with abuse of notation) we also write $x \mapsto x \bmod d$. The ring homomorphism

$$\mathbf{Z}_n \to \prod_{i=1}^{s} \mathbf{Z}_{n_i} \qquad \text{(direct product of rings)} \tag{2}$$

given by $x \mapsto (x \bmod n_i)_i$, where $n_i = r_i^{k_i}$ for $i = 1,\ldots,s$, clearly has kernel 0. By counting the number of elements on the two sides we see that the mapping is also surjective, and so (2) is an isomorphism. This is a version of the *Chinese Remainder Theorem* which is often stated in the form: If $m_1,\ldots,m_t$ are integers which are relatively prime in pairs, and their product is m, then for all integers $x_1,\ldots,x_t$ there is a unique integer $x \in [0, m-1]$ such that $x \equiv x_i (\bmod m_i)$ for each i. From the computational point of view it is important to know how to construct x from the values of the x_i. This can be done using the Euclidean algorithm, and an efficient method is described in [**21**, §4.3.2].

The units (elements with multiplicative inverses) in $\mathbf{Z}_n$ are exactly the elements $x \bmod n$ for which the greatest common divisor $\mathrm{GCD}(n, x)$ equals 1. The units form a multiplicative group which we shall denote by U_n, and its order is $\phi(n)$, the Euler phi-function. From the isomorphism (2) we obtain the group isomorphism

$$U_n \simeq \prod_{i=1}^{s} U_{n_i}. \tag{3}$$

(Note: $\phi(n) = \prod_i \phi(n_i) = n\prod_i(1 - (1/r_i))$.)

For an odd prime r it can be proved that for each $k \geqslant 1$ the group U_{r^k} is cyclic. An integer x for which $x \bmod r^k$ is a generator of this group is called a *primitive root modulo* r^k (see, for example, [**60**, Chap. VI]). Since we are assuming that n is odd, (3) shows that U_n can be generated by a set of s elements.

Exercise 1. Show that U_n cannot be generated by fewer than s elements.

The following elementary property that holds for groups in general will be used repeatedly in what follows. Suppose that x is an element of finite order h in a group G, and that m is a positive integer. Then:

(4) If $x^m = 1$ and $x^{m/q} \neq 1$ for some prime $q|m$, then the largest power of q dividing m also divides h; moreover, if $x^{m/q} \neq 1$ for *all* primes $q|m$, then $m = h$.

Finally, we recall some basic facts about quadratic residues. Let a be an integer and p be a prime with $p \nmid a$. Then a is a *quadratic residue* for p if for some integer c, $c^2 \equiv a (\bmod p)$; otherwise a is a *quadratic nonresidue*. This is summarized by the *Legendre symbol* $\left(\frac{a}{p}\right)$ which (for

$p \nmid a$) takes the values 1 or -1 depending on whether a is a quadratic residue or not. Euler proved that $\left(\frac{a}{p}\right) \equiv a^{(p-1)/2} (\mathrm{mod}\, p)$. It is useful to introduce the more general *Jacobi symbol* $\left(\frac{a}{n}\right)$. This is defined for all relatively prime a and n with n odd and greater than 1. If n has the prime factorization (1), then $\left(\frac{a}{n}\right)$ is defined in terms of the Legendre symbol as the product $\Pi_i \left(\frac{a}{r_i}\right)^{k_i}$. Again, the Jacobi symbol takes the values 1 and -1, and it is readily seen that $\left(\frac{a}{n}\right) = \left(\frac{b}{n}\right)$ whenever $a \equiv b (\mathrm{mod}\, n)$, and that $\left(\frac{ac}{n}\right) = \left(\frac{a}{n}\right)\left(\frac{c}{n}\right)$ whenever $\mathrm{GCD}(ac, n) = 1$. Note, however, that $\left(\frac{a}{n}\right) = 1$ does not in general imply that $a \bmod n$ is a square. The important theorem which is basic to many results on quadratic residues is the *Quadratic Reciprocity Law*. It states that if a and n are both odd and relatively prime, then:

$$\left(\frac{-1}{n}\right) = (-1)^{(n-1)/2}, \qquad \left(\frac{2}{n}\right) = (-1)^{(n^2-1)/8},$$

and

$$\left(\frac{a}{n}\right) = (-1)^{(a-1)(n-1)/4}\left(\frac{n}{a}\right).$$

Proofs may be found, for example, in [**60**, Chap. V].

3. Basic Computational Operations. We shall want to analyze the time that it will take to execute various algorithms. The internal hardware of most computers will perform the basic arithmetic operations on integers of some fixed size (single precision), frequently up to about 10^{10}, but operations on larger integers must be handled by a multiprecision package (perhaps written by the programmer). An extensive discussion of such a package is given in [**21**, §4.3.1]. We only observe here that multiprecision operations for integers about the size of n will be slower than single precision operations by factors of up to $c(\log n)^2$ for some constant c. As usual we express this by saying that these multiprecision operations require $O(\log n)^2$ single precision operations.* Among these operations we include addition, subtraction, multiplication and integer division with quotient and remainder, so ring operations in $\mathbf{Z}_n$ will take a comparable time. It is perhaps worthwhile noting that the time taken by a computer to carry out a single precision operation may vary from around 10^{-3} seconds for a small microcomputer, through about 10^{-6} seconds of central processing time for a medium-sized computer, to perhaps 10^{-10} seconds for the fastest computers.

As well as these basic arithmetic operations there are two other important basic number theoretic algorithms (see [**21**] and [**28**]). The first of these is the *Extended Euclidean Algorithm* which, for integers a and b not both 0, computes $d = \mathrm{GCD}(a, b)$ together with integers u and v such that $au + bv = d$. In particular, this permits us to compute the multiplicative inverse of a unit in $\mathbf{Z}_n$. If a and b are bounded in size by n, then the number of multiprecision operations on numbers of this size which are required to execute this algorithm is bounded by a multiple of $\log n$ (see [**21**, §4.5.2]). Almost the same method allows us to compute the Jacobi symbol via the reciprocity law.

The second important algorithm is the *Power Algorithm*. It is useful to describe this in a more general situation. Let R be a ring and suppose that we want to compute x^m where $x \in R$ and m is a positive integer. If we write m in terms of its binary expansion,

$$m = m_0 + m_1 2^1 + \cdots + m_t 2^t$$

with each $m_i = 0$ or 1, then we can compute $z = x^m$ using at most $2t$ multiplications as follows.

*A real-valued function g is said to be of order f at infinity (written $g(n) = O(f(n))$, if $|g(n)|$ is bounded by some constant multiple of $f(n)$ as $n \to \infty$.

Compute $x_0 = x$ and $x_i = x_{i-1}^2$ for $i = 1, \ldots, t$; then z is the product of all x_i for which $m_i = 1$ (the calculations can be carried out without intermediate storage of the m_i and x_i). The algorithm is evidently related to the curiosity which is sometimes referred to as the Russian peasant method of multiplication using successive doubling; and according to [**21**, §4.6.3] the power algorithm appears in Hindu manuscripts which predate 200 BC. In particular, this algorithm permits us to compute x^m in $\mathbf{Z}_n$ using $O(\log m)$ multiprecision operations on integers the size of n. The power algorithm is also useful for computations in algebraic number fields, polynomial rings, matrix rings, and (using matrices) for computing high order terms in sequences given by linear recurrence relations with constant coefficients.

4. Probabilistic Algorithms. A (deterministic) algorithm A accepts as input an element u from a set I of inputs and, after some finite time $t(u)$, produces an output $A(u)$ lying in a specified set of possible outputs. A probabilistic algorithm is a generalization of this concept. A *probabilistic algorithm* P accepts an input u from a set I of inputs together with an element ω selected from a set Ω_u according to a specified probability distribution. Then, with probability 1, the algorithm will, after some finite time $t(u, \omega)$, produce an output $P(u, \omega)$; and moreover the average $t(u)$ of the times $t(u, \omega)$ over Ω_u is finite. If the algorithm is to be useful, then the output $P(u, \omega)$ should have some property independent of ω, and the elements of Ω_u should be easy to generate with the correct distribution. Probabilistic algorithms have been used informally for many years, but the first formal description (which is slightly different from the one we have given) seems to have been given by M. O. Rabin in [**47**]. The advantage of a formal definition is that it allows us to carry out a rigorous analysis of such algorithms. Note that for a fixed input u, and given $\lambda > 1$, the probability that $t(u, \omega)$ exceeds $\lambda t(u)$ is less than $1/\lambda$; hence an estimate of $t(u)$ gives a good idea of how long the algorithm P is likely to run. For almost all practical purposes (excluding certain real-time situations where a strictly bounded running time is vital), a probabilistic algorithm is as satisfactory as a deterministic algorithm with a comparable running time, and frequently the former is more amenable to analysis.

In the situations which we consider below Ω_u will be the set of all sequences of integers from a fixed finite interval. Although ω is an infinite sequence, only the first few terms are computed, each term being selected independently and uniformly at random from a given finite set (this defines the probability distribution on Ω_u). Actually, in practice, we only approximate this procedure since we are likely to use a pseudo-random number generator rather than make truly random choices (see [**21**, Chap. 3]).

EXAMPLE. No deterministic algorithm for finding a quadratic nonresidue for a prime p is known to have a running time bounded by a power of $\log p$ (compare [**9**], but see Lemma 2 below). However, it is easy to *recognize* when an integer a is a quadratic nonresidue by either computing the Legendre symbol using the quadratic reciprocity law or using Euler's criterion ($a^{(p-1)/2} \equiv -1 (\mathrm{mod}\ p)$). Since, for $p > 2$, half of the integers in $[1, p-1]$ are nonresidues, we can find a quadratic nonresidue for p by a probabilistic algorithm simply by trying successive values of a chosen independently and uniformly at random from $[1, p-1]$ until one satisfies the condition. It is easy to see that on the average only two values of a will be tried, so the average number of single precision operations needed to find a quadratic nonresidue of p is $O(\log p)^3$. Note that the output (the nonresidue) depends on the particular random sequence.

REMARK. We should probably not use this algorithm in practice since for many special classes of primes there are easier ways to find quadratic nonresidues. For example, the quadratic reciprocity law shows that we can always take $a = -1$ if $p \equiv 3 (\mathrm{mod}\ 4)$, $a = 2$ if $p \equiv 5 (\mathrm{mod}\ 8)$, and $a = 3$ if $p \equiv 17 (\mathrm{mod}\ 24)$. This covers all odd primes $p \not\equiv 1 (\mathrm{mod}\ 24)$, and the rules can be extended in an obvious way.

In the problems of factorization and primality testing the size of the integer n is measured by $\log n$ which is roughly proportional to the number of digits of n. One of the principal theoretical

questions is whether there are *polynomial-time* algorithms to solve either of these problems. In other words, are there (perhaps probabilistic) algorithms which for each odd integer $n > 1$ can (i) find a proper factor of n if n is composite, or (ii) find a proof that n is prime if n is prime, such that the execution times are bounded by some power of $\log n$. As we see below, these questions are open problems. The evidence that we have suggests that there may be a polynomial-time algorithm for (ii), but perhaps not one for (i). At the same time there are the practical problems of finding and implementing algorithms to solve these problems for general values of n up to a certain size. At the time of writing, composite numbers of around 50 digits are routinely factored and proofs of primality found for primes up to 200 digits.

Exercise 2. Let p and q be primes and suppose that $q - 1 = p^t m$ where $t \geqslant 1$ and $p \nmid m$. Describe a probabilistic algorithm to find an element $b \in U_q$ of order p^t. Use this to give an efficient algorithm which, for each $a \in U_q$, decides whether $x^p = a$ has a solution $x \in U_q$ and if so finds such an x. [Hint: a is a pth power in U_q if and only if a^m has order dividing p^{t-1}. Show that in the latter case x can be written in the form $a^{mi}b^j$ (compare with [**28**, p. 133] and [**56**]).]

Exercise 3. If g is a primitive root modulo p for an odd prime p, and $g^{p-1} \not\equiv 1 \pmod{p^2}$, show that g is also a primitive root modulo p^k for all $k > 1$. Note that at least one of g^{p-1} and $(g + p)^{p-1}$ is not congruent to 1 modulo p^2, and so either g or $g + p$, is a primitive root modulo p^k.

Exercise 4. Suppose that p is a prime and that $p - 1$ can be completely factored. Describe an efficient probabilistic algorithm to find a primitive root modulo p^k for all $k \geqslant 1$.

5. Certificates of Primality. A classical algorithm which both tests n for primality and produces a proper factor of n if n is composite is based on the fact that either n is prime or it has a prime factor $r \leqslant \sqrt{n}$. However, the prime number theorem shows that the number of primes less than $\sqrt{n}$ tends asymptotically to $2\sqrt{n}/\log n$, so it is out of the question to check this criterion directly once n becomes reasonably large. Indeed, you might like to consider the following problem (using whatever computing resources you have available). It was posed in a well-known puzzle book [**15**] published in 1907.

Exercise 5. Is the number $n = 111\ldots1$ (19 ones) prime?

As far back as 1877 E. Lucas and T. Pepin showed, in special cases, how we can prove primality indirectly (see [**12**, p. 376]). The basic criterion follows from (4) and the fact that U_n is cyclic of order $n - 1$ if and only if n is prime. In this simple form the criterion is useful only when $n - 1$ can be completely factored:

(5) n is prime if and only if for some integer a, $a^{n-1} \equiv 1 \pmod{n}$ and $a^{(n-1)/q} \not\equiv 1 \pmod{n}$ for each prime $q | n - 1$.

Exercise 6 (T. Pepin 1877). Let n be the Fermat number $F_k = 2^{2^k} + 1$ ($k \geqslant 2$). If n is prime show that 5 is a quadratic nonresidue for n, and hence (in this case!) that 5 is a primitive root. Deduce that n is a prime if and only if $5^{(n-1)/2} \equiv -1 \pmod{n}$.

Exercise 7. Show that an odd integer $n > 1$ is prime if and only if for each prime $q | n - 1$ there exists an integer a_q such that $a_q^{n-1} \equiv 1 \pmod{n}$ but $a_q^{(n-1)/q} \not\equiv 1 \pmod{n}$.

Exercise 8 (H. C. Pocklington 1914). Suppose that $n - 1 = ml$, and that for each prime $q | m$ there exists an integer a_q such that $a_q^{n-1} \equiv 1 \pmod{n}$ and $\mathrm{GCD}(a_q^{(n-1)/q} - 1, n) = 1$. Show that each prime $r | n$ satisfies $r \equiv 1 \pmod{m}$. Thus, if $m \geqslant l$, then n is prime.

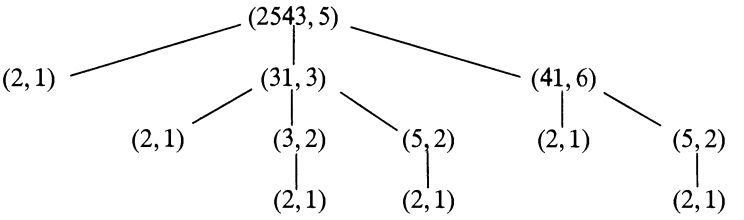
Exercise 9 (Analogue of (5)). Let $f(X)$ be a monic polynomial over a field $\mathbf{Z}_p$. Show that $f(X)$ is a product of distinct monic irreducible polynomials in $\mathbf{Z}_p[X]$ all of whose degrees divide a given positive integer d if and only if $f(X) | X^{p^d} - X$. Hence show that a monic polynomial $f(X)$ of

degree d is irreducible if and only if $X^{p^d} \equiv X \pmod{f(X)}$ but $X^{p^{d/q}} \not\equiv X \pmod{f(X)}$ for each prime $q|d$. (The latter criterion may be checked in $O(d \log p)^3$ single precision operations using the power algorithm in the ring $\mathbf{Z}_p[X]/(f(X))$. Since approximately $1/d$ of the monic polynomials of degree d in $\mathbf{Z}_p[X]$ are irreducible, the criterion can be used in a probabilistic algorithm to construct an irreducible polynomial of specified degree over $\mathbf{Z}_p$. See also [**10**], [**19**] and [**32**].)

If n is composite, then a short proof of this may be given by simply writing n as a product of two proper factors and checking the multiplication. E. T. Bell [**5**] recounts an anecdote in which F. N. Cole did precisely this for the Mersenne number $M_{67} = 2^{67} - 1$ at a meeting of the American Mathematical Society in 1903. Such a proof may be verified by a computation involving $O(\log n)^2$ single precision operations. The catch, of course, is to find the proof—a common difficulty in mathematics.

In the case that n is prime, it is not so obvious that a similarly short proof of primality exists, although proofs in special cases were already given in the last century. However, in 1975 V. P. Pratt [**46**] observed that (5) implies that "every prime has a succinct certificate." Again this proof may be difficult to find, but once written down its validity can be checked easily (Pratt showed that it can be checked in $O(\log n)^5$ single precision operations). The proof can be written as a finite tree whose vertices are labeled by pairs (p, g_p), where p is a prime and g_p is a primitive root modulo p. The root of the tree is labeled (n, g_n), and each vertex labeled (p, g_p) with $p > 2$ has as its children vertices with labels (q, g_q) as q ranges over the primes dividing $p - 1$. The leaves of the tree are all labeled $(2, 1)$.

EXAMPLE*. A certificate of primality for $n = 2543$ is given by



The proof should be checked as follows. For each vertex (k, g_k) with $k > 2$, we should have: (i) $g_k^{k-1} \equiv 1 \pmod{k}$, (ii) $g_k^{(k-1)/l} \not\equiv 1 \pmod{k}$ whenever (l, g_l) is the label of a child of the vertex (k, g_k), and (iii) k is the product of the l (taken to suitable powers) as (l, g_l) runs over the labels of the children of (k, g_k). If these conditions hold, then starting from the leaves $(2, 1)$ the criterion (5) shows successively that the label of every vertex consists of a prime together with an associated primitive root. In particular, n is prime.

R. P. Brent [**7**] has written down proof trees like this for prime factors of the Fermat numbers F_6, F_7 and F_8, but it is not easy to find such proofs since it involves complete factorizations of $p - 1$ for all the primes appearing in the tree. It has been pointed out in [**37**] that, on the other hand, it is rather easy to construct artificially very large primes together with their proof trees by starting out at the leaves of the tree and working up to the root.

The question remains: how do you *find* a proof of primality? Until quite recently, this seemed only a little easier than the factorization problem. Results such as Exercise 8 show that a complete factorization of $n - 1$ can be avoided. E. Lucas showed that in some cases it is possible to use a factorization of $n + 1$ in place of $n - 1$. His idea was generalized by D. H. Lehmer (see [**28**, p. 128]), and more recently it has been shown that partial factorizations of $n^2 \pm 1$ and $n^2 \pm n + 1$ can all be combined to prove primality of n. An extensive survey of these methods is given by H. C. Williams in [**61**]. It now appears, however, that except for special classes of numbers such as the Mersenne numbers, the recent algorithm of L. M. Adleman, R. S. Rumely and C. Pomerance

*As is customary, we have drawn our tree "upside down" with its leaves at the bottom.

is superior on both theoretical and practical grounds for finding proofs of primality. We shall deal with this algorithm in Section 13.

Note: As a curiosity we note that, as at time of writing, the largest known prime is the 39,751-digit Mersenne number $M_{132049} = 2^{132049} - 1$. It was proved prime by David Slowinski in September 1983, using the fact that in this case $n + 1$ is a power of 2.

6. Pseudoprimes and Proofs of Compositeness. To many people it comes as a surprise that it is often easy to show that an integer is composite without having any idea what proper factors it might have. If n is prime, then $|U_n| = n - 1$, and so (as Fermat knew)

$$b^{n-1} \equiv 1 \pmod{n} \tag{6}$$

holds whenever n is prime and $n \nmid b$. In general, if (6) holds, then we say that n passes the *pseudoprime test to base b*; if, in addition, n is composite, then we call n a *pseudoprime to base b*. (Some people say when (6) holds that n is a "probable prime to base b.") Thus, if for some $b \in [1, n - 1]$, n does not pass the pseudoprime test to base b, then we have a proof that n is composite. This is a simple and often effective way to prove compositeness.

Historically, there was some confusion about the converse: if (6) holds for some b and n, can we deduce that n is prime? It is reported in [**12**, p. 91] that at one stage Leibniz believed that any integer n passing the pseudoprime test to base 2 is prime, and perhaps the fact that each Fermat number $F_k = 2^{2^k} + 1$ passes the pseudoprime test to base 2 misled Fermat into believing that all these numbers are prime (actually numerous Fermat numbers are now known to be composite, but no prime beyond F_4 has been found). The fact is that for each $b > 1$ there are n which are pseudoprime to base b, and it is even true that for some n, n is pseudoprime to base b for all integers b relatively prime to n. On the other hand, for a fixed $b > 1$, the number of pseudoprimes to base b is very small compared with the number of primes (see [**16**] and [**42**]).

Exercise 10. Show that F_5 is composite by proving that it does not pass the pseudoprime test to base 3.

Exercise 11 (R. D. Carmichael 1912). Suppose that n is composite with the factorization (1). Show that (6) holds for all b relatively prime to n if and only if for each i we have $k_i = 1$ and $r_i - 1 | n - 1$. (The latter condition implies that $s \geqslant 3$.) Find examples of such "Carmichael numbers." (D. Shanks notes that if $p > 3$, $2p - 1$ and $3p - 2$ are all primes, then their product is a Carmichael number.)

The existence of Carmichael numbers shows that in some cases it will be difficult to find proofs of compositeness using a pseudoprime test of the type (6). A slightly stronger form of the test does much better. Write $n - 1 = 2^h m$ with m odd. If n is prime, then U_n is cyclic and so has a unique subgroup of order 2 which is generated by -1. Also, for each $x \in U_n$, the order of x divides $n - 1$ and so the order of x^m divides 2^h; hence either $x^m = 1$ or, for some $i \in [0, h - 1]$, x^{m2^i} has order 2 and so equals -1. Thus the condition

$$\text{either } b^m \equiv 1 \pmod{n} \text{ or } b^{m2^i} \equiv -1 \pmod{n} \text{ for some } i \in [0, h - 1] \tag{7}$$

holds whenever n is prime and $n \nmid b$. Clearly (7) implies (6), and if (7) holds then we shall say that n passes the *strong pseudoprime test to base b*; if, in addition, n is composite, then we say n is a *strong pseudoprime to base b*. Note that using the power algorithm to check (7) will take approximately the same amount of work as checking (6), even though the former test is stricter. As we see in the theorem below there are no analogues of the Carmichael numbers for the test (7), and the power of the strong pseudoprime test is illustrated by computations of [**44**] which show that every odd $n < 25 \cdot 10^9$ which satisfies (7) for $b = 2$, 3, 5, 7 and 11 is prime. The proof of Theorem 1 is based on the following result about the structure of U_n.

LEMMA 1. *Let n be an odd integer > 1, and let B consist of all elements $u \in U_n$ such that the cyclic subgroup $\langle u \rangle$ either has odd order or contains -1. If B generates U_n, then n is a prime power.*

Proof. Assume that n has the factorization (1). Then (3) shows that each element of U_n has order dividing the least common multiple

$$\mathrm{LCM}\{\phi(n_i) | i = 1, \ldots, s\} = 2^t l,$$

say, where l is odd. Choose j so that 2^t divides $\phi(n_j)$. We consider the group homomorphism $\psi : U_n \to U_n$ defined by $\psi(x) = x^{2^{t-1}l}$. For each $x \in B$, $\psi(x) \in \langle x \rangle$ and $\psi(x)^2 = 1$; thus, either $\psi(x) = 1$ or $\psi(x)$ is the (unique) element of order 2 in $\langle x \rangle$, namely -1. If B generates U_n, then $\psi(x) = \pm 1$ for all $x \in U_n$. On the other hand, if n were not a prime power, then $s > 1$ in (3), and so U_n contains an element v such that $v \bmod n_j$ has order 2^t in U_{n_j} and $v \bmod n_i = 1$ for all $i \neq j$; evidently $\psi(v) \neq \pm 1$. Thus we conclude that if B generates U_n, then $s = 1$ and n is a prime power as asserted.

THEOREM 1. (a) *Suppose that $n - 1 = 2^h m$ with m odd, and set*

$$B' = \left\{ x \in U_n | x^m = 1 \text{ or } x^{m2^i} = -1 \text{ for some } i \in [0, h-1] \right\}.$$

If B' generates U_n, then n is prime.

(b) (M. O. Rabin [47]). *If n is composite, then (7) fails to hold for at least half of the integers $b \in [1, n-1]$.*

Proof. (a) With the notation of Lemma 1 we have $B' \subseteq B$ and $x^{n-1} = 1$ for all $x \in B'$. Thus, if B' generates U_n, then Lemma 1 shows that $n = r^k$ for some prime r, and $x^{n-1} = 1$ for all $x \in U_n$. But then U_n is cyclic of order $r^{k-1}(r-1)$, and so $r^{k-1}(r-1)$ divides $r^k - 1$, which shows that $k = 1$ as required.

(b) Since B' consists of all elements $b \bmod n$ such that $b \in [1, n-1]$ and (7) holds for b, part (a) shows that B' generates a proper subgroup of U_n. Therefore $|B'| \leqslant \frac{1}{2}|U_n|$, and the result follows.

Note: Actually Rabin uses a slightly more complicated, but equivalent, formulation of the condition (7). In [**49**] he shows that "half" in the theorem can be replaced by "one quarter," and he gives examples of composite n where this is essentially best possible.

The property proved in Theorem 1(b) can be used to give a compositeness test for integers. (In [**47**] and elsewhere it is referred to as a primality test, but this is misleading.) Fix an integer $k \geqslant 1$ and choose k integers independently and uniformly at random from the interval $[1, n-1]$. If n fails the strong pseudoprime test to any of these bases, then declare n to be composite; otherwise the test is inconclusive, but there is very strong evidence that n is prime. There has been some confusion about what this test indicates when the second of these alternatives holds. As Rabin emphasizes, it is nonsense to say that n is "probably prime" (it is either prime or not), and there is no value of k for which this probabilistic algorithm will ever prove that n is prime. On the other hand, if n is composite, then, independently of n, the probability that the test will show that n is composite is at least $1 - 2^{-k}$ ($1 - 4^{-k}$ on the basis of the stronger result). Thus, if $k = 100$, say, then it is extremely unlikely that the test will fail to show that n is composite. So, if n passes the strong pseudoprime test for 100 randomly chosen bases, then there is an extremely strong presumption that n is prime, but the proof is lacking. Note that each test of (7) requires $O(\log n)^3$ single precision operations.

Independently, at about the same time as [**47**], a slightly different probabilistic compositeness test was proposed by R. Solovay and V. Strassen [**59**] (see also the paper [**29**] of D. H. Lehmer). In this test we compute the value of the Jacobi symbol $(\frac{a}{n})$ for randomly chosen integers $a \in [1, n-1]$ and compare this with the value of $a^{(n-1)/2} \bmod n$; if n is prime, the two values are the same. However, L. Monier [**34a**] shows that compared to the Rabin test the Solovay-Strassen test is slightly inferior: it requires more computation and is sometimes less effective in detecting a nonprime. More recent papers [**1**], [**24**], [**44**] (and others) describe related tests.

Is it possible to strengthen Theorem 1 in some way to obtain a result which can be used to prove primality? The difficulty seems to be at about the same level as the problem of recognizing when a subset of U_n generates U_n without knowing the prime factors of n—which seems to be a very difficult problem. However there is a conditional solution based on the unproved Extended Riemann Hypothesis (ERH). The ERH is that whenever χ is a character modulo m, then the L-function $L(s, \chi) = \sum k^{-s}\chi(k)$ (summed over all positive integers k relatively prime to m) can be extended by analytic continuation to a meromorphic function *without zeros* in the half plane $\operatorname{Re} s > \frac{1}{2}$. The famous Riemann Hypothesis is the special case where $m = 1$ and χ is identically 1. Since the Riemann Hypothesis has resisted efforts to either prove or disprove it for over a century, it is difficult to evaluate results conditional on the ERH. Still, in absence of anything better we have the following result (see [**35**, p. 120] for a proof).

LEMMA 2 (N. C. Ankeny, H. L. Montgomery). *Assuming* ERH *there exists a constant* $C > 0$ *such that, for all integers* $n > 1$ *and each nontrivial group homomorphism* $\psi : U_n \to G$ *into some group* G, *there exists a prime* q *such that* $1 < q \leqslant C(\log n)^2$ *and* $\psi(q \bmod n) \neq 1$.

As a simple corollary we get useful information about a generating set for U_n.

LEMMA 3. *Assuming* ERH *and taking* C *as the constant in Lemma* 2, *the group* U_n *is generated by the set* B *consisting of the elements* $q \bmod n$ *as* q *ranges over the primes in* $[1, C(\log n)^2]$.

Proof. Let S be the subgroup of U_n generated by B and put $G = U_n/S$. Then the canonical homomorphism $\psi : U_n \to G$ is surjective and has the kernel S. Since $\psi(q \bmod n) = 1$ for all $q \in B$, Lemma 2 shows that ψ is trivial. Hence $G = 1$, and so $S = U_n$.

Together Lemma 3 and Theorem 1(a) imply at once the following conditional result.

THEOREM 2 (G. L. Miller). *Assuming* ERH *and taking* C *as the constant in Lemma* 2, *each odd integer* $n > 1$ *which passes the strong pseudoprime test to base* q *for each prime* $q \in [1, C(\log n)^2]$ *is prime.*

Evidently Theorem 2 can be used to establish (conditionally on ERH) a polynomial-time primality test, and this was done by Miller in [**34**]. Miller's original result is slightly different from our Theorem 2, but it was his work which led to the fruitful idea of a strong pseudoprime test and Rabin's compositeness test.

Exercise 12. Let m and k be positive integers. Show that if k integers $b_1, \ldots, b_k$ are chosen independently and uniformly at random from $[1, m]$, then the probability that at least one composite odd integer $n \in [1, m]$ is a pseudoprime to all the bases $b_1, \ldots, b_k$ is less than $m2^{-k-1}$. In particular, there is a set $T \subseteq [1, m]$ consisting of at most $\log m/\log 2$ integers such that every odd composite $n \in [1, m]$ fails (7) for at least one $b \in T$. (Compare with [**2**]. Unfortunately, this result gives no idea how to find such a set T.)

7. Factorization. In practice, if we want to factor n into its prime factors, then we begin by removing its small prime factors (up to perhaps 10^4) by trial division. We then make one or more pseudoprime tests (7) on the unfactored part until we either discover that it is composite or are strongly convinced that it must be prime. In the latter case we must look for a proof of primality, and that process we shall describe in Section 13. In the former case we must search for a proper factor; when we have found one we have reduced the problem into two simpler problems and can repeat the process on the smaller numbers. At present, the most powerful known methods of factoring routinely succeed in factoring integers of 50 digits in a few hours, but factoring larger integers involves some measure of luck (see [**45**]). Thus successful factorizations of large integers have been achieved by a set of rather ad hoc methods, some of which may be more or less appropriate when faced with a specific integer. In Sections 8–11 we give a summary of the principal methods which have had some success in practice; further details may be found in [**20**] and [**21**]. In Section 12 we discuss some theoretical results.

8. Direct Methods and Sieves. Once we have eliminated small prime divisors, the straightforward method of trial division is usually unfruitful. A sieve method is a refinement which eliminates various classes of possible divisors. For example, suppose n is a Fermat number $F_k = 2^{2^k} + 1$ such that the prime $r|n$. Then the element $x = 2 \bmod r$ in U_r satisfies $x^{2^k} = -1$, and so x has order 2^{k+1} by (4). Hence $2^{k+1}|r - 1$; that is, all prime divisors r of n lie in the arithmetic progression $2^{k+1}m + 1$ $(m = 1, 2, \ldots)$.

Exercise 13. (a) Show that if $k \geqslant 2$ and the prime $r|F_k$, then, in fact, $2^{k+2}|r - 1$. [Hint: $r \equiv 1 \pmod 8$ and so $(\frac{2}{r}) = 1$.]

(b) (Euler 1747). Find a prime factor of F_5.

Clearly the method just described depends very heavily on the special form of n. A more generally applicable sieve method uses quadratic residues. In 1798 Legendre noted that if for some integers a and c relatively prime to n we have $c^2 \equiv a \pmod n$, then for each prime $r|n$ the same congruence holds and so $(\frac{a}{r}) = 1$. If a is odd, then the quadratic reciprocity law shows that $(\frac{r}{|a|})$ can be computed, and this implies that r must lie in certain arithmetic progressions with common difference $|a|$ if $4|a - 1$ or $4|a|$ otherwise. Knowledge of a number of small squares modulo n can lead to useful restrictions on the possible values of r. (See [**23**]. A similar method appears in [**17**, §322].)

EXAMPLE. As a rather artificial example consider $n = 1711$. Note that $16^2 \cdot 7 - 9^2 = 1711$ and that $37^2 \cdot 5 - 1 = 4 \cdot 1711$. Thus 7 and 5 are both squares modulo n, and so for each prime $r|n$ we have $(\frac{r}{5}) = (\frac{5}{r}) = 1$ and

$$\left(\frac{r}{7}\right) = \left(\frac{7}{r}\right)(-1)^{(r-1)/2} = (-1)^{(r-1)/2}.$$

Thus r is congruent to $\pm 1 \pmod 5$ and to ± 1, ± 9 or $\pm 25 \pmod{28}$. The smallest integers which satisfy these two conditions are $1, 9, 19, 29, \ldots$ and indeed $r = 29$ divides n.

The example shows the two difficulties which arise when this method is applied. Firstly, we need to be able to find relatively small integers which are squares modulo n, and secondly, we need an efficient way of combining the information on r obtained from the different moduli. Legendre suggested using the continued fraction approximations to $(kn)^{1/2}$ for various integers k to solve the first problem (see Section 11 below), and over the past 50 years D. H. Lehmer and his associates have constructed a series of special purpose computers ("delay line sieves") which handle the second problem (see [**27**] and [**30**]). Asymptotically, the method is poor; its running time grows like $cn^{1/2}$ although the constant c may be rather small. However, in the past it has been competitive because the delay line sieves were much faster (for their specific task) than the contemporary general purpose computers. The SRS-181 built in Berkeley in the early 1970s processes $2 \cdot 10^7$ integers per second, and faster sieves have been built since then (for example, by H. C. Williams in Winnipeg).

9. Pollard's Monte Carlo Method (or Rho Method). Let S be a finite set and consider the semigroup $\mathrm{Map}(S)$ of all functions of S into itself under the operation of composition. If $\psi \in \mathrm{Map}(S)$ and $x_0 \in S$, then (since S is finite) there exists an integer $l \geqslant 1$ such that the elements $x_0 = \psi^0(x_0), \psi(x_0), \ldots, \psi^{l-1}(x_0)$ are all distinct, but $\psi^l(x_0) = \psi^{l'}(x_0)$ for some $l' \in [0, l - 1]$. We shall call l the *length of the orbit* of x_0 under ψ, and note that once $k \geqslant l'$ the sequence $\{\psi^k(x_0)\}$ is periodic with period $l - l'$.

Now suppose S has m elements and $x_0 \in S$. The length of the orbit of x_0 under ψ is at least l provided $x_0, \psi(x_0), \ldots, \psi^{l-1}(x_0)$ are all distinct. Thus the number of $\psi \in \mathrm{Map}(S)$ for which x_0 has an orbit of length at least l is

$$m^{m-l}\prod_{i=0}^{l-1}(m - i) < m^m \exp\left(-\sum_{i=0}^{l-1} i/m\right) = m^m \exp(-l(l-1)/2m).$$

Thus, for each $\lambda > 0$, the proportion of $\psi \in \mathrm{Map}(S)$ under which x_0 has an orbit of length at least $(2\lambda m)^{1/2} + 1$ is less than $\exp(-\lambda)$. Hence there will be relatively very few pairs x_0, ψ where the orbit is greater than, say, $5m^{1/2}$; we shall call such pairs "exceptional" (compare [**21**, Ex. 3.1.12]).

J. M. Pollard in [**39**] and [**40**] applied this observation to obtain the following factorization method. Suppose that n is composite and the prime $r|n$. Set $S = \mathbf{Z}_r$, let y_0 be an integer and ψ be a polynomial function. If the pair $y_0 \bmod r$, ψ is not exceptional, then the length of the orbit of $y_0 \bmod r$ under ψ is bounded by a small multiple of $r^{1/2}$. We do not know r so we cannot compute the elements $x_k = \psi^k(y_0 \bmod r)$ in $\mathbf{Z}_r$, but we can compute the sequence $\{y_k\}$ where, for each $k > 0$, $y_k \in [0, n-1]$ and $y_k \equiv \psi(y_{k-1}) \pmod n$; and the latter satisfies $x_k = y_k \bmod r$ for all k. Now, with the notation above, we know that $\{\psi^k(x_0)\}$ is periodic with period $d = l - l'$ once $k \geqslant l'$. This implies that r divides $\mathrm{GCD}(y_k - y_h, n)$ whenever $k > h > l'$ and $d|k - h$. This is a special case of finding cycles in sequences of iterates. It was first solved by R. W. Floyd who noted that we can look for solutions with $k = 2h$; and later R. P. Brent [**6**] showed that it is often better to look for solutions where k and h have the forms $h = 2^i - 1$ and $k = 2^i + j$ with $0 \leqslant j < 2^i$. The point is that by searching for suitable y_h and y_k in this way it is only necessary to retain in memory two values of the sequence, so Pollard's method requires very little storage. Unless we are very unlucky, the result of this search will be a proper factor of n in the form $\mathrm{GCD}(y_k - y_h, n)$.

EXAMPLE. Let $n = 30623$ and take $\psi(x) = x^2 + 1$ and $y_0 = 1$. Then the successive values of y_k such that $y_k \equiv \psi(y_{k-1}) \pmod n$ for all $k > 0$ are: $1, 2, 5, 26, 677, 29608, 19667, 22400, \ldots$. In practice, as k ranges over the interval $[2^i, 2^{i+1} - 1]$ only the current value of y_k and the value of y_{2^i-1} are stored. We find that $y_7 - y_3 = 22374$ has a common divisor of 113 with n, and hence 113 is a proper division of n.

How should we choose ψ and x_0 so that they form a nonexceptional pair for $\mathbf{Z}_r$? Nothing positive seems to be known, in general, although it can be shown that ψ should not be linear, nor of the form $x^2 - 2$. Numerical evidence for the primes up to 10^6 (see [**20**]) indicates that $\psi(x) = x^2 + 1$ or $x^2 - 1$ are, as well as simple to compute, quite good; and these functions are most generally used, regularly producing prime factors in $O(r^{1/2})$ steps. The most notable success of this method was to factor the 78-digit Fermat number F_8 whose compositeness had been proved by J. C. Morehead and A. E. Western in 1909 (see [**8**]). For details and implementation of Pollard's method see [**6**], [**8**], [**20**], [**21**] and [**40**].

Since each composite number n has a prime factor $r \leqslant n^{1/2}$, the Monte Carlo method appears to have its execution time bounded by $O(n^{1/4})$, but this conclusion is conditional on the unproved assumption that the particular x_0, ψ chosen are nonexceptional. It is an open problem whether the method (or some modification) is susceptible to a complete analysis.

Exercise 14. Show that, if $x_0 \in S$ and $|S| = m$, then the average length of the orbit of x_0 under ψ as ψ runs over the *bijections* of S into itself is $(m + 1)/2$. (This can be used to explain why a linear polynomial should not be used in Pollard's method.)

Exercise 15 (J. M. Pollard). Let p be prime and let g be a primitive root modulo p. Use the ideas above to give an algorithm which, for any integer a with $p \nmid a$, finds the index i for which $g^i \equiv a \pmod p$, and is likely to have its execution time bounded by $O(p^{1/2}(\log p)^3)$. (For related results see [**3**] and [**38**].)

10. Using the Group Structure of U_n to Find Factors of n. If n has s distinct prime factors, then it follows from (3) that the subgroup $T = \{x \in U_n | x^2 = 1\}$ has order 2^s since each U_{n_i} has a unique subgroup of order 2. Conversely, if for some integer c, $c^2 \equiv 1 \pmod n$ and $c \not\equiv \pm 1 \pmod n$, then $\mathrm{GCD}(c - 1, n)$ is a proper factor of n. Thus every element $x \neq \pm 1$ in T yields a proper factor of n, and many factoring methods are based on this observation.

Exercise 16. Suppose that for some integer $d > 1$ you have $d + 1$ distinct roots in $\mathbf{Z}_n$ for a polynomial in $\mathbf{Z}_n[X]$ of degree d. How can you find a proper factor of n?

Now suppose that we know an integer $m > 0$ with the property that $x^m = 1$ for all $x \in U_n$. Write $m = 2^h l$ with l odd. If n is not a prime power, then Lemma 1 shows that for at least half of the elements $x \in U_n$, the cyclic group $\langle x \rangle$ has even order but does not contain -1. In this latter case there is an index $t \in [0, h-1]$ such that $y = x^{l2^t} \neq \pm 1$ but $y^2 = 1$, and so we can obtain a proper factor of n. This leads to a probabilistic algorithm in which a sequence of values of x are chosen independently and uniformly at random from U_n, and the algorithm halts when it finds a value of x which gives a proper factor of n in the manner above. The average number of values of x which must be chosen is at most 2, so the average number of single precision operations required to find a factor of n is $O((\log n)^2 \log m)$. For example, if we know $\phi(n)$, the order of U_n, then $m = \phi(n)$ will factor n very quickly. The choice of $m = n!$ is also valid but then $\log m$ is too large.

In general, it will not be possible to find a suitable value of m which is small enough to make the method above feasible, unless we have further information on the arithmetic structure of n (which is what we are trying to find!) However, variations of this idea have been suggested by D. N. and D. H. Lehmer (unpublished) and by J. M. Pollard [**39**], and have been used with some success. Fix a constant $c > 0$ (around 10^5 in some applications), and let m be the least common multiple of all prime powers less than c. If n has the factorization (1) and $\phi(r_i^{k_i}) | m$ for all i, then $x^m = 1$ for all $x \in U_n$ by (3), and so the method above applies. However, we still obtain a factorization if there is at least one $\phi(r_j) | m$, since in this case we always have $x^m \bmod r_j = 1$ and so $\mathrm{GCD}(x^m - 1, n) \neq 1$. Further modifications (see [**20**], [**39**] and [**62**]) extend the utility of this method, but its success clearly depends to a certain amount on a fortunate choice of n.

EXAMPLE. Let $n = 1711$ and take $c = 10$. Then $m = 2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. Choose $x = 2$. Then $x^m = 523 \pmod{n}$ and $\mathrm{GCD}(522, n) = 29$ which is a proper factor of n.

Exercise 17. If k is relatively prime to the order of U_n, show that $x \mapsto x^k$ is a bijection of U_n onto itself, and that the inverse mapping also has the form $x \mapsto x^{k'}$ for some k'. Explain how to compute k' from k if the prime factors of n are known. Conversely, show that n may be easily factored if we know a single pair of such integers k, k' (of a size comparable to n) with $k > 1$. (Thus computing the inverse function is as difficult as factoring n into primes. These mappings have been used as "trapdoor functions;" see, for example, [**13**].)

11. Factor Bases and the Continued Fraction Method. As noted in the previous section, we can find a proper factor of n whenever we can find $x \in U_n$ such that $x^2 = 1$ and $x \neq \pm 1$; and, conversely, such an element exists provided n is not a prime power. Evidently, it is enough to find integers a and b such that $a^2 \equiv b^2 \pmod{n}$ and $a \not\equiv \pm b \pmod{n}$, since again $\mathrm{GCD}(a - b, n)$ is a proper factor of n. In 1643 Fermat factored integers by making a direct search for a and b such that $n = a^2 - b^2$ (see [**12**, p. 357]). More recently R. S. Lehman [**25**] showed that a modification of Fermat's idea using Farey fractions gives a factoring algorithm whose running time is $O(n^{1/3})$, but this does not seem to be competitive with the known alternatives. A better approach is the following.

We define a *factor base* to be a set B of nonzero integers $\{b_0, b_1, \ldots, b_h\}$ (for example, -1 and the first h primes). Call an integer a a *B-number* if the integer c defined by $c \equiv a^2 \pmod{n}$ and $c \in [-n/2, n/2]$ can be written as a product of factors from B. Thus, if a is a B-number, then we can compute integers $e(a, i) \geq 0$ such that $a^2 \equiv \prod_{i=0}^{h} b_i^{e(a,i)} \pmod{n}$. To each B-number a we associate the $(h+1)$-vector $e(a) = (e(a,0), \ldots, e(a,h))$. If we can find a sufficiently large set A of B-numbers ($|A| = h + 2$ is enough), then the vectors $e(a)$ $(a \in A)$ are linearly dependent when considered modulo 2. So using ordinary linear algebra (over $\mathbf{Z}_2$) we can find a nonempty subset A' of A such that

$$\sum_{a \in A'} e(a) \equiv (0, \ldots, 0) \pmod{2},$$

and then $h_i = \frac{1}{2}\Sigma_{a\in A'} e(a,i)$ is an integer for each i. If we now define

$$u \equiv \prod_{a\in A'} a \pmod{n} \text{ and } v \equiv \prod_{i=0}^{h} b_i^{h_i} \pmod{n},$$

then $u^2 \equiv v^2 \pmod{n}$. If we are lucky, then $u \not\equiv \pm v \pmod{n}$ and so we obtain a proper factor of n.

EXAMPLE. Let $n = 1711$. Then

$41^2 - 1711 = -30, 83^2 - 4 \cdot 1711 = 45, 124^2 - 9 \cdot 1711 = -23$ and $455^2 - 121 \cdot 1711 = -6$. Using the base $B = \{-2, 3, 5\}$ we obtain

$$41^2 \equiv (-2) \cdot 3 \cdot 5, \quad 83^2 \equiv 3^2 \cdot 5 \text{ and } 455^2 \equiv (-2) \cdot 3 \pmod{n}.$$

Although we have only three congruences, it turns out that the three vectors of exponents, namely (1 1 1), (0 2 1) and (1 1 0) are linearly dependent modulo 2 since the sum is 2(1 2 1). Thus

$$(41 \cdot 83 \cdot 455)^2 \equiv \left((-2) \cdot 3^2 \cdot 5\right)^2 \pmod{n},$$

but we are unlucky since

$$41 \cdot 83 \cdot 455 \equiv -(-2) \cdot 3^2 \cdot 5 \pmod{n}.$$

However, with the additional equation $185^2 - 20 \cdot 1711 = 5$ we have the further exponent vector (0 0 1), and since

$$(0\,2\,1) + (0\,0\,1) = 2(0\,1\,1)$$

we obtain $(83 \cdot 185)^2 \equiv (3 \cdot 5)^2 \pmod{n}$ and from this get the proper factor $\mathrm{GCD}(83 \cdot 185 - 3 \cdot 5, n) = 29$.

This idea can be traced back to M. Kraitchik [**22**] in 1926, but it is not described in any systematic way. In 1931 D. H. Lehmer and R. E. Powers [**31**] suggested a similar idea in which a method of Legendre was used to generate relatively small quadratic residues (which might be expected to factor into products of small primes more often). Their method was impractical until the advent of fast computers with large memory resources. In the late 1960s, using [**31**] as a basis, J. Brillhart and M. A. Morrison systematized and implemented a computer version of the method. Almost at once they had their first major success: they factored the 39-digit Fermat number F_7 which had been known to be composite since 1897 (see [**36**]). The *continued fraction method* as it is now called is presently the most consistently successful method for factoring large numbers up to about 50 digits.

What then was Legendre's idea for finding small quadratic residues? In general, for any irrational number, the continued fraction expansion gives a sequence of increasingly close rational approximations (see, for example, [**60**, Chap. 1]). In particular, if n is not a square, then, using only integer arithmetic, we can compute two increasing integer sequences $\{u_k\}$ and $\{v_k\}$ such that, for each k,

$$|n^{1/2} - u_k/v_k| < 1/v_k v_{k+1},$$

and hence $u_k^2 - nv_k^2 = w_k$, say, with $|w_k| < 2n^{1/2}$. Thus $\{w_k\}$ is a sequence of quadratic residues for n whose sizes are $O(n^{1/2})$. Moreover, n is a quadratic residue for any prime dividing w_k. Taking our factor base B to consist of -1 and the first h primes p for which $(\frac{n}{p}) = 1$ (for a suitable value of h), the relatively small size of the w_k increases the chance that u_k is a B-number, and in this way we may hope to generate enough B-numbers to make the method work. We should note several points which are important in practice. There is no need to compute the integers v_k, and the integers u_k are only computed modulo n; and it is sometimes advantageous to work with the continued fraction for $(dn)^{1/2}$ for some small integer d in place of $n^{1/2}$. A lot of work has been done to ensure that the implementation is efficient as possible, and some of this is described in the

original paper [**36**]. Data given in the paper [**63**] suggests that in the range 10^{30} to 10^{50} the average running time is roughly proportional to $n^{1/7}$. The recent paper [**45**] discusses several devices which have had the effect of speeding up the algorithm by a factor of 10–15; one of these is an "early abort strategy" which discards terms w_k at an early stage when there is evidence that they do not have enough small factors. No one has given a complete theoretical analysis of the effectiveness of the continued fraction algorithm, but two recent papers have shown that on plausible (but unproved) assumptions about the distribution of primes, the asymptotic running time is of the form

$$\exp\{c(\log n \log\log n)^{1/2}\} = n^{c(\log\log n/\log n)^{1/2}}$$

(see [**43**] and [**51**]).

12. Theoretical Estimates for the Factoring Problem. It is unfortunately true that none of the factoring methods which are most effective in practice has had an adequate analysis of its performance. As noted above, the continued fraction method is well supported by empirical evidence (see [**63**] and [**45**]); [**62**] gives some statistics on the method discussed in Section 10; [**20**] reports computational evidence in a limited range which supports the Monte Carlo method; and both [**43**] and [**51**] give analyses of the continued fraction method and its variants based on plausible but unproved conjectures about the distribution of primes and prime divisors. Still there is no assurance that for some (perhaps most!) values of n these methods might all work quite badly. It is therefore of interest to know that there is a method which, although undoubtedly poor from a practical point of view, can be completely analysed and shown to have a running time which is $O(n^{\varepsilon})$ for each $\varepsilon > 0$.

This asymptotically fast factoring algorithm is a probabilistic algorithm based on a naive use of a factor base. Using the notation of Section 11 consider the base $B = \{b_0, \ldots, b_h\}$ consisting of -1 and the first h primes. The basic step in the algorithm consists of choosing independently and uniformly at random enough integers from $[1, n-1]$ until we have a set A of B-numbers whose exponent vectors $e(a)$ $(a \in A)$ are linearly dependent modulo 2. As was shown in Section 11 this leads to a pair of integers u, v such that $u^2 \equiv v^2 \pmod{n}$, and we obtain a proper factor of n provided $u \not\equiv \pm v \pmod{n}$. It is now not too difficult to prove the following theorem.

THEOREM 3 (J. D. Dixon). *Suppose that n is not a prime power. Then the basic step described above will find a proper factor of n with probability at least $\frac{1}{2}$. Moreover, there exist absolute constants $c_1, c_2 > 0$ such that if $h = [\exp\{c_1(\log n \log\log n)^{1/2}\}]$ and the basic step is repeated until a proper factor has been found, then the average running time is $O(\exp\{c_2(\log n \log\log n)^{1/2}\})$.*

REMARK. In the original paper [**14**] the values $c_1 = \sqrt{2}$ and $c_2 = 3\sqrt{2}$ were obtained. These were improved in [**21**] and [**51**], and in [**43**] it is shown that the average running time for an optimal choice of h is of the form $\exp\{(2 + \varepsilon_n)(\log n \log\log n)^{1/2}\}$, where $\varepsilon_n \to 0$ as $n \to \infty$. Thus the running time grows subexponentially in $\log n$ but faster than any power of $\log n$.

The papers [**43**] and [**51**] carefully analyze a whole family of factoring algorithms which are basically of the type above but which use various ways to try to choose the set A of B-numbers more effectively. These papers make precise some very rough arguments which had been given to support proposed methods. The analyses are not complete (they are based on clearly stated, but unproved, heuristic assumptions), but their value lies in suggesting how various modifications might be expected to affect the performance of the algorithms. As well as dealing with the continued fraction method, both papers include interesting details of factoring methods which we have not discussed here. Among these are the linear sieve of R. Schroeppel (see [**43**]); the quadratic sieve of Pomerance (see [**43**] and [**18**]); and variants of a method of D. Shanks (see [**55**], [**51**] and [**52**]) which works in the group of equivalence classes of binary quadratic forms with discriminant $-n$ (and brings us back to Gauss's original investigations [**17**]). We also mention another method of Shanks christened SQUFOF which appears quite effective for certain ranges of n (see [**58**]).

13. Finding Primality Proofs in Almost Polynomial Time. The final stage in a complete factorization of an integer is to prove primality of its factors. Suppose that n is a large integer which we suspect of being prime. We should certainly carry out enough pseudoprime tests of type (7) until we are convinced that there is a strong presumption that n is prime, but at that point we are faced with having to find a proof. Until recently, there was no reasonably fast, uniform method of proceeding, and although it was frequently possible to provide certificates of primality for some primes of 80 digits and above, often even smaller primes caused trouble (see [**61**]). However, in 1980, L. M. Adleman and R. S. Rumely proposed a general algorithm for proving primality, and using work of K. Prachar and P. X. Gallagher, A. Odlyzko and C. Pomerance proved that the algorithm would run in close to polynomial-time, namely $O((\log n)^{c \log\log\log n})$ for a suitable constant $c > 0$ (see [**4**]). H. W. Lenstra, Jr., and H. Cohen have produced a streamlined version of this algorithm and Cohen, A. K. Lenstra and D. T. Winter have carefully programmed it (see [**11**] and [**33**]). The most recent report is that this program can routinely provide proofs of primality for primes up to 200 digits in less than 10 minutes. However, the question of whether there exists an algorithm which finds proofs of primality in strictly polynomial-time remains open.

The basic idea behind the Adleman-Rumely algorithm can be described as follows. Working in cyclotomic fields, we carry out a series of generalized pseudoprime tests on n. If any of these tests fails, then n is not prime; if they succeed, then we use the information obtained from the tests to construct a sieve restricting the possible prime divisors of n (compare with Section 8). If n is a prime, then, because we know a great deal about the arithmetic structure of the rings in which we are working in this case, we can choose the tests so as to produce a very simple sieve. The result is that, if the tests are satisfied, then we can prove that each prime divisor r of n must lie in one of a relatively small set of arithmetic progressions which are of the form $t^i \pmod w$ for $i = 0, 1, \ldots$ where $w > n^{1/2}$. Moreover, the choice of w ensures that each element of U_w has order at most

$$z = O\big((\log n)^{c \log\log\log n}\big);$$

so only values of i less than z need to be considered. Thus, to prove that n is prime (or discover that after all it is not), it only remains to check that n has no proper divisor among this latter set. (In the probabilistic version considered below we may take $t = n$.) A critical part of the algorithm is how it is possible to transfer so much information from the pseudoprime tests over to the sieve. In the original version [**4**] this was done via a general reciprocity law (the Eisenstein reciprocity theorem is adequate), but in the version we give below we follow [**33**] which replaces the use of a reciprocity law by an elementary argument on Gaussian sums.

Theorem 4 below describes the basic generalized pseudoprime tests to which we have referred. To explain these tests and to prepare for the proof of the theorem we begin with a discussion of some elementary properties of Gaussian sums (see [**20a**]).

Let p and q be primes not dividing n such that $p|q-1$, and let ζ_p and ζ_q be primitive pth and qth roots of unity in $\mathbf{C}$. Let $g = g_q$ be a generator of the cyclic group U_q (of order $q-1$). Since $p|q-1$, there exists a homomorphism χ_{pq} of U_q into the cyclic group $\langle \zeta_p \rangle$ defined by $\chi_{pq}(g^j) = \zeta_p^j$ for all j (χ_{pq} is called a "character of degree p with conductor q"). Now for each integer t we define the *Gaussian sum*

$$\tau\big(\chi_{pq}^t\big) = \sum_u \chi_{pq}(u)^t \zeta_q^u$$

where u runs over U_q (the expression ζ_q^u has the obvious meaning; the value of ζ_q^j depends only on the value of $j \bmod q$ so there is no ambiguity). Note that $\tau(\chi_{pq}^t)$ lies in the ring $R_{pq} = \mathbf{Z}[\zeta_p, \zeta_q]$.

LEMMA 4. *Writing χ for χ_{pq} we have*:
(a) $\tau(\chi)\tau(\chi^{-1}) = \chi(-1)q$;
(b) *if n is prime, then* $\tau(\chi)^{n^{p-1}-1} \equiv \chi(n) \pmod n$ *in* R_{pq};
(c) *if for some prime $r \neq p$ we have* $\zeta_p^i \equiv \zeta_p^j \pmod r$, *then* $\zeta_p^i = \zeta_p^j$.

Proof. In all the sums below the variables range over U_q. Several times we use the fact that if v

is a fixed element of U_q (or an integer relatively prime to q), then uv ranges over U_q as u ranges over U_q.

(a)
$$\begin{aligned}\tau(\chi)\tau(\chi^{-1}) &= \sum_u \sum_v \chi(u)\chi(v)^{-1}\zeta_q^{u+v} \\ &= \sum_u \sum_v \chi(uv)\chi(v^{-1})\zeta_q^{uv+v} \\ &= \sum_u \chi(u) \sum_v \zeta_q^{v(u+1)}.\end{aligned}$$

Since $\zeta_q^q = 1$,

$$\sum_v \zeta_q^{vw} = -1 + (\zeta_q^{qw} - 1)/(\zeta_q^w - 1) = -1$$

if $q \nmid w$, and the lefthand sum is $q - 1$ otherwise. On the other hand,

$$\sum_u \chi(u) = \sum_u \chi(gu) = \chi(g)\sum_u \chi(u),$$

so $\Sigma_u \chi(u) = 0$ since $\chi(g) \neq 1$. Therefore

$$\tau(\chi)\tau(\chi^{-1}) = -\sum_u \chi(u) + q\chi(-1) = q\chi(-1)$$

as asserted.

(b) To simplify notation we shall temporarily put $m = n^{p-1}$. Since n is prime, we have

$$(\alpha_1 + \cdots \alpha_t)^{n^i} \equiv \alpha_1^{n^i} + \cdots + \alpha_t^{n^i} (\mathrm{mod}\ n)$$

for all $\alpha_1, \ldots, \alpha_t \in R_{pq}$ and $i > 0$. Thus

$$\tau(\chi)^m = \left(\sum_u \chi(u)\zeta_q^u\right)^m \equiv \sum_u \chi(u)^m \zeta_q^{um} (\mathrm{mod}\ n).$$

Now $\chi(u)^m = \chi(u)$ since $p|m - 1$, and there exists $v \in U_q$ such that $vm = 1$ since $q \nmid m$. Thus

$$\begin{aligned}\sum_u \chi(u)^m \zeta_q^{um} &= \sum_u \chi(u)\zeta_q^{um} = \sum_w \chi(vw)\zeta_q^w \\ &= \chi(v)\tau(\chi) = \chi(n)^{1-p}\tau(\chi) = \chi(n)\tau(\chi)\end{aligned}$$

because $q|p - 1$ and $vn^{p-1} = 1$. Finally, since $\chi(-1)q = \pm q$ is relatively prime to n, there exists an integer a such that $\chi(-1)qa \equiv 1 \pmod{n}$. So multiplying the congruence

$$\tau(\chi)^m \equiv \chi(n)\tau(\chi) (\mathrm{mod}\ n)$$

through by $a\tau(\chi^{-1})$ and using (a) we obtain the desired result.

(c) Set $k = i - j$. Then the hypothesis implies that $\zeta_p^k \equiv 1 \pmod{r}$, and hence

$$f(\zeta_p^k) \equiv f(1) = p (\mathrm{mod}\ r)$$

where $f(X)$ is the cyclotomic polynomial

$$(X^p - 1)/(X - 1) = X^{p-1} + X^{p-2} + \cdots + 1.$$

Since $r \nmid p$, $f(\zeta_p^k) \neq 0$ and so ζ_p^k is not a primitive pth root of 1. Hence $\zeta_p^k = 1$ and so $\zeta_p^i = \zeta_p^j$. This completes the proof of the lemma.

Now let P and Q be finite sets primes not dividing n where, for all $q \in Q$, $q - 1$ is a product of distinct primes from P. Put $z = \prod_{p \in P} p$ and $w = \prod_{q \in Q} q$. Then $q - 1|z$ for each $q \in Q$, and so the order of each element in U_w (compare with (3)) must divide z. Thus $a^z \equiv 1 \pmod{w}$ for each integer a which is relatively prime to w.

THEOREM 4 (H. W. Lenstra Jr.). *Suppose that n is an odd integer satisfying the following conditions*:

(i) *For all primes p and q with $q \in Q$ and $p|q-1$, $\tau(\chi_{pq})^{n^{p-1}-1} \equiv \chi_{pq}(n) \pmod{n}$;*

(ii) *For each prime $p \in P$ and each prime $r|n$, $p^{e_p}|r^{p-1}-1$ where p^{e_p} is the largest power of p dividing $n^{p-1}-1$.*

Then each prime $r|n$ is congruent to $n^i \pmod{w}$ for some $i \in [0, z-1]$.

REMARK. If n is prime, then part (b) of Lemma 4 shows that (i) is satisfied, and (ii) is trivially satisfied. There are probably very few other integers which satisfy the conditions.

Proof. Condition (ii) shows that, for each $p \in P$ and each prime $r|n$, there exist integers a_p and $b_p(r)$ such that

$$n^{p-1} = 1 + a_p p^{e_p},\ r^{p-1} = 1 + b_p(r)p^{e_p}$$

and $p \nmid a_p$. Thus for some integer $l_p(r)$ we have $a_p l_p(r) \equiv b_p(r) \pmod{p}$.

Now, for each pair p, q with $q \in Q$ and $p|q-1$, Lemma 4(b) shows that

$$\chi(r) \equiv \tau(\chi)^{r^{p-1}-1} \pmod{r}$$

for each prime $r|n$ (where again we have put χ for χ_{pq}). However,

$$(r^{p-1}-1)a_p = (n^{p-1}-1)b_p(r),$$

and so using the congruence in (i) taken modulo r we conclude that

$$\chi(r)^{a_p} \equiv \chi(n)^{b_p(r)} \pmod{r},$$

and so $\chi(r)^{a_p} = \chi(n)^{b_p(r)}$ by Lemma 4(c). Since both sides of this equation are pth roots of 1, and $p \nmid a_p$, we conclude that $\chi(r) = \chi(n)^{l_p(r)}$ by the choice of $l_p(r)$. This holds for each prime $r|n$ and all $p \in P$ and $q \in Q$ with $p|q-1$.

Finally, we use the Chinese Remainder Theorem to find an integer $l(r) \in [0, z-1]$ such that $l(r) \equiv -l_p(r) \pmod{p}$ for each $p \in P$. Then, since $\chi(n)$ is a pth root of 1,

$$\chi(rn^{l(r)}) = \chi(r)\chi(n)^{l(r)} = 1,$$

and so $rn^{l(r)}$ lies in the kernel of $\chi = \chi_{pq}$ for all $q \in Q$ and all $p|q-1$. Since $q-1$ is squarefree, the intersection of these kernels as p runs over the prime divisors of $q-1$ is trivial. Thus $rn^{l(r)} \equiv 1 \pmod{q}$ for all $q \in Q$, and so $rn^{l(r)} \equiv 1 \pmod{w}$. Hence $r \equiv n^i \pmod{w}$ with $i = z - l(r)$, and the theorem is proved.

Exercise 18. Suppose that for some prime q (not necessarily in Q) we have $p|q-1$, $\chi_{pq}(n) \neq 1$ and the congruence in condition (i) of Theorem 4 holds. Show that condition (ii) of Theorem 4 will then hold for p. [Hint: Let h be the order of $\tau(\chi_{pq}) \bmod r$ in the group of units of the ring R_{pq}/rR_{pq}, and show that $p^{e_p+1}|h$. Then use Lemma 4(b) to show that $h|(r^{p-1}-1)p$.]

To apply Theorem 4 we have to be able to verify the hypotheses. One way to do this is to first verify (i) using the power algorithm, and then use Exercise 18. The latter requires a search for a prime $q \equiv 1 \pmod{p}$ with $\chi_{pq}(n) \neq 1$. In practice this is quite fast since (asymptotically) $(p-1)/p$ of all primes q satisfying $q \equiv 1 \pmod{p}$ also satisfy $\chi_{pq}(n) \neq 1$. However, a proof that a small "good" prime q exists seems to require the ERH. An alternative approach given in [**11**] avoids this problem.

The remaining question is how we should make appropriate choices for the sets P and Q. In order that a primality proof based on Theorem 4 should be efficient, we should like P and Q small so that the conditions (i) and (ii) can be quickly verified, but at the same time like z small and w large so that the potential prime divisors can be disposed of quickly. By choosing $w > n^{1/2}$ we ensure that each congruence class $n^i \bmod w$ contains at most one potential prime divisor $r \leqslant n^{1/2}$, and then the final step can be executed in $O(z)$ multiprecision operations on numbers the size of n. Moreover, if z and w are both bounded by n, then $|P|$ and $|Q|$ are both $O(\log n)$ which has the

effect that (i) and (ii) may both be verified within a time polynomial in $\log n$. Thus, the following theorem (see [4] for a proof) shows that a suitable choice of P and Q gives an algorithm with running time $O((\log n)^{c \log\log\log n})$ for any $c > c_2$.

THEOREM 5 (C. Pomerance, A. Odlyzko). *Let P, Q, z and w be defined as above such that Q consists of all primes q with $q - 1 | z$. Then there are absolute computable constants $c_1, c_2 > 0$ such that, if $w > n^{1/2} \geqslant 10$, then the least possible value of z satisfies*

$$(\log n)^{c_1 \log\log\log n} < z < (\log n)^{c_2 \log\log\log n}.$$

REMARK. Computations in [4] show that if P is taken as the set of the first 6 primes, then $z = 30030$, Q consists of 21 primes and $w > (5 \cdot 10^{89})^{1/2}$. Similarly, if P is the set of the first 13 primes, then z is about $3 \cdot 10^{14}$, Q consists of 807 primes and $w > (10^{11356})^{1/2}$. These are optimal choices in the sense of Theorem 5.

The current implementation [11] differs from the algorithm described above in several ways. Firstly, it uses Jacobi sums (which were introduced in the original paper [4]) in place of Gauss sums. The computations required to verify condition (i) of Theorem 4 can then be replaced by computations in a smaller ring $\mathbf{Z}_n[\zeta_p]$ and the exponent $n^{p-1} - 1$ reduced to one about the size of n. Secondly, a generalization of Theorem 4 allows multiple prime factors in z and w and this has a significant effect on the practical running time, although not on its asymptotic form. The result is an efficient primality proving program based on an algorithm which is a major theoretical and practical advance on any earlier method. Finally, we mention that [4] and [33] also give completely deterministic algorithms for finding proofs of primality, and these algorithms have the same kind of asymptotic running-time as the version described above. However, the probabilistic version turns out to be the better choice for implementation.

14. What of the Future? On the theoretical side there still remain the questions as to whether there exist polynomial-time algorithms for proving primality or for factoring integers. The prospect for a polynomial-time algorithm for proving primality seems fairly good, but it may turn out that, on the contrary, factoring is NP-hard.*

The following are some directions which might lead to better theoretical or practical algorithms.

1. Do there exist "highly composite" integer-valued functions which can be computed quickly? In a sense this is the idea behind the method described in Section 10 where $x^m - 1$ is divisible by every prime r such that $r - 1 | m$. For example, if $k! \bmod n$ could be computed quickly for $k \in [1, n]$, then we should be able to factor n quickly (see [54] for a mathematical pun on this theme).

2. Can we speed up the method described in Section 11? For example, are there better ways than the use of continued fractions to produce small quadratic residues modulo n (smaller residues should increase the likelihood of a larger portion being B-numbers). Are there better choices for the factor base B than small primes? See [43] for descriptions of other variants of the basic method of Section 11.

3. The Adleman-Rumely-Pomerance algorithm of Section 13 uses the fact that n passes certain kinds of pseudoprime tests to set up an especially attractive form of sieve. Is an analogous idea possible when n is not prime? Specifically, can we construct a sieve so that the potential divisors which remain lie in a simply describable set?

4. What about the use of rings and groups other than $\mathbf{Z}_n$ and U_n? There is a "$p + 1$ method" of factoring which is based on computations in the ring $\mathbf{Z}_n[X]/(X^2 - c)$ (see [20, p. 73] and [62] where the idea appears in a slightly disguised form in terms of Lucas series). D. Shanks has shown how to use the class groups of binary quadratic forms (see [51], [53], [55], [57] and [58]). As Section 13 suggests, the extra freedom might be very helpful.

5. Can the theoretical bases of the currently used methods of factorization be tidied up?

*The important concepts of NP-hard and NP-complete problems are discussed, for example, in [16a].

Perhaps there are probabilistic versions which might be more amenable to analysis.

The bibliography below lists only the books and papers which have been mentioned in this survey. Further references may be found in [**20**], [**41**], [**43**], [**61**] and the Math. Centrum Tracts in which the articles [**43**] and [**53**] appear.

Acknowledgements. I am indebted to R. P. Brent, D. A. Buell, H. W. Lenstra, Jr., C. Pomerance and C. P. Schnorr, all of whom supplied me with source material and references. Very special thanks are due to Carl Pomerance who also read through an earlier version of the manuscript and offered many helpful corrections and criticisms (but, of course, is not responsible for errors which remain). I also wish to acknowledge the hospitality which I have enjoyed both at the Department of Pure Mathematics, I.A.S., Australian National University and at the Mathematics Institute, University of Warwick, during the time that I prepared this paper.

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128. MISCELLANEA

Proper integrals are just like sums, we always say. Are they? Could the equation

$$\int_0^1 x^{-x} dx = \sum_{n=1}^{\infty} n^{-n}$$

be true? See p. 357.

4. Newton and Secant Methods. We may paraphrase the foregoing equation (6) as follows: each of three numbers x_{n-r}, x_n and x_{n+r} is the quotient of consecutive Fibonacci numbers, with denominators respectively u_{n-r}, u_n and u_{n+r}, and the resulting Aitken number is itself the quotient of consecutive Fibonacci numbers. This beautiful result encourages us to check whether any other iterative methods of solving the equation $x^2 - x - 1 = 0$ yield sequences of quotients of Fibonacci numbers.

Let us consider Newton's method: given an initial approximation t_0 to the golden number, we compute the Newton sequence (t_n) from

$$t_{n+1} = t_n - \frac{(t_n^2 - t_n - 1)}{2t_n - 1}, \quad n \geqslant 0.$$

If $t_n = u_{k+1}/u_k$ for some value of k, we readily find that $t_{n+1} = u_{2k+1}/u_{2k}$. Thus, if we choose as the initial approximant $t_0 = u_2/u_1 = 1$, we see by induction that

$$t_n = u_{2^n+1}/u_{2^n}. \tag{15}$$

We now turn to the secant method, in which we approximate to a root of the equation $f(x) = 0$ as follows: we choose two initial approximants t_1 and t_2 and compute the sequence (t_n) from

$$t_{n+1} = t_n - \frac{f(t_n)(t_n - t_{n-1})}{f(t_n) - f(t_{n-1})}, \quad n \geqslant 2.$$

(The reader will recall that the secant method is transmuted into Newton's method on replacing the difference quotient

$$(f(t_n) - f(t_{n-1}))/(t_n - t_{n-1})$$

by the derivative $f'(t_n)$.) In this case $f(x) = x^2 - x - 1$ and, if we choose $t_1 = u_{p+1}/u_p$ and $t_2 = u_{q+1}/u_q$ for some $p \neq q$, we find that $t_3 = u_{p+q+1}/u_{p+q}$. An induction argument shows that, if we choose as initial values $t_1 = 1$ and $t_2 = 2$, the secant method gives

$$t_n = u_{u_{n+1}+1}/u_{u_{n+1}}$$

as the nth approximant to the golden number.

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ANSWER TO QUESTION ON PAGE 352

Yes, it's true. It appears in G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, vol. 1, Springer, Berlin, 1972 (Part I, Chapter 4, no. 160).

ANSWER TO PHOTO ON PAGE 353

A. N. Tihonov or Tychonoff depending on your transliterational convictions. Best known for his theorem about compact product spaces with many factors.



His products are big. (See p. 357.)

4. Newton and Secant Methods. We may paraphrase the foregoing equation (6) as follows: each of three numbers x_{n-r}, x_n and x_{n+r} is the quotient of consecutive Fibonacci numbers, with denominators respectively u_{n-r}, u_n and u_{n+r}, and the resulting Aitken number is itself the quotient of consecutive Fibonacci numbers. This beautiful result encourages us to check whether any other iterative methods of solving the equation $x^2 - x - 1 = 0$ yield sequences of quotients of Fibonacci numbers.

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$$t_n = u_{u_{n+1}+1}/u_{u_{n+1}}$$

as the nth approximant to the golden number.

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ANSWER TO PHOTO ON PAGE 353

A. N. Tihonov or Tychonoff depending on your transliterational convictions. Best known for his theorem about compact product spaces with many factors.

AITKEN SEQUENCES AND FIBONACCI NUMBERS

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1. Introduction. Some years ago I observed that if Aitken acceleration is applied to the sequence (x_n) defined by

$$x_{n+1} = 1 + 1/x_n, \quad \text{with } x_1 = 1, \tag{1}$$

the resulting Aitken sequence (x_n^*) satisfies the simple relation

$$x_n^* = x_{2n}. \tag{2}$$

Table 1 displays the first few members of the sequences (x_n) and (x_n^*), where

$$x_n^* = \frac{x_{n+1}x_{n-1} - x_n^2}{x_{n+1} - 2x_n + x_{n-1}}, \quad n \geqslant 2. \tag{3}$$

Table 1. Aitken acceleration.

n	x_n	x_n^*
1	1.000 000	
2	2.000 000	1.666 667
3	1.500 000	1.625 000
4	1.666 667	1.619 048
5	1.600 000	1.618 181
6	1.625 000	1.618 056
7	1.615 385	1.618 037
8	1.619 048	

An account of Aitken acceleration may be found in any introductory text on numerical analysis. (From a wider viewpoint, Aitken acceleration can be regarded as part of Wynn's ε-algorithm. See, for example, Brezinski [**1**].)

It is easily verified by induction that

$$x_n = u_{n+1}/u_n, \quad n \geqslant 1, \tag{4}$$

where the u_n are the Fibonacci numbers, defined by

$$u_{n+1} = u_n + u_{n-1}, \quad n \geqslant 2, \tag{5}$$

and $u_1 = u_2 = 1$. Thus the limit of the sequence (x_n) is

$$\lim_{n\to\infty} u_{n+1}/u_n = \tfrac{1}{2}(1 + \sqrt{5}),$$

which is the "golden number," the positive root of the equation $x^2 - x - 1 = 0$. See, for example, Vorob'ev [**3**].

In §2 we verify (2), as a special case of a more general result, using certain properties of the Fibonacci numbers. Then in §3 we examine the effect of applying Aitken acceleration repeatedly to our original sequence (x_n). Finally, in §4 we also find that the Fibonacci numbers give rise to equally nice results when either Newton's method or the secant method is used to compute the golden number iteratively.

G. M. Phillips: My first degree and master's degree were completed in Aberdeen, Scotland, followed by a few years abroad (in England), where I taught at the University of Southampton. I returned to my native land in 1967 to teach in Scotland's oldest university and to pursue my main interests of numerical analysis and approximation theory. My remedy for the midlife crisis has been to continue with mathematics and to take up marathon (26 miles and 385 yards) running. If my progress over the past two years in the latter activity can be maintained, I will match the current world best performance by the age of 95.

2. Aitken Acceleration. In this section we establish the result

$$\frac{x_{n+r}x_{n-r} - x_n^2}{x_{n+r} - 2x_n + x_{n-r}} = x_{2n}, \quad 1 \leqslant r < n, \tag{6}$$

where (x_n) is the sequence defined by (1). The special case of (6) with $r = 1$ then verifies (2). The full generality of (6) will be employed in §3.

First we require three lemmas.

LEMMA 1. *For all* $n > m \geqslant 1$,

$$u_{n+m}u_{n-m} - u_n^2 = (-1)^{n+m-1}u_m^2. \tag{7}$$

LEMMA 2. *For all* $n > m \geqslant 1$,

$$u_{n+1}u_m - u_nu_{m+1} = (-1)^{m+1}u_{n-m}. \tag{8}$$

LEMMA 3. *For all* $n \geqslant 1$ *and* $m \geqslant 2$,

$$u_{n+1}u_m + u_nu_{m-1} = u_{n+m}. \tag{9}$$

Proof. Induction. Lemmas 1 and 3 are both quoted in Hoggath [2]. Note that Lemma 1 with $m = 1$ and Lemma 2 with $m = n - 1$ both reduce to the more familiar result

$$u_{n+1}u_{n-1} - u_n^2 = (-1)^n.$$

We also require one further Fibonacci identity which follows readily from the addition of (8) and (9):

$$u_{n+m} + (-1)^{m+1}u_{n-m} = u_m(u_{n+1} + u_{n-1}). \tag{10}$$

Before leaving the lemmas above we remark that if the definition of the Fibonacci number u_n is extended to integers n less than 1 in an obvious way, so that the recurrence relation (5) holds for all integers n, then $u_{-n} = (-1)^{n-1}u_n$ and the two formulas (8) and (9) are seen to be the same.

We now express the numerator in (6) in terms of the Fibonacci numbers. We obtain

$$x_{n+r}x_{n-r} - x_n^2 = \frac{u_{n+r+1}u_{n-r+1}u_n^2 - u_{n+r}u_{n-r}u_{n+1}^2}{u_{n+r}u_{n-r}u_n^2}. \tag{11}$$

The numerator in (11) may be replaced by

$$\left(u_{n+r+1}u_{n-r+1} - u_{n+1}^2\right)u_n^2 - \left(u_{n+r}u_{n-r} - u_n^2\right)u_{n+1}^2$$

and, on using Lemma 1 followed by Lemma 3, the latter expression becomes

$$(-1)^{n+r}u_r^2\left(u_n^2 + u_{n+1}^2\right) = (-1)^{n+r}u_r^2u_{2n+1}.$$

Thus

$$x_{n+r}x_{n-r} - x_n^2 = (-1)^{n+r}\frac{u_r^2u_{2n+1}}{u_{n+r}u_{n-r}u_n^2}. \tag{12}$$

For the denominator in (6) we first obtain

$$\begin{aligned} x_n - x_{n-r} &= \frac{u_{n+1}u_{n-r} - u_{n-r+1}u_n}{u_nu_{n-r}} \\ &= (-1)^{n+r-1}\frac{u_r}{u_nu_{n-r}}, \end{aligned}$$

on using Lemma 2 with $m = n - r$. We deduce that

$$x_{n+r} - 2x_n + x_{n-r} = (-1)^{n+r}\frac{u_r}{u_{n+r}u_n u_{n-r}}\left(u_{n+r} + (-1)^{r-1}u_{n-r}\right)$$

and, on using (10), we obtain

$$x_{n+r} - 2x_n + x_{n-r} = (-1)^{n+r}\frac{u_r^2(u_{n+1} + u_{n-1})}{u_{n+r}u_n u_{n-r}}. \tag{13}$$

Finally, on combining (12) and (13), we derive the required result

$$\frac{x_{n+r}x_{n-r} - x_n^2}{x_{n+r} - 2x_n + x_{n-r}} = x_{2n}.$$

3. Repeated Acceleration. We began with a sequence (x_n) and applied Aitken acceleration to give a sequence (x_n^*). We now investigate the application of Aitken acceleration to (x_n^*), and so on, repeatedly. It is helpful to use a different notation. Let us write

$$x_n^{(k+1)} = \frac{x_{n+1}^{(k)}x_{n-1}^{(k)} - \left(x_n^{(k)}\right)^2}{x_{n+1}^{(k)} - 2x_n^{(k)} + x_{n-1}^{(k)}}, \tag{14}$$

for $k = 0, 1, \ldots$, where $x_n^{(0)} = x_n$. Thus $(x_n^{(0)})$ is our original sequence and $(x_n^{(1)})$ is the sequence which we have hitherto called (x_n^*). For $k \geqslant 0$, the $(k+1)$th sequence $(x_n^{(k+1)})$ is obtained by using Aitken acceleration on the kth sequence $(x_n^{(k)})$. Note that, just as we saw earlier that our (first) accelerated sequence $(x_n^{(1)})$ is defined for $n = 2$ onwards, the kth accelerated sequence $(x_n^{(k)})$ is defined for $n = k + 1$ onwards.

We have already seen from (6) that

$$x_n^{(1)} = x_{2n}.$$

It follows from this and (14) that

$$x_n^{(2)} = \frac{x_{n+1}^{(1)}x_{n-1}^{(1)} - \left(x_n^{(1)}\right)^2}{x_{n+1}^{(1)} - 2x_n^{(1)} + x_{n-1}^{(1)}} = \frac{x_{2n+2}x_{2n-2} - x_{2n}^2}{x_{2n+2} - 2x_{2n} + x_{2n-2}}.$$

Using (6), with n replaced by $2n$ and with $r = 2$, we deduce that

$$x_n^{(2)} = x_{4n}.$$

Finally, it follows by induction that

$$x_n^{(k)} = x_{n\cdot 2^k},$$

which holds for each $k \geqslant 0$ and all $n \geqslant k + 1$. To illustrate this, Table 2 shows the members of the sequences $(x_n^{(k)})$ which may be computed beginning with the elements $x_1, \ldots, x_7$ of the original sequence. All of the numbers in Table 2 are approximations to $\frac{1}{2}(1 + \sqrt{5})$, the limit of the sequence (x_n).

Table 2. Repeated acceleration.

x_1			
x_2	$x_2^{(1)}$		
x_3	$x_3^{(1)}$	$x_3^{(2)}$	
x_4	$x_4^{(1)}$	$x_4^{(2)}$	$x_4^{(3)}$
x_5	$x_5^{(1)}$	$x_5^{(2)}$	
x_6	$x_6^{(1)}$		
x_7			

If we begin with the elements $x_1, \ldots, x_{2k+1}$ of the original sequence, and carry out k repeated Aitken accelerations, the "final number" (corresponding to $x_4^{(3)}$ in Table 2) will be $x_{(k+1)\cdot 2^k}$.

4. Newton and Secant Methods. We may paraphrase the foregoing equation (6) as follows: each of three numbers x_{n-r}, x_n and x_{n+r} is the quotient of consecutive Fibonacci numbers, with denominators respectively u_{n-r}, u_n and u_{n+r}, and the resulting Aitken number is itself the quotient of consecutive Fibonacci numbers. This beautiful result encourages us to check whether any other iterative methods of solving the equation $x^2 - x - 1 = 0$ yield sequences of quotients of Fibonacci numbers.

Let us consider Newton's method: given an initial approximation t_0 to the golden number, we compute the Newton sequence (t_n) from

$$t_{n+1} = t_n - \frac{(t_n^2 - t_n - 1)}{2t_n - 1}, \quad n \geqslant 0.$$

If $t_n = u_{k+1}/u_k$ for some value of k, we readily find that $t_{n+1} = u_{2k+1}/u_{2k}$. Thus, if we choose as the initial approximant $t_0 = u_2/u_1 = 1$, we see by induction that

$$t_n = u_{2^n+1}/u_{2^n}. \tag{15}$$

We now turn to the secant method, in which we approximate to a root of the equation $f(x) = 0$ as follows: we choose two initial approximants t_1 and t_2 and compute the sequence (t_n) from

$$t_{n+1} = t_n - \frac{f(t_n)(t_n - t_{n-1})}{f(t_n) - f(t_{n-1})}, \quad n \geqslant 2.$$

(The reader will recall that the secant method is transmuted into Newton's method on replacing the difference quotient

$$(f(t_n) - f(t_{n-1}))/(t_n - t_{n-1})$$

by the derivative $f'(t_n)$.) In this case $f(x) = x^2 - x - 1$ and, if we choose $t_1 = u_{p+1}/u_p$ and $t_2 = u_{q+1}/u_q$ for some $p \neq q$, we find that $t_3 = u_{p+q+1}/u_{p+q}$. An induction argument shows that, if we choose as initial values $t_1 = 1$ and $t_2 = 2$, the secant method gives

$$t_n = u_{u_{n+1}+1}/u_{u_{n+1}}$$

as the nth approximant to the golden number.

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ANSWER TO QUESTION ON PAGE 352

Yes, it's true. It appears in G. Pólya and G. Szegő, *Problems and Theorems in Analysis*, vol. 1, Springer, Berlin, 1972 (Part I, Chapter 4, no. 160).

ANSWER TO PHOTO ON PAGE 353

A. N. Tihonov or Tychonoff depending on your transliterational convictions. Best known for his theorem about compact product spaces with many factors.

PROGRESS REPORTS

EDITED BY THOMAS BANCHOFF AND RICHARD MILLMAN

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

EMPIRICAL MEASURES

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The estimation of unknown probabilities forms a theme central to much statistical thinking: its importance is paramount in the study of pattern recognition and nonparametric statistics, and in one form or another, the subject has occupied the attention of mathematicians from the time of Laplace. This report will discuss a few of the major advances in this area over the past two decades.

A *probability measure* on a set X (the "sample space") is an assignment to certain subsets C of X of a real number $P(C) \geqslant 0$, read "the probability of C," with these properties:

(1) $P(X) = 1$,
(2) if $C = C_1 \cup C_2 \cup \cdots$ for a sequence of pairwise disjoint sets C_n, then $P(C) = P(C_1) + P(C_2) + \cdots$.

The simplest example is formed by fixing a point x in X and letting $\delta(x)$ be the probability measure whose value on the subset C is 1 if $x \in C$ and 0 otherwise. Another example is length (Lebesgue) measure on the interval $[0, 1]$.

How does a statistician estimate an unknown probability measure P? If a "sample" of n "independent observations" $y_1, \ldots, y_n$ of P is available, then a standard procedure involves the formation of the *empirical measure* P_n defined by

$$P_n(C) = \frac{\delta(y_1)(C) + \cdots + \delta(y_n)(C)}{n},$$

the proportion of the points $y_1, \ldots, y_n$ falling into C. Exactly how well P_n approximates or approaches P as $n \to \infty$ is a matter of primary importance.

At this point, it might be well (although not essential) to represent the situation in the language of probability: for each fixed C, the value $P_n(C)$ is the average of "independent and identically distributed random variables" $\delta(y_1)(C), \ldots, \delta(y_n)(C)$, each of which has an "average" or "expected value" $P(C)$. Without dwelling on the exact interpretation of these phrases, we simply note

that such conditions signal the invocation of the Strong Law of Large Numbers, a result near center stage within the body of probabilistic theory. It implies the following:

(A) *For each fixed* C, $P_n(C) \to P(C)$ *as* $n \to \infty$ *with probability one.*

Notice that phrase "for fixed C." There's something unsatisfying and unanswered here; surely one can do better. Exactly how much better isn't fully known. Perhaps (A) could be strengthened to a convergence of $|P_n(C) - P(C)|$ to zero that is uniform across some class $\mathscr{C}$ of subsets of the space X, as in

(A +) *with probability one,* $\sup\{|P_n(C) - P(C)|: C \in \mathscr{C}\}$ *tends to zero as* $n \to \infty$.

Results such as (A +) have been known for a while: if P is taken to be a probability measure on the line, and if $\mathscr{C}$ is the collection of all intervals of the form $(-\infty, a]$, then result (A +) holds, as was shown by Glivenko and Cantelli in 1933. For this reason, classes of sets satisfying (A +) are called *Glivenko-Cantelli* (GC) *classes*. Proving more general classes $\mathscr{C}$ to have the GC property meets two major obstacles, one of a combinatorial nature, the other measure-theoretic. For an example, suppose that P is the uniform (Lebesgue) measure on $[0,1]$ and that $\mathscr{C}$ is the collection of all finite subsets of $[0,1]$. If $C = \{y_1,\ldots,y_n\}$, then $P_n(C) = 1$, whereas the Lebesgue measure of C is zero. Thus the supremum in (A +) is always unity, and uniform convergence of $P_n(C)$ to $P(C)$ fails miserably.

What's going on? Well, somehow the combinatorial structure of the class of intervals $(-\infty, a]$ is tamer than the corresponding structure for the class of all finite sets. Both classes are uncountable, but the latter collection is in some sense too large. How can this distinction be formulated? The dramatic breakthrough came with the work of the Soviet mathematicians Vapnik and Červonenkis. In their effort to understand GC classes, they discovered a very elegant combinatorial criterion on $\mathscr{C}$ which in most cases implies $\sup|P_n(C) - P(C)| \to 0$ as $n \to \infty$. If $x_1,\ldots,x_n$ are n points of the space X, and $\mathscr{C}$ is a class of subsets, define $\Delta^{\mathscr{C}}(x_1,\ldots,x_n)$ to be the number of different sets of the form $\{x_1,\ldots,x_n\} \cap C$ as $\mathscr{C}$ ranges over $\mathscr{C}$. Clearly $\Delta^{\mathscr{C}}(x_1,\ldots,x_n) \leqslant 2^n$. Define the *growth function* of $\mathscr{C}$ by

$$m^{\mathscr{C}}(n) = \max \Delta^{\mathscr{C}}(x_1,\ldots,x_n),$$

where the maximum is taken over all subsets of X of size n. Vapnik and Červonenkis proved that for a given $\mathscr{C}$, the growth function is either the biggest it can be (identically $m^{\mathscr{C}}(n) = 2^n$ for all n), or else is dominated by a polynomial ($m^{\mathscr{C}}(n) \leqslant n^s + 1$, where s is the least number such that $m^{\mathscr{C}}(s) < 2^s$). In the latter case, $\mathscr{C}$ is termed a *Vapnik-Červonenkis* (VC) *class*, and as the Soviet researchers showed, VC classes are usually GC classes as well.

If $\mathscr{C}$ is the class of intervals $(-\infty, a]$ in the line, then for any two reals $x_1 < x_2$, $\Delta^{\mathscr{C}}(x_1, x_2) = 3$ (the sets ϕ, $\{x_1\}$, and $\{x_1, x_2\}$ are "picked out" by $\mathscr{C}$). Similarly, $m^{\mathscr{C}}(n) = n + 1$, so that $\mathscr{C}$ is a VC class. If $\mathscr{C}$ is the collection of all finite linear sets, then $\Delta^{\mathscr{C}}(x_1,\ldots,x_n) = 2^n$ for any n numbers $x_1,\ldots,x_n$: in this case, $\mathscr{C}$ is not a VC class. The success of (A +) for the former class and its failure for the latter are partly explained by the VC property.

The second principal difficulty occurs when $\mathscr{C}$ is somehow pathological, running against the grain of the ambient measurable structure. For example, assuming the continuum hypothesis, it is possible to well-order the set $[0,1]$ by a relation $<$ such that each section $\{x: x \leqslant a\}$ is countable. Let $\mathscr{C}$ be the collection of all such sections. As in the example of the finite sets, $\sup|P_n(C) - P(C)| = 1$; yet $\mathscr{C}$ *is* a VC class. The problem here lies with the bizarre nature of such a well-ordering, in particular the nonmeasurability of the set $\{(x, y): x < y\}$. The question of measurability is at least partly laid to rest with recent work of Dudley (1978). He proposes to exclude classes $\mathscr{C}$ that cannot be parametrized by a suitably well-behaved subset of $\mathbb{R}$ (or some other metric space) in such a way that if $t \to C_t$ is the parametrization, then $\{(x, t): x \in C_t\}$ is measurable.

Looking back for a moment, we may view these results as attempts to improve upon the Strong

Law of Large Numbers as it appears in (A). A similar line of thought, running quite parallel to this one, substitutes at (A) the central limit theorem and considers formulations such as:

(B) *As a stochastic process,* $\sqrt{n}(P_n - P)$ *converges to a Gaussian process indexed by* $\mathscr{C}$.

A general review of the subject may be found in the survey article of Gaenssler and Stute (1979).

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1. R. M. Dudley, Central limit theorems for empirical measures, Ann. Probab., 6 (1978) 899–929; correction 7 (1979) 909–911.

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NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

A RECURSION ON QUADRUPLES

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Let (a, b, c, d) be a quadruple of real numbers and consider the sequence of the quadruples $P^n(a, b, c, d)$, where $P : \mathbb{R}^4 \mapsto \mathbb{R}^4$ denotes the operation

$$P(a, b, c, d) = (|a - b|, |b - c|, |c - d|, |d - a|).$$

By a well-known problem, if a, b, c, d are integers, then $P^n(a, b, c, d) = (0, 0, 0, 0)$ if n is large enough. (See, e.g., Honsberger: *Ingenuity in Mathematics*, Chapter 10.) If we start from arbitrary real numbers, this will not always be true. Consider the polynomial $p(x) = x^3 + 2x^2 - 2$. The roots of $p'(x) = 0$ are $-4/3$ and 0, and $p(-4/3) < 0$, $p(0) < 0$ while $p(1) > 0$. Therefore p has exactly one real root, λ_0, and $0 < \lambda_0 < 1$. If we start from the quadruple

$$z = \left(1, 1 + \lambda_0, (1 + \lambda_0)^2, (1 + \lambda_0)^3\right), \tag{1}$$

then an immediate computation shows that $Pz = \lambda_0 z$ and therefore $P^n z = \lambda_0^n z$; that is, we will never reach the quadruple $(0, 0, 0, 0)$. A similar case is obtained for any initial quadruple $v = kz$, where $k > 0$.

We will show that for every quadruple (a, b, c, d), except for these quadruples kz above, there exists an n for which $P^n(a, b, c, d) = (0, 0, 0, 0)$. In order to formulate our statement more precisely, we need some notation. If $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, then let

$$lv = (v_2, v_3, v_4, v_1)$$

$$tv = (v_4, v_3, v_2, v_1).$$

Let G be the group generated by l and t. Since $lt = tl^3$ and $t^2 = l^4 = 1$, G has 8 elements:

$$G = \{t^i l^j; \quad i = 0, 1, \quad j = 0, 1, 2, 3\}.$$

Let $u \sim v$ if there is a $g \in G$ for which $u = gv$; this is an equivalence relation. An immediate computation shows that for every $v \in \mathbb{R}^4$

$$Plv = lPv \quad \text{and} \quad Ptv = tl^3 Pv.$$

CENTER SECTION
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Telegraphic Reviews

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books submitted for review should be sent to Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

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General, S?(17-18). Perspectives of Elementary Mathematics. Gerhard P. Hochschild. Springer-Verlag, 1983, vii + 120 pp, $20 (P). [ISBN: 0-387-90848-X] Summary of key definitions and results mostly from algebra and number theory, amplified by exercise sets and programming projects. This reviewer could not determine what the "Perspectives" of the title are, nor what they are on. JRG

General, S. Problem Solving and Comprehension, Third Edition. Arthur Whimbey, Jack Lochhead. Franklin Inst, 1982, v + 343 pp, $10.95 (P) [ISBN: 0-89168-048-9]; Instructor's Guide, 37 pp (P). A collection of intelligence-test-type questions, with extensive discussion, designed to improve performance on tests like the SAT, the GRE, the LSAT. Includes sections on verbal reasoning, analogy, trends and patterns, and mathematical word problems. LCL

General, S?, P? Blackout in Mathematical Manuals. Joseph Burstein. Metrics Pr, 1983, vii + 61 pp, (P). [ISBN: 0-9607126-0-7] Reminiscences of mathematics in the Soviet Union in the first half of the century, with disparaging remarks about present trends in mathematics education (obscure, excessive jargon, etc.). Unfortunately, his "lucid and instantly revealing" examples of how it used to be done are incomprehensible. LCL

General, S(18), P*. Studies in Pure Mathematics: To the Memory of Paul Turán. Ed: Paul Erdős. Birkhauser Boston, 1983, 773 pp, $49.95. [ISBN: 3-7643-1288-2] This volume, written by his friends, collaborators and students, is offered to the memory of Paul Turán. Includes over sixty papers from a wide range of topics in which his influence has been felt. A fitting tribute to a great mathematician. CEC

General. College Mathematics in Texas, 1983-1984, Fourth Edition. Texas Section MAA (College Mathematics in Texas, Math. Dept., U. of Houston, 4800 Calhoun, Houston, TX 77004), 1984, 476 pp, $18 (P). An annotated list of undergraduate mathematics courses offered by all institutions of higher education in Texas, including course descriptions and current textbooks. Intended as an aid in student advising (especially for transfer credits), it is also a useful resource for those engaged in curriculum planning and course revisions. Even those in other states may find the information in this volume useful--for comparison, for ideas, and for contacts. LAS

Precalculus, T(13: 1). Algebra and Trigonometry, Second Edition. Walter Fleming, Dale Varberg. Prentice-Hall, 1984, xiv + 573 pp, $25.95. [ISBN: 0-13-021535-X] Changes from the First Edition (TR, November 1980) include more problems, earlier treatment of squares and cube roots, and addition of "caution boxes" warning students about commonly made errors. Fairly relaxed writing style. BK

History, S(16-18), P, L.** Quantum Theory and Measurement. Ed: John Archibald Wheeler, Wojciech Hubert Zurek. Ser. in Physics. Princeton U Pr, 1983, xxviii + 811 pp, $75; $19.50 (P). [ISBN: 0-691-08325-8; 0-691-08320-7] A magnificent anthology of 49 seminal papers highlighting the major philosophical issues of measurement in quantum physics, from the Bohr-Einstein papers of the '20's to contemporary research in quantum "non-demolition" experiments. Numerous commentaries (also reprinted from various original documents) fill the interstices between sections and provide useful perspective on one of the most fundamental yet perplexing problems of modern science. LAS

Foundations, P. Southeast Asian Conference on Logic. Ed: C.-T. Chong, M.J. Wicks. Stud. in Logic & the Found. of Math., V. 111. Elsevier Sci Pub, 1983, xiv + 210 pp, $38.25. [ISBN: 0-444-86706-6] Proceedings of the conference held at Singapore, November 1981. Recursion theory and model theory are chiefly represented. Most contributors are internationally known. GHM

Graph Theory, P. Lecture Notes in Mathematics-1018: Graph Theory. Ed: M. Borowiecki, J.W. Kennedy, M.M. Syslo. Springer-Verlag, 1983, x + 289 pp, $15 (P). [ISBN: 0-387-12687-2] Proceedings of the International Graph Theory Conference, Łagów, Poland, February 1981. Thirty-five contributed papers representing many areas of graph theory. In addition contains the first English translation of

Kuratowski's classic 1930 paper characterizing nonplanar graphs, the most frequently cited paper in graph theory. GHM

Algebra, P. Near-Rings: The Theory and its Applications, Revised Edition. Günter Pilz. Math. Stud., No. 23. Elsevier Sci Pub, 1983, xv + 470 pp, $59.50 (P). [ISBN: 0-7204-0566-1] Revised edition (First Edition TR, June-July 1977), containing numerous minor additions and corrections, as well as four new chapters. Presents a systematic account of theory of near rings. Rather dry theorem-proof presentation. BK

Algebra, P. Contributions to Lattice Theory. Ed: A.P. Huhn, E.T. Schmidt. North-Holland, 1983, 782 pp, $102. [ISBN: 0-444-86507-1] Proceedings of an August 1980 colloquium held in Szeged, Hungary. Contains 33 research and survey papers, as well as a list of unsolved problems. LAS

Algebra, P. Lecture Notes in Mathematics-1038: Algebra in a Localic Topos with Applications to Ring Theory. Francis Borceux, Gilberte Van den Bossche. Springer-Verlag, 1983, ix + 240 pp, $11 (P). [ISBN: 0-387-12711-9] A Franco-Flemish collaboration in which categorical and topos-theoretic techniques are applied to deduce results in ring theory. Not for the novice. DD

Calculus, S(13-14), L. Elementary Integral Calculus. Delmar C. Crabill, John J. Neitzke. U Pr of America, 1983, viii + 202 pp, $11.50 (P). [ISBN: 0-8191-3516-X] Six pages on the evolution of integration from the method of exhaustion of Eudoxus to the nonstandard analysis of Abraham Robinson, followed by about 160 pages of integral calculus, without application except to area. There are no surprises. Appendices include reviews of trigonometry and differentiation plus programs in BASIC for Riemann sums, the trapezoidal rule and Simpson's rule. Exercises, with answers to about two-thirds. Useful for review but unsuitable as a text in a usual calculus sequence. JK

Differential Equations, P. Nonlinear Partial Differential Equations and Their Applications, Collège de France, Seminar Volume V. Ed: H. Brezis, J.L. Lions. Research Notes in Math., No. 93. Pitman Pub, 1983, 370 pp, $24.95 (P). [ISBN: 0-273-08620-0] Written versions of ten sets of lectures on theory and applications of nonlinear partial differential equations from the 1981-82 Seminar on Applied Mathematics at the Collège de France. Brief abstracts are included. PZ

Differential Equations, P*. Lecture Notes in Mathematics-1037: Non-linear Partial Differential Operators and Quantization Procedures. Ed: S.I. Andersson, H.-D. Doebner. Springer-Verlag, 1983, vii + 334 pp, $15 (P). [ISBN: 0-387-12710-0] Several lectures containing new results on differential geometrical and topological properties of non-linear physical systems given in July 1981 at the Technical University in Clausthal in a summer workshop and a parallel conference on mathematical physics. Edited and up-dated. In two parts: Non-linear partial differential equations, and quantization procedures. JK

Numerical Analysis, P, L. Computational Mathematics: Worked Examples and Problems with Elements of Theory. N.V. Kopchenova, I.A. Maron. MIR Pub, 1981, 395 pp, $6.95. Reprint of 1975 original edition (TR, August-September 1977). A real bargain. RWN

Functional Analysis, P. Spectral Theory of Hyponormal Operators. Daoxing Xia. Operator Theory, V. 10. Birkhauser Verlag, 1983, xiv + 241 pp, $29.95. [ISBN: 3-7643-1541-5] A study of the spectral theory of certain operators which are unitarily equivalent to singular integral operators which are pseudo-differential operators of 0-order. Primary concern is with the operators in this class that are not normal. Related to various problems in classical mechanics, quantum mechanics, etc. DD

Functional Analysis, P. Control and Observation of Neutral Systems. D. Salamon. Research Notes in Math., No. 91. Pitman, 1984, 207 pp, $19.95 (P). [ISBN: 0-273-08618-9] Intended for specialists in functional differential equations. Chapter 2 purports to fill a gap previously described as an open problem in the product space framework of linear functional differential equations, namely the lack of a satisfactory duality theory. AWR

Functional Analysis, S(18), P. Theory of Function Spaces. Hans Triebel. Mono. in Math., V. 78. Birkhauser Boston, 1983, 284 pp, $34.95. [ISBN: 3-7643-1381-1] By dealing with two classes of distribution spaces, it is the author's intent to demonstrate the power of Fourier analysis (using multipliers and maximal functions) in giving a unified treatment applicable to a variety of function spaces including Hölder, Sobolev, and Hardy spaces. Applications to differential equations. References, index. JS

Analysis, P. Lecture Notes in Mathematics-1024: Lie Group Representations I. Ed: R. Herb, R. Lipsman, J. Rosenberg. Springer-Verlag, 1983, ix + 369 pp, $17 (P). [ISBN: 0-387-12725-9] First of three Lecture Notes in Mathematics volumes from the 1982-83 special year in Lie Group Representations at Maryland, this one focused on the algebraic and analytic aspects of semi-simple theory. LAS

Geometry, P. Convexity and Its Applications. Ed: Peter M. Gruber, Jörg M. Wills. Birkhauser Boston, 1983, 421 pp, $59.95. [ISBN: 3-7643-1384-6] A collection of 16 surveys, many of them written by authors whose names are synonomous with convexity. Many such collections, purporting to be overviews of a field, turn out to be quite narrow articles focused around the work of the writer. This book is different. The articles really do survey, and so the book would be a valuable addition to any library used by workers in the field. AWR

Topology, T(15-16: 1), L.** Basic Topology. M.A. Armstrong. Undergrad. Texts in Math. Springer-Verlag, 1983, xii + 251 pp, $19.80. [ISBN: 0-387-90839-0] Reprint, with minor corrections, of a 1979 text published in the United Kingdom by McGraw-Hill. An elegant, inviting, undergraduate introduction, emphasizing geometric insights in a remarkably wide variety of contexts: Euler's theorem, point set theory, fundamental groups, triangulations, simplical homology, knots, fixed point theorems, and more. LAS

Operations Research, P. Progress in Modelling and Simulation. Ed: Francois E. Cellier. Academic Pr, 1982, x + 466 pp, $55.50. [ISBN: 0-12-164780-3] Collection of invited lectures presented during the third IASTED conference on simulation techniques, describing recent research efforts in modelling and simulation. JRG

Optimization, T(15-17: 1), L. Linear Programming. Katta G. Murty. Wiley, 1983, xix + 482 pp, $37.95. [ISBN: 0-471-09725-X] An extensively revised edition of the author's Linear and Combinatorial Programming. Much new material has been added to make the book complete and comprehensive. AO

Dynamical Systems, S(17). Nonnegative Matrices in Dynamic Programming. W.H.M. Zijm. Math. Centre Tracts, No. 167. Math Centrum, 1983, ii + 186 pp, Dfl. 25,30 (P). [ISBN: 90-6196-264-1] This version of a doctoral thesis studies dynamic programming models in which the transition law is specified by a set of nonnegative matrices. It necessarily contains a lot of information about such matrices which is itself a convenient summary. AWR

Statistics, S(14). Graphical Methods for Data Analysis. John M. Chambers, et al. Wadsworth, 1983, xiv + 395 pp, $13 (P). [ISBN: 0-534-98052-X] This book presents graphical methods for analyzing data, including scatterplots, box plots, histograms, stem-and-leaf diagrams, plots for multivariate data, plots to assess distributional assumptions, and plots for developing and assessing regression models. The book assumes knowledge of elementary statistics and an ability to think quantitatively. MT

Statistics, P. Lecture Notes in Control and Information Sciences-57: Instrumental Variable Methods for System Identification. T. Söderström, P.G. Stoica. Springer-Verlag, 1983, vii + 243 pp, $16 (P). [ISBN: 0-387-12814-X] Treatise on "instrumental variable methods" (IVM) for parameter estimation in dynamical (linear) models. IVM tries to forge a compromise between the algorithmically simple least-squares methods and the bias-free but computationally intensive prediction error methods. Of importance to control engineering, econometrics, signal processing and other applications involving stochastic systems and time series analysis. GHM

Statistics, P. Lecture Notes in Mathematics-1021: Probability Theory and Mathematical Statistics. Ed: K. Itô, J.V. Prokhorov. Springer-Verlag, 1983, viii + 747 pp, $30 (P). [ISBN: 0-387-12718-6] Proceedings of a USSR-Japan symposium held at Tbilisi, USSR, August 1982. Contains 73 contributed papers. LAS

Computer Literacy, S. CP/M Primer, Second Edition. Stephen M. Murtha, Mitchell Waite. Howard W Sams & Co, 1983, 150 pp, $16.95 (P). [ISBN: 0-672-22170-5] Minor revision of the best-selling 1980 original edition, stressing simplicity over comprehensive detail. An excellent first book on CP/M. LAS

Computer Programming, T?, S(13-16), L. Programming the 6502, Third Edition. Rodnay Zaks. Sybex, 1980, ix + 386 pp, $14.95 (P) [ISBN: 0-89588-046-6]; 6502 Applications, 1979, 278 pp, $13.95 (P) [ISBN: 0-89588-015-6]; Advanced 6502 Programming, 1982, viii + 292 pp, $13.95 (P). [ISBN: 0-89588-089-X] The first volume is a rather standard text for assembly language programming, including necessary background on binary arithmetic and data representation. This book, however, is seriously flawed as a general text by being very chip-specific. The latter two volumes consist primarily of discussions of a wide variety of sample programs. Thus, while this drawn-out three-volume format is slow enough for beginners, its narrowness seems more appropriate for a processor handbook. The books could be handy as a reference. JAS

Computer Programming, S(13). 8080/8085 Assembly Language Subroutines. Lance A. Leventhal, Winthrop Saville. Osborne/McGraw-Hill, 1983, xii + 476 pp, $15.95 (P). [ISBN: 0-931988-58-6] Written as a reference for assembly language programmers using an 8080/8085-based computer, this book presents an overview of assembly language programming and computer code for subroutines including arithmetic, bit manipulation, code conversion, sorting, and searching. MT

Computer Programming, T(13), S. Microcomputer Programming with Microsoft BASIC. Robert G. Crawford, David T. Barnard. Reston Pub, 1984, xv + 407 pp, $15.95 (P); $20.95. [ISBN: 0-8359-4356-9; 0-8359-4357-7] A careful, comprehensive introduction to programming in Basic, beginning with algorithms and concluding with a wide variety of applications (e.g., sorting, linear equations, linked lists). Uses a major subset of Microsoft Basic, and includes in an appendix special syntax details for Apple, Pet, TRS-80, and IBM PC Basics. LAS

Computer Programming, S(13-14). Graphics Primer for the IBM PC. Mitchell Waite, Christopher L. Morgan. Osborne/McGraw-Hill, 1983, vi + 440 pp, $21.95 (P). [ISBN: 0-931988-99-3] A thorough introduction to the graphics commands of IBM Basic Version 2.0 (including painting, rotation, scale, and animation). Illustrated in full color, with dozens of sample programs. LAS

Computer Science, T(15: 1). Fundamentals of Systems Analysis, Second Edition. Jerry FitzGerald, Ardra F. FitzGerald, Warren D. Stallings, Jr. Wiley, 1981, xvii + 590 pp, $30.95. [ISBN: 0-471-04968-9] Introductory chapters on feasibility studies followed by thorough discussion (in ten chapters) of the ten steps of computer systems analysis, from problem definition, through system design, and implementation. A third section covers the tools of systems analysis, e.g., charting, records retention, procedure writing. Each chapter includes a substantial bibliography, questions for the student, and situation cases for discussion. JRG

Computer Science, S(14-16). Interfacing to the IBM Personal Computer. Lewis C. Eggebrecht. Howard W Sams & Co, 1983, 246 pp, $15.95 (P). [ISBN: 0-672-22027-X] The book is serious about its topic; it serves as a go-between for the PC technical manuals and the serious user of the PC electronics. Provides an exegesis of the bus, timing, peripheral cards and the like together with chip layouts for useful circuits (chips only, not other components). Essentially no programming, just hardware and digital logic considerations. Reasonably well written (some grammatical errors), with much of its information valuable for a wider range of machines. Unfortunately the author doesn't bring this out. Good index. JAS

Computer Science, P. Lecture Notes in Computer Science-159: CAAP '83. Ed: G. Ausiello, M. Protasi. Springer-Verlag, 1983, vi + 416 pp, $18 (P). [ISBN: 0-387-12727-5] Proceedings of the 8th colloquium on trees in algebra and programming held in March, 1983 in L'Aquila, Italy. Five invited lectures plus 21 selected short papers. LAS

Computer Science, L.** Encyclopedia of Computer Science and Engineering, Second Edition. Ed: Anthony Ralston, Edwin D. Reilly, Jr. Van Nostrand Reinhold, 1983, xxix + 1664 pp, $87.50. [ISBN: 0-442-24496-7] A significant revision and enlargement of the 1976 first edition: 40% of the text is new, including nearly 100 new articles. Covers hardware, computer systems, information and data, software, mathematics and theory of computing, and applications. Appendices cover abbreviations and acronyms, journals, departments, high-level languages, and a glossary of major terms in English, French, German, Spanish and Russian. LAS

Control Theory, T(16-17: 1), L. Elements of Computer Process Control With Advanced Control Applications. Pradeep B. Deshpande, Raymond H. Ash. Prentice-Hall, 1983, xiv + 382 pp, $47.95 (P). [ISBN: 0-13-264093-7] Introduction to direct digital process control (computer simulation of analog controller using sampled inputs, Z-transforms replacing the Laplace transform) including some advanced applications. Assumes knowledge of linear control theory and Laplace transforms, but the introductory chapters should suffice for this. RM

Systems Theory, P. Adequate Modeling of Systems. Ed: Horst Wedde. Springer-Verlag, 1983, xi + 336 pp, $25. [ISBN: 0-387-12567-1] Proceedings of the International Working Conference on Model Realism, Bad Honnef, 1982. Considers the ways in which different methodologies deal adequately with large scale real systems. Papers are divided into three problem studies: decentralized control in a technical system; reorganization of a socio-technical system; a balanced system of social groups in a new suburb. JRG

Applications (Biology), P. Lecture Notes in Biomathematics-52: Population Biology. Ed: H.I. Freedman, C. Strobeck. Springer-Verlag, 1983, xvii + 440 pp, $24 (P). [ISBN: 0-387-12677-5] Proceedings of a July 1981 conference held at the University of Alberta. Papers (both invited and contributed) are organized into eight chapters. LAS

Applications (Engineering), S(16), L. The Finite Element Method: A Basic Introduction, Second Edition. K.C. Rockey, et al. Halsted Pr, 1983, x + 239 pp, $17.95 (P). [ISBN: 0-470-27459-X] Written for engineers, the text assumes no familiarity with the finite-element method or with matrices. Starts with simple skeletal structures and advances to continuous structures, approximated by finite elements. Includes a chapter on programming the finite element method. Briefly discusses the use of isoparametric and higher-order elements. Clearly written, with many helpful diagrams. BH

Applications (Engineering), S(16). The Mathematical Theory of Plasticity. R. Hill. Engineering Sci. Ser. Oxford U Pr, 1983, ix + 355 pp, $26.95 (P). [ISBN: 0-19-856162-8] Written to attract engineers and applied mathematicians to the study of plastic straining of metals. Readers are expected to have an acquaintance with the theory of elasticity. Mathematical demands are not heavy, and the content offers nice insights into engineering applications. AWR

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Law of Large Numbers as it appears in (A). A similar line of thought, running quite parallel to this one, substitutes at (A) the central limit theorem and considers formulations such as:

(B) *As a stochastic process,* $\sqrt{n}(P_n - P)$ *converges to a Gaussian process indexed by* $\mathscr{C}$.

A general review of the subject may be found in the survey article of Gaenssler and Stute (1979).

References

1. R. M. Dudley, Central limit theorems for empirical measures, Ann. Probab., 6 (1978) 899–929; correction 7 (1979) 909–911.

2. P. Gaenssler and W. Stute, Empirical processes: a survey of the results for independent and identically distributed random variables, Ann. Probab., 7 (1979) 193–243.

NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

A RECURSION ON QUADRUPLES

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Let (a, b, c, d) be a quadruple of real numbers and consider the sequence of the quadruples $P^n(a, b, c, d)$, where $P: \mathbb{R}^4 \mapsto \mathbb{R}^4$ denotes the operation

$$P(a, b, c, d) = (|a - b|, |b - c|, |c - d|, |d - a|).$$

By a well-known problem, if a, b, c, d are integers, then $P^n(a, b, c, d) = (0, 0, 0, 0)$ if n is large enough. (See, e.g., Honsberger: *Ingenuity in Mathematics*, Chapter 10.) If we start from arbitrary real numbers, this will not always be true. Consider the polynomial $p(x) = x^3 + 2x^2 - 2$. The roots of $p'(x) = 0$ are $-4/3$ and 0, and $p(-4/3) < 0$, $p(0) < 0$ while $p(1) > 0$. Therefore p has exactly one real root, λ_0, and $0 < \lambda_0 < 1$. If we start from the quadruple

$$z = \left(1, 1 + \lambda_0, (1 + \lambda_0)^2, (1 + \lambda_0)^3\right), \tag{1}$$

then an immediate computation shows that $Pz = \lambda_0 z$ and therefore $P^n z = \lambda_0^n z$; that is, we will never reach the quadruple $(0, 0, 0, 0)$. A similar case is obtained for any initial quadruple $v = kz$, where $k > 0$.

We will show that for every quadruple (a, b, c, d), except for these quadruples kz above, there exists an n for which $P^n(a, b, c, d) = (0, 0, 0, 0)$. In order to formulate our statement more precisely, we need some notation. If $v = (v_1, v_2, v_3, v_4) \in \mathbb{R}^4$, then let

$$lv = (v_2, v_3, v_4, v_1)$$

$$tv = (v_4, v_3, v_2, v_1).$$

Let G be the group generated by l and t. Since $lt = tl^3$ and $t^2 = l^4 = 1$, G has 8 elements:

$$G = \{t^i l^j; \quad i = 0, 1, \quad j = 0, 1, 2, 3\}.$$

Let $u \sim v$ if there is a $g \in G$ for which $u = gv$; this is an equivalence relation. An immediate computation shows that for every $v \in \mathbb{R}^4$

$$Plv = lPv \quad \text{and} \quad Ptv = tl^3 Pv.$$

This implies that

$$u \sim v \Rightarrow Pu \sim Pv. \tag{2}$$

THEOREM. *With "essentially one" exception, for every $v \in \mathbb{R}^4$ we have $P^n v = 0$ if n is large enough. More precisely, $P^n v \neq 0$ $(n = 1, 2, \ldots)$ if and only if there is a $k > 0$ such that*

$$Pv \sim k \cdot z,$$

where z is the quadruple in (1). *In this case $P^n v \sim \lambda_0^{n-1} \cdot k \cdot z$ $(n = 1, 2, \ldots)$.*

Proof. Denote by Av the nondecreasing permutation of v, where $v \in \mathbb{R}^4$.

LEMMA. *If $P^6 v \neq 0$, then $v \sim Av$ and the elements of v are different.*

The lemma is proved indirectly by showing that $P^6 v = 0$ whenever v has two equal elements or when $Av \not\sim v$. It is easy to see that $P^4(t, p, t, q) = 0$ for any $p, q, t \in \mathbb{R}$. Thus it is sufficient to show that whenever v has two equal elements or when $Av \not\sim v$, then $P^2 v$ has two opposite elements that are equal:

$$P^2 v = (t, p, t, q) \text{ or } (p, t, q, t) \text{ (which is } l(t, p, t, q)),$$

which is summarized by saying

$$P^2 v \sim (t, p, t, q) \quad \text{for suitable } p, q, t.$$

Let r and s denote the permutations, which carry $v = (v_1, v_2, v_3, v_4)$ to $rv = (v_1, v_2, v_4, v_3)$ and $sv = (v_1, v_3, v_2, v_4)$, respectively. The permutations r, s and rs^{-1} carry two opposite elements (that is, 1st and 3rd or 2nd and 4th) to two adjacent ones, while the elements of G preserve the opposition. It follows that G, Gr and Gs are disjoint, and therefore their union constitutes the whole 24 permutations of S_4.

Thus, letting $Av = (a, b, c, d)$, each permutation of (a, b, c, d) which includes v is equivalent to one of Av, rAv, sAv. Thus, on the one hand, if $v \not\sim Av$, then $v \sim rAv$ or $v \sim sAv$. On the other hand, suppose v has two equal elements. If it is the case that $a = b$ or $c = d$, it is easy to see that $Av \sim rAv$, and when $b = c$, that $Av = sAv$. Because $a \leqslant b \leqslant c \leqslant d$, the other three possible cases $(a = c, a = d, b = d)$ each implies one of the three just discussed. We conclude that whenever $v \not\sim Av$ or v has two equal elements, we have $v \sim rAv$ or $v \sim sAv$. A simple computation shows that both $P^2 rAv$ and $P^2 sAv$ have two opposite elements that are equal, completing the proof. (For example,

$$PrAv = P(a, b, d, c) = (b - a, d - b, d - c, c - a),$$

and thus in $P^2 rAv$ both of the 2nd and 4th elements equal $c - b$.)

Now we turn to the proof of the theorem. It is sufficient to prove the "only if" part of the statement since we have already discussed the exceptional case in which $P^n v \sim \lambda_0^{n-1} kz$ for v such that $Pv \sim kz$, $k > 0$.

Suppose that for a fixed $v_0 \in \mathbb{R}^4$ we have $P^n v_0 \neq 0$ for each n. Let $AP^n v_0$ be denoted by (a_n, b_n, c_n, d_n). Then by the lemma, a_n, b_n, c_n, d_n are different, and for $n \geqslant 1$ therefore $|a_{n-1} - b_{n-1}| = a_n > 0$. Also, for all n we have $P^{n+6} v_0 \neq 0$, implying that $P^n v_0 \sim AP^n v_0$, and, using "$u \sim v \Rightarrow Pu \sim Pv$", we obtain

$$\begin{aligned}(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) &= AP^{n+1} v_0 \sim P^{n+1} v_0 \sim PAP^n v_0 \\ &= (b_n - a_n, c_n - b_n, d_n - c_n, d_n - a_n),\end{aligned} \tag{3}$$

which we denote by $(\alpha, \beta, \gamma, \delta)$. That is to say, for such a quadruple v_0 $(\alpha, \beta, \gamma, \delta)$ and $(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1})$ are arrangements of the same four numbers of which d_{n+1} is the greatest. Since $\delta = \alpha + \beta + \gamma$ is also the greatest number, we have $d_{n+1} = \delta$, that is

$$d_{n+1} = d_n - a_n \quad \text{for all } n. \tag{4}$$

Since α, β, γ are merely $a_{n+1}, b_{n+1}, c_{n+1}$ in some order, then $\delta = \alpha + \beta + \gamma$ yields $d_{n+1} =$

$a_{n+1} + b_{n+1} + c_{n+1}$ for all n or

$$d_n = a_n + b_n + c_n \quad \text{for } n \geqslant 1. \tag{5}$$

Since " $\sim$ " preserves opposition, the elements opposite the equal d_{n+1} and δ must be the same, and we have $b_{n+1} = \beta = c_n - b_n$. Finally, for $n \geqslant 1$ it follows from (5) that $\gamma = d_n - c_n = a_n + b_n > b_n - a_n = \alpha$, and therefore it must be that the smaller of α, γ is a_{n+1} and the larger c_{n+1}: $a_{n+1} = b_n - a_n$ and $c_{n+1} = a_n + b_n$. Thus

$$(a_{n+1}, b_{n+1}, c_{n+1}, d_{n+1}) = (b_n - a_n, c_n - b_n, a_n + b_n, d_n - a_n),$$

and we have $h_{n+1} = Th_n$ for $n \geqslant 1$ where $h_n = (a_n, b_n, c_n) \in \mathbb{C}^3$ and

$$T = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

By (4) $d_1 > d_2 > \cdots > d_n > c_n > b_n > a_n$ and therefore

$$\|T^n h_1\|^2 = a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 < 3d_1^2 \tag{6}$$

for all n.

Computing the characteristic polynomial of T, we obtain $p(x) = x^3 + 2x^2 - 2$. We saw that p has exactly one real root, λ_0, and $0 < \lambda_0 < 1$. Denote by λ_1, λ_2 the complex roots of p; then $\lambda_2 = \bar{\lambda}_1$ and, since the constant member of p is -2, thus $2 = \lambda_0\lambda_1\lambda_2 = \lambda_0|\lambda_1|^2$, from which it follows that $|\lambda_1| = |\lambda_2| > 1$, for $0 < \lambda_0 < 1$.

Let w_0, w_1, w_2 be the eigenvectors of T in $\mathbb{C}^3$ corresponding to the eigenvalues $\lambda_0, \lambda_1, \lambda_2$, respectively; we can choose $w_0 = (1, 1 + \lambda_0, (1 + \lambda_0)^2)$. Since the λ_i are different, the w_i are independent, and therefore there exist $\alpha_i \in \mathbb{C}$ such that

$$h_1 = \sum_{i=0}^{2} \alpha_i w_i.$$

Since w_i are independent, each of them has positive distance from the subspace generated by the other two. Therefore $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$ would imply $\|T^n h_1\| \to \infty$ (recall that $|\lambda_1| = |\lambda_2| > 1$). But by (6) $\|T^n h_1\|$ is bounded. Consequently $\alpha_1 = \alpha_2 = 0$ and $h_1 = \alpha_0 w_0$. Since both h_1 and w_0 have only positive elements, we have $\alpha_0 \in \mathbb{R}$ and $\alpha_0 > 0$.

Finally, from $d_1 = a_1 + b_1 + c_1$, we obtain $d_1 = \alpha_0(1 + \lambda_0)^3$ (using $\lambda_0^3 + 2\lambda_0^2 - 2 = 0$). Thus we have $Pv_0 \sim APv_0 = \alpha_0 z$ and the proof is complete.

HELLY'S THEOREM WITH VOLUMES

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This paper establishes a Helly type result for convex sets in Euclidean d-dimensional space $\mathbb{R}^d$. For many other Helly type results consult [**1**] or [**2**].

Helly's well-known theorem [**3**] states that if $\mathscr{C}$ is a finite family of convex sets in $\mathbb{R}^d$ with the property that any $d + 1$ of them have a nonempty intersection, then $\cap\mathscr{C} \neq \varnothing$. What happens if we assume somewhat more, namely that the intersection of any $d + 1$ sets is of sufficiently large volume? Is it then true that the volume of $\cap\mathscr{C}$ is at least 1? The following example answers this

question in the negative. Let $\mathscr{C}$ consist of the $2d$ supporting halfspaces of the d-dimensional cube with sides $1/2$. Then $\mathrm{Vol}(\cap\mathscr{C}) = 2^{-d}$, though the intersection of any $2d-1$ ($\geqslant d+1$) members is of infinite volume. This construction shows that the best result to be expected is the following.

THEOREM. *There exists a constant $V(d) > 0$ such that if $\mathscr{C}$ is a finite family of convex sets in $\mathbb{R}^d$, with the property that if $\mathscr{C}'$ is a $2d$-membered subfamily of $\mathscr{C}$ then $\mathrm{Vol}(\cap\mathscr{C}') \geqslant V(d)$ is valid, then $\mathrm{Vol}(\cap\mathscr{C}) \geqslant 1$.*

Proof. We recall Steinitz' lemma ([**4**] or [**1**, Theorem 2.3]) which states that if V is a set of points in $\mathbb{R}^d$ whose convex hull $\mathrm{conv}(V)$ contains a ball around 0, then there is a subset $V' \subseteq V$, $|V'| \leqslant 2d$, with the same property. This has the immediate consequence that if P is a convex polytope in $\mathbb{R}^d$, then one can choose at most $2d$ of its faces so that the intersection of the corresponding supporting halfspaces is bounded. (Use the lemma for the normal vectors of the faces.)

First we prove the theorem in the case when all members of $\mathscr{C}$ are halfspaces. Suppose, indirectly, that there is an infinite sequence $\mathscr{H}_1, \mathscr{H}_2, \ldots$ of finite families of halfspaces satisfying

(i) $\mathrm{Vol}(\cap\mathscr{H}_1) = \mathrm{Vol}(\cap\mathscr{H}_2) = \cdots = 1$,

(ii) the intersection of any $2d$ members of $\mathscr{H}_n$ has volume at least n ($n = 1, 2, \ldots$).

For each n take a simplex S_n contained in $\cap\mathscr{H}_n$ such that S_n has the largest possible volume. Suppose, without loss of generality, that each S_n is regular with center at the origin. (Otherwise, apply to $\mathscr{H}_n$ a suitable volume-preserving affine transformation. Clearly, this will not affect the validity of (i) and (ii).) Let $c_0, c_1, \ldots, c_d$ denote the vertices of S_n. Define another simplex S_n' by

$$S_n' = -dS_n = \mathrm{conv}\{-dc_0, -dc_1, \ldots, -dc_d\},$$

and denote the faces of S_n' by $F_0, F_1, \ldots, F_d$, where $c_i \in F_i$ for all i ($i = 0, 1, \ldots, d$). Write G_i for the open halfspace whose boundary is the hyperplane supporting F_i and for which $0 \notin G_i$. Thus, S_n' is exactly the complement of $\cup_{i=0}^{d} G_i$.

We claim that $\cap\mathscr{H}_n \subseteq S_n'$. Suppose, to the contrary, that there exists a point $c \in \cap\mathscr{H}_n$ which is outside of S_n', that is, $c \in G_i$ for some i. Now the simplex spanned by the vertices $c_0, c_1, \ldots, c_{i-1}, c, c_{i+1}, \ldots, c_d$ is also contained in $\cap\mathscr{H}_n$ and has obviously larger volume than S_n, contradicting the definition. Hence,

$$S_n \subseteq \cap\mathscr{H}_n \subseteq S_n', \tag{1}$$

which implies that

$$\mathrm{Vol}\, S_n' = d^d \mathrm{Vol}\, S_n \leqslant d^d \mathrm{Vol}(\cap\mathscr{H}_n) = d^d. \tag{2}$$

For each $i = 0, 1, \ldots, d$, there exists a d-membered subfamily of $\mathscr{H}_n$ whose intersection is disjoint from G_i. (This follows from Helly's theorem applied to the family $\mathscr{H}_n \cup \{G_i\}$.) Take the union of these subfamilies for all i, to obtain a system $\mathscr{H}_n' \subseteq \mathscr{H}_n$ consisting of at most $d(d+1)$ halfspaces. Put $P_n = \cap\mathscr{H}_n'$. We clearly have

$$S_n \subseteq P_n \subseteq S_n', \tag{1'}$$

where P_n is a convex polytope with $f(n) \leqslant d(d+1)$ faces. P_n can be written in the following form

$$P_n = \{x \in \mathbb{R}^d \mid \langle a_{ni}, x\rangle \leqslant 1; \quad i = 1, 2, \ldots, f(n)\},$$

where $a_{n1}, a_{n2}, \ldots, a_{nf(n)}$ are the so-called *outer normals* to the faces of P_n. Observe that the set of vectors $\cup_{n=1}^{\infty} \{a_{n1}, \ldots, a_{nf(n)}\}$ is necessarily bounded, because each P_n contains a regular simplex S_n with center in the origin, satisfying

$$\mathrm{Vol}\, S_n = d^{-d} \mathrm{Vol}\, S_n' \geqslant d^{-d} \mathrm{Vol}(\cap\mathscr{H}_n) = d^{-d}, \tag{2'}$$

that is, each P_n contains also a small fixed ball B around 0. This means that $\|a_{ni}\|$ cannot exceed the reciprocal of the radius of B. Since $f(n)$ is also bounded, one can choose a subsequence

$n_j \to \infty$, such that $f(n_1) = f(n_2) = \cdots = f$ and

$$\lim_{j\to\infty} a_{n_j i} = a_i$$

exists for each $i = 1, 2, \ldots, f$. Put

$$P = \left\{ x \in \mathbb{R}^d | \langle a_i, x\rangle \leqslant 1; \quad i = 1, 2, \ldots, f \right\}.$$

Take a large ball B' which contains all S_n'. (This is possible, by (2).) It is now clear that

$$B \subseteq P \subseteq B',$$

which shows that P is a nonsingular convex polytope.

We are going to show that the intersection of any $2d$ or fewer halfspaces supporting faces of P is unbounded, contradicting the consequence of Steinitz' lemma which we mentioned above. Assume indirectly that, say,

$$Q = \bigcap_{i=1}^{k} \left\{ x \in \mathbb{R}^d | \langle a_i, x\rangle \leqslant 1 \right\} \qquad (k \leqslant 2d)$$

is bounded, i.e., $\max_{q\in Q} \|q\| < R$. Consider now the sets

$$Q_{n_j} = \bigcap_{i=1}^{k} \left\{ x \in \mathbb{R}^d | \langle a_{n_j i}, x\rangle \leqslant 1 \right\}, \quad j = 1, 2, \ldots.$$

By condition (ii) we know that $\operatorname{Vol} Q_{n_j} \geqslant n_j$, hence we can choose points $q_{n_j} \in Q_{n_j}$ satisfying $\|q_{n_j}\| = R$, for each sufficiently large j. Let t be the limit point of any convergent subsequence of $\{q_{n_j}\}$. Since $\|t\| = R$, we have $t \notin Q$, in other words,

$$\langle a_i, t\rangle > 1 \quad \text{for some } i \quad (1 \leqslant i \leqslant k).$$

On the other hand

$$\langle a_{n_j i}, q_{n_j}\rangle \leqslant 1 \qquad (j = 1, 2, \ldots)$$

obviously holds, and from here, taking the limit, we obtain

$$\langle a_i, t\rangle \leqslant 1.$$

This contradiction proves the theorem for systems of halfspaces.

The rest of the proof is entirely routine. Let $\mathscr{C} = \{C_1, C_2, \ldots, C_m\}$ be an arbitrary family of convex sets satisfying the requirements of the Theorem. By using Helly's theorem for the interior sets $\operatorname{int} C_i$, we obtain $\operatorname{Vol}(\cap \mathscr{C}) > 0$. Further, we may suppose that $\cap \mathscr{C}$ is bounded, otherwise $\operatorname{Vol}(\cap \mathscr{C}) = \infty$ and our theorem holds. For every C_i consider the system $\mathscr{H}_i$ of all closed halfspaces containing C_i. It is obvious that the intersection of any $2d$ halfspaces belonging to $\mathscr{H} = \cup_{i=1}^{m} \mathscr{H}_i$ has volume at least $V(d)$, and $\cap \mathscr{H} = \cap \mathscr{C}$.

Let $\varepsilon > 0$ and let U_ε be defined as the set of those points of $\mathbb{R}^d$ whose distance from $\cap \mathscr{C}$ is at most ε. The boundary of U_ε is compact and is covered by the complements of the closed halfspaces belonging to $\mathscr{H}$. It follows from the Borel covering theorem that there exists a finite covering of the boundary. This, in turn, implies that there is a finite subfamily $\mathscr{H}'$ of $\mathscr{H}$ such that $\cap \mathscr{H}' \subseteq U_\varepsilon$. If we apply our theorem to $\mathscr{H}'$, we get $\operatorname{Vol} U_\varepsilon \geqslant \operatorname{Vol}(\cap \mathscr{H}') \geqslant 1$. Taking into account that

$$\lim_{\varepsilon\to 0} \left[\operatorname{Vol} U_\varepsilon - \operatorname{Vol}(\cap \mathscr{C})\right] = 0,$$

this implies $\operatorname{Vol}(\cap \mathscr{C}) \geqslant 1$. This completes the proof.

REMARKS. One can easily see that the condition of finiteness of $\mathscr{C}$ is inessential in the Theorem, and can be replaced for instance by the condition that $\mathscr{C}$ consists of convex closed sets at least one of which is bounded (compact). Concerning the exact value of $V(d)$ we know only $V(d) \leqslant d^{2d^2}$,

but we conjecture $V(d) \leqslant d^{cd}$ with a suitable constant c. Similar results are valid for the diameter and the surface area of $\cap \mathscr{C}$.

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THE TEACHING OF MATHEMATICS

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AN ALTERNATE METHOD FOR FINDING THE PARTIAL FRACTION DECOMPOSITION OF A RATIONAL FUNCTION

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The so-called partial fraction decomposition method is fundamental for the integration of rational functions. Furthermore, as Chrystal points out, this decomposition is "a fruitful source of complicated algebraical identities." In this note we describe an efficient method for determining the partial fraction decomposition of a rational function. Although this method can be found in several classical algebra texts such as Chrystal [**1**] and van der Waerden [**3**], it seems to have disappeared from current calculus texts.

The usual methods found in modern calculus textbooks essentially require the solution of a linear algebraic system. As such, they involve a good deal of algebra and take roughly n^3 steps to determine the constants in the decomposition, where n is the degree of the denominator of the rational function. As will be shown, the alternative method we recommend requires roughly only n^2 steps to find the necessary constants. Even more important to the student, this method, being recursive, is algebraically simpler and is to a large extent self-checking.

We begin with an example in order to contrast our method with those found in current calculus texts. Consider the problem of determining the decomposition of $r(x) = p(x)/q(x)$, where $p(x) = 4x^4 + 5x^3 - 25x^2 - 5x + 12$ and $q(x) = (x + 2)^2(x - 1)^3$. As usual, we assume the existence of constants A_i $(i = 1, 2)$ and B_j $(j = 1, 2, 3)$ such that

$$r(x) = \sum_i \frac{A_i}{(x+2)^i} + \sum_j \frac{B_j}{(x-1)^j}.$$

Thus

$$p(x) = \sum_i A_i(x+2)^{2-i}(x-1)^3 + \sum_j B_j(x+2)^2(x-1)^{3-j}.$$

Replacing x first by -2, then by 1, yields $A_2 = 2$ and $B_3 = -1$, so that

$$4x^4 + 3x^3 - 18x^2 - 7x + 18 = A_1(x+2)(x-1)^3 + B_1(x+2)^2(x-1)^2 + B_2(x+2)^2(x-1). \tag{1}$$

At this point modern calculus books will recommend one of two methods. The first method is to expand the right-hand side of (1), then equate coefficients of like-degree terms. The second method involves choosing three distinct values for x (other than -2 and 1). Substituting these values into (1) yields a 3 by 3 linear system in the unknowns A_1, B_1 and B_2. Thus either method leads to a 3 by 3 linear system. Each of them involves a good deal of algebra, both in obtaining the linear system, and then in finding the solution. There is ample opportunity for error.

Using the alternate method we note that $(x+2)(x-1)$ is a factor of the right-hand side of (1), hence also a factor of the left-hand side. (As Chrystal points out, if at this point $(x+2)(x-1)$ was not a factor, then a mistake has been made.) Dividing (1) through by $(x+2)(x-1)$ gives

$$4x^2 - x - 9 = A_1(x-1)^2 + B_1(x+2)(x-1) + B_2(x+2). \tag{2}$$

Note that the division of $4x^4 + 3x^3 - 18x^2 - 7x + 18$ by $(x+2)(x-1)$ may be accomplished very quickly by two successive synthetic divisions. Next, evaluation of (2) at $x = -2$ and 1 quickly gives $A_1 = 1$, $B_2 = -2$; thus

$$3x^2 + 3x - 6 = B_1(x+2)(x-1). \tag{3}$$

Equation (3) immediately implies that $B_1 = 3$ and we are done.

To describe the alternate method in general, let $r(x) = p(x)/q(x)$, where $p(x)$, $q(x)$ are relatively prime, deg $p(x) <$ deg $q(x)$ and

$$q(x) = (x-a)^m t(x),$$

where $t(a) \neq 0$ and $m \geqslant 1$. Then for any constant c,

$$r(x) = \frac{c}{(x-a)^m} + \frac{p(x) - ct(x)}{(x-a)^m t(x)}.$$

In particular, when $c = p(a)/t(a)$, $p(x) - ct(x) = (x-a)p_1(x)$ and

$$r(x) = \frac{c}{(x-a)^m} + \frac{p_1(x)}{(x-a)^{m-1} t(x)}.$$

Now consider $p_1(x)/(x-a)^{m-1}t(x)$. To quote from van der Waerden, "we may proceed to treat this fraction in a similar fashion, until $r(x)$ is completely resolved into partial fractions."

This method, besides providing a nice application of synthetic division, involves less algebra than those methods which require one to find and then solve a linear system. It can be described recursively, and also has the advantage of providing a check at each stage that the last constant computed was computed correctly. (Namely, $x - a$ must be a factor of $p(x) - ct(x)$.)

If $q(x) = (x-a_1)^{n_1}(x-a_2)^{n_2}\cdots(x-a_k)^{n_k}$ and $n = n_1 + \cdots + n_k$, then decomposition of $r(x)$ into partial fractions involves finding the n constants c_{ij}, $1 \leqslant i \leqslant k$, $1 \leqslant j \leqslant n_i$, where $c_{ij}/(x-a_i)^j$ is the general term in the decomposition. Those methods which require solution of an n by n linear system are $O(n^3)$, provided one of the usual methods, such as elimination, is used. (Even the best known methods for solving an n by n linear system are $O(n^d)$, $d > 2.5$; see for

example Strang [2], p. 5.) On the other hand, the alternate method is easily seen to be $O(n^2)$, since computation of each c_{ij} is $O(n)$ and there are n constants to be found. In fact, developing a computer program for the alternate method provides several nice examples of subprograms and the application of one and two-dimensional arrays.

One final comment concerning the case where $q(x)$ contains a single irreducible quadratic factor $x^2 + ax + b$; this case occurs frequently in examples in calculus. To handle it using our method we save determining the corresponding constants c and d for last. At that stage we will have

$$p_1(x) = cx + d,$$

where $p_1(x)$ has degree at most one, and thus c and d can be found by comparing coefficients.

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POWER SERIES WITHOUT TAYLOR'S THEOREM

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It is typical in today's calculus texts to develop the full theory of sequences and infinite series before deriving the power series expansions for any of the standard transcendental functions. The Taylor polynomials and Taylor's theorem are usually presented for arbitrary functions, and only after considerable development do students ever see, for example, the application of computing the number e to five decimal places. We suggest here reversing the approach by first deriving the power series for e^x, $\sin x$, and $\cos x$, giving some numerical applications, and then using the examples to motivate the general theory behind Taylor's theorem.

The usual infinite series representation for e, for example, can be derived quite easily as a wonderful application of integration by parts. For $n \geq 0$, let $E_n = \int_0^1 t^n e^{-t}\,dt$. The integrands are easily graphed and the definite integrals E_n can be interpreted as areas. Integration by parts gives the reduction formula

$$E_n = -e^{-1} + nE_{n-1}, \qquad n \geq 1, \tag{1}$$

and E_0 satisfies the equation

$$e = 1 + eE_0. \tag{2}$$

Multiplication by $e/n!$ transforms (1) into the nice recursion relation

$$\frac{eE_{n-1}}{(n-1)!} = \frac{1}{n!} + \frac{eE_n}{n!}, \qquad n \geq 1, \tag{3}$$

and repeated substitution of (3) into (2) gives

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!} + R_n, \qquad n \geq 1, \tag{4}$$

where the remainder term $R_n = eE_n/n!$.

We are thus led directly to the notion of convergence of an infinite sum. Since the areas represented by E_n approach zero as n gets large, the R_n's approach zero very rapidly because of the presence of the $n!$ in the denominator. Hence the long sums give very good approximations to e, and these approximations get dramatically better as we take n larger.

More precisely, for $0 \leqslant t \leqslant 1$, we have $e^{-1} \leqslant e^{-t} \leqslant 1$, and so

$$\frac{1}{e(n+1)} \leqslant E_n \leqslant \frac{1}{n+1}.$$

Hence R_n satisfies the inequality

$$\frac{1}{(n+1)!} \leqslant R_n \leqslant \frac{e}{(n+1)!}, \qquad n \geqslant 1, \tag{5}$$

and the error term, while not less than the next term in the sum (which makes perfectly good sense), is no larger than e times the next term in the sum (which is a nontrivial result). We can now compute e to five decimal places if we wish, but more importantly, the concept of convergence of an infinite sum to a particular number is beautifully and simply illustrated with a nontrivial example.

To derive the power series for e^x, we simply replace e^{-t} by e^{-xt} in the integrand for E_n. The reduction formula

$$xE_n = -e^{-x} + nE_{n-1}, \qquad n \geqslant 1, \tag{1'}$$

can be put into the nicer form

$$\frac{x^n e^x E_{n-1}}{(n-1)!} = \frac{x^n}{n!} + \frac{x^{n+1} e^x E_n}{n!}, \qquad n \geqslant 1. \tag{3'}$$

Repeatedly applying (3′) to the equation $e^x = 1 + xe^x E_0$, we arrive at the power series expansion for e^x and an exact expression for the remainder term of the finite partial sums. Convergence follows by fixing x, bounding the integrals E_n, and letting n go to infinity.

The power series for the functions $\sin x$ and $\cos x$ can also be derived directly. One way is to consider the definite integrals

$$S_n = \int_0^1 (1-t)^n \cos xt \, dt \quad \text{and} \quad C_n = \int_0^1 (1-t)^n \sin xt \, , \qquad n \geqslant 0. \tag{6}$$

Direct integration gives

$$\sin x = xS_0 \quad \text{and} \quad \cos x = 1 - xC_0. \tag{7}$$

Integration by parts gives the reduction formulas

$$xS_n = nC_{n-1} \quad \text{and} \quad xC_n = 1 - nS_{n-1}, \qquad n \geqslant 1. \tag{8}$$

Combining these two reduction formulas, we obtain

$$x^2 S_n = n - n(n-1)S_{n-2} \quad \text{and} \quad x^2 C_n = x - n(n-1)C_{n-2}, \qquad n \geqslant 2, \tag{9}$$

which can be put into the forms

$$\frac{x^{n-1} S_{n-2}}{(n-2)!} = \frac{x^{n-1}}{(n-1)!} - \frac{x^{n+1} S_n}{n!} \quad \text{and} \quad \frac{x^{n-1} C_{n-2}}{(n-2)!} = \frac{x^n}{n!} - \frac{x^{n+1} C_n}{n!}, \qquad n \geqslant 2. \tag{10}$$

Repeated substitution of (10) into (7) yields the power series for $\sin x$ and $\cos x$, respectively, and an easy analysis of the error terms proves convergence.

At this point it can be pointed out that, by substitution, e^{kx} also has a power series development, and that the coefficient of x^n in this series is of the form $f^{(n)}(0)/n!$. Also, the Taylor series for e^x, $\sin x$, and $\cos x$ about the point a, say, can be derived from the relations

$$e^x = e^a \cdot e^{x-a},$$

$$\sin x = \sin a \cos(x-a) + \cos a \sin(x-a),$$

$$\cos x = \cos a \cos(x-a) - \sin a \sin(x-a),$$

and it is easily seen that the coefficient of $(x - a)^n$ in these expansions always has the proper form: $f^{(n)}(a)/n!$.

Students are now prepared for Taylor's theorem and its proof. Integration by parts leads to the integral form of the remainder term, which, by bounding the integral as we did in inequality (5), is all that is needed to prove convergence for many specific functions.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS), H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk () indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the address given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by November 30, 1984. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3046. *Proposed by Steven C. Althoen and Lawrence D. Kugler, The University of Michigan, Flint, MI.*

It is easy to check that if $\|(x, y)\|_1$ and $\|(x, y)\|_2$ are norms on R^2, then so is their arithmetic mean $\frac{1}{2}(\|(x, y)\|_1 + \|(x, y)\|_2)$. Let

$$\|(x, y)\|_1 = \left(a^2x^2 + b^2y^2\right)^{1/2} \quad \text{and} \quad \|(x, y)\|_2 = \left(c^2x^2 + d^2y^2\right)^{1/2},$$

where a, b, c, and d are reals. Find necessary and sufficient conditions on a, b, c, and d so that the geometric mean $\sqrt{\|(x, y)\|_1\|(x, y)\|_2}$ will also be a norm.

E 3047. *Proposed by C. Douglas Harper and Bruce Reznick, University of Illinois at Urbana-Champaign.*

Suppose f is a convex function defined on R and let $U = \{(x, y) : f(x) - 1 \leqslant y \leqslant f(x)\}$. Must U contain line segments of arbitrary length?

SOLUTIONS OF ELEMENTARY PROBLEMS

Simultaneous Rational Bilinear Diophantine Equations

E 2913 [1981, 705]. *Proposed by John J. Wahl, Mt. Pocono, PA.*

Given rational N and P, find all integers a, b, x, y satisfying $a + y = b + x$, for which $N = (ay - bx)/(xy - ab)$, $P = (xy - ab)/(ax - by)$.

Solution by Charles R. Wall, Trident Technical College. Substituting $y = b + x - a$ reduces the restrictions to $N = (a - b)/(x + b)$ and $P = (x - a)/(a - b)$. Suppose that in lowest terms $N = c/d$ and $P = h/j$. Let $e =$ l.c.m. $[c, j]$ and write $N = e/f$ and $P = g/e$. Then there exists k such that

$$\begin{cases} a - b = ke \\ x + b = kf \\ x - a = kg. \end{cases}$$

This system of equations is easily solved:

$$x = k(f + e + g)/2$$
$$a = k(f + e - g)/2$$
$$b = k(f - e + g)/2$$

whence

$$y = k(f - e + 3g)/2.$$

If one of a, b, x, y is non-integral, all are, since the numbers in parentheses above are of the same parity, so we reject as non-integral the solutions above for which k is odd and $\{e, f g\}$ includes an odd number of odd integers.

Also solved by J. C. Binz (Switzerland), P. Bruckman, E. C. Buissant des Amone (The Netherlands), L. Kuipers (Switzerland), M. Pasteka (Bratislava), K. H. Sze, E. Trost (Switzerland), J. Ward, D. G. Weinman, and the proposer.

Partially solved by M. Bencze (Rumania) and O.P. Lossers (The Netherlands).

The Intersection of a Nested Sequence of Polygons

E 2917 [1981, 763]. *Proposed by F. W. Luttman, Sonoma State University.*

Let P_0 be a convex polygon of n sides and let $0 < f < 1$. Let $P_0, P_1, P_2, \ldots$ be a nested sequence of polygons similar to P_0 with the following properties:

(1) P_{k+1} is a linear contraction of P_k by the factor f.

(2) Two adjacent sides of P_{k+1} lie on P_k. (Necessarily P_k and P_{k+1} share a single vertex.)

(3) The vertex which P_k shares with P_{k+1} lies next clockwise from the vertex it shares with P_{k-1}.

There is precisely one point lying inside all P_k's. Construct it. (See H. S. M. Coxeter, *Introduction to Geometry*, p. 164.)

I. *Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, The Netherlands.* Let C be the point that lies inside all P_k's. Obviously after n steps one is back in the original situation except for the multiplication by a factor f^n with C as center.

Let $A_1, A_2, \ldots, A_n$ be the vertices of P_0 and let $B_1, B_2, \ldots, B_n$ be the corresponding vertices of P_n. It follows that C is the intersection point of the lines through A_i and B_i, $1 \leqslant i \leqslant n$.

II. *Solution by Richard A. Vitale, Claremont Graduate School, Claremont, CA.* The desired point w^* is the limit of the sequence of shared vertices.

Let $v_0, v_1, \ldots, v_{n-1}$ be a clockwise enumeration of the vertices of P_0; extend this to an infinite periodic sequence $\{v_k\}$. Let $w_0 = v_0$ be the shared vertex of P_0 and P_1, and, generally, let w_k be the shared vertex of P_k and P_{k+1}. Then

$$w_{k+1} - w_k = f^{k+1}(v_{k+1} - v_k).$$

Solving this with the initial condition yields

$$w_N = f^N v_N + (1 - f) \sum_0^{N-1} f^k v_k,$$

which tends to $w^* = (1 - f)\Sigma_0^\infty f^k v_k$. The periodicity of the v sequence then gives

$$w^* = [(1 - f)/(1 - f^n)] \sum_0^{n-1} f^k v_k.$$

Also solved by J. C. Binz (Switzerland), J. Dou (Spain), R. B. Eggleton (Australia), W. Janous (Austria), L. Kuipers (Switzerland), R. E. Megginson, W. A. Newcomb, D. A. Rawsthorne, D. B. Shapiro, D. L. Shell, J. Ward and the proposer.

Shapiro pointed out that "the statement of the problem has some ambiguity, since it is not specified whether the n-gon P_{k+1} is to be traced in the same sense or in the opposite sense as P_k." The original intention was for P_{k+1} to be traced in the same sense as P_k.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by November 30, 1984. The solver's full post-office address should be on each sheet.

6460. *Proposed by L. Richard Duffy (student), University of Chicago.*

It is well known that a discontinuous *linear* real function (i.e., one satisfying $f(x + y) = f(x) + f(y)$, all $x, y \in R$) must be unbounded on any open interval. Is there a linear function which is so pathological that it actually assumes *all* real values on any open interval?

6461. *Proposed by L. E. Mattics, University of South Alabama.*

Let p be a prime and suppose $1 \leqslant h < p$ where the order of h modulo p is $(p - 1)/\nu$. Prove that

$$(\nu - 1)\sqrt{p} + 1 \geqslant \left| \sum_{m=1}^{p-1} e^{2\pi i h^m/p} \right|.$$

6462. *Proposed by Randall J. LeVeque and Lloyd N. Trefethen, Courant Institute of Mathematical Sciences, New York University.*

Let S be the unit circle (or any other circle) in the complex plane. Let r be a rational function of type (n, n) with no poles on S.

(a) Show that $\|r'\|_1 \leqslant 4\pi n \|r\|_\infty$ where $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are the L_1 and L_∞ norms on S. Note that $\|r'\|_1$ is the arclength of the image of S under r.

(b)* Show that $\|r'\|_1 \leqslant 2\pi n \|r\|_\infty$.

SOLUTIONS OF ADVANCED PROBLEMS

The Derivatives of x^x

6392 [1982, 429; 1984, 60]. *Proposed by C. Ward Henson, Bruce Reznick and Lee A. Rubel, University of Illinois.*

Is there some $x_0 > 0$ such that if $f(x) = x^x$, then $f^{(n)}(x) \geqslant 0$ for all $x \geqslant x_0$ and all $n = 0, 1, 2, \ldots$?

Correction to solution. G. G. Bilodeau, Boston College, Massachusetts, has pointed out that the published solution [1984, 60] is incorrect in assuming that a Taylor series having infinite radius of convergence necessarily converges to the function generating the series. He also observed (as did some of the listed solvers) that a result due to Bernstein on absolutely monotonic functions [see D. V. Widder, *The Laplace Transform*, Princeton 1946, p. 146] shows that if $f^{(n)}(x) \geqslant 0$ for all $x \geqslant x_0$ and all $n \geqslant 0$, then $f(x) = x^x$ can be extended analytically to an entire function. This yields the desired contradiction.

A Partition of the Positive Integers

6413 [1982, 788]. *Proposed by P. Erdős, Hungarian Academy of Sciences.*

The positive integers are partitioned into two sets, A, B so that $A + A$ and $B + B$ each represent no more than finitely many primes. Show that the partition is unique. (This generalizes E 2977.)

Solution by Zun Shan, University of Science and Technology of China, Hefei, Anhui, China. Let $x \sim y$ denote that both x and y are either in A or in B, and let a, n denote positive integers. To show that one of the sets A, B must necessarily consist of all the even positive integers, it suffices to prove that, for any n, $a \sim a + 2$ whenever $a + 2 < n$. Suppose that n is larger than every prime in $(A + A) \cup (B + B)$. Let $p_1, p_2, \ldots$ be the primes larger than n in ascending order, and let

$$d_i = d_i^{(0)} = p_{i+1} - p_i, \qquad d_i^{(j)} = \left(d_i^{(j-1)}, d_{i+1}^{(j-1)}\right) = \left(d_i, d_{i+1}, \ldots, d_{i+j}\right).$$

We may assume n to be so large that, by Chebyshev's theorem, $p_{i+1} < 5p_i/4$; so that $d_i < p_i/4$ and hence

$$d_i + d_{i+1} < p_{i+1}, \quad i = 1, 2, \ldots. \tag{1}$$

PROPOSITION 1. *If $a + d_i < p_{i+1}$, then $a \sim a + d_i$.*

Proof. Suppose that $a \in A$ and $a + d_i < p_{i+1}$. Since $a < p_{i+1} - d_i = p_i$, we obtain $p_i - a \in B$ and hence $p_{i+1} - (p_i - a) = a + d_i \in A$.

PROPOSITION 2. *Let x, y, p denote positive integers with $x + y < p$ and let $z = (x, y)$. Suppose that $a \sim a + x$ whenever $a + x < p$ and that $a \sim a + y$ whenever $a + y < p$. Then $a \sim a + z$ whenever $a + z < p$.*

Proof. Suppose that $a \in A$ and $a + z < p$. We may assume that $a \leqslant y$ (otherwise we can replace a by $a - kx$ for a suitable positive integer k). We have $z = ux - vy$ where u, v are positive integers. Add x's to a until the sum is $> y$ (but $\leqslant x + y < p$); then subtract y's until the difference is $\leqslant y$. Continue the process until u x's have been added. The resultant expression will then be $a + ux - v'y$. Adding or subtracting the appropriate number of y's, we obtain the required number $a + ux - vy = a + z < p$. Since the numbers arising throughout the above procedure are $< p$, they are all in A. In particular, $a + z \in A$.

It follows from (1) and Propositions 1 and 2, by taking $p = p_{i+1}$, $x = d_i$, $y = d_{i+1}$, $z = d_i^{(1)}$,

that $a \sim a + d_i^{(1)}$ whenever $a + d_i^{(1)} < p_{i+1}$. Repeating this argument, we conclude that $a \sim a + d_i^{(j)}$ whenever $a + d_i^{(j)} < p_{i+1}$. In particular, $a \sim a + d_1^{(j)}$ whenever $a + d_1^{(j)} < p_2$. Since $d_1^{(j)}$ is even and $d_1^{(j)} \geqslant d_1^{(j+1)}$, there is an even integer d such that $d_1^{(j)} = d$ for all j sufficiently large. It follows that all primes greater than n are members of the arithmetic progression $\{p_1 + id\}$. Now, by Dirichlet's theorem, there are primes $p, p' > n$ such that $p = jd + 1$, $p' = j'd - 1$. Hence, there are integers i, i' such that $jd + 1 = p_1 + id$, $j'd - 1 = p_1 + i'd$; so that $(i - i' - j + j')d = 2$ and consequently $d = 2$. We have thus shown that $a \sim a + 2$ whenever $a + 2 < n$.

Also solved by Roger B. Eggleton & Donald K. Skilton (Australia), L. E. Mattics, and the proposer.

On Conservative Matrices

6414 [1982, 788]. *Proposed by A. Wilansky, Lehigh University.*

Suppose the matrix $A = (a_{nk})$ transforms the sequence $x = \{x_n\}$ into the sequence $y = \{y_n\}$, so that

$$y_n = \sum_{k=1}^{\infty} a_{nk} x_k \quad \text{for} \quad n = 1, 2, \ldots .$$

If

$$y_n = \begin{cases} x_1 & n \text{ even}, \\ x_{(n+1)/2} & n \text{ odd}, \end{cases}$$

observe that (i) $a_{nk} = 0$ for $k > n$; (ii) x bounded implies y bounded; (iii) y convergent implies x convergent; (iv) there is a convergent x for which y is not convergent.

Is there a matrix A with properties (i), (ii), (iii), and (iv) for which $a_{nn} \neq 0$ for all n?

Solution by Jeff Connor, Kent State University. The answer is NO. We use the following theorem of I. D. Berg [A Banach algebra criterion for Tauberian theorems, *Proc. Amer. Math. Soc.*, 15 (1964) 648–52].

THEOREM. *Let S be a conservative matrix. If S transforms a bounded divergent sequence into a convergent sequence, then for any $\varepsilon > 0$ there is a convergent sequence x such that $\|x\| = \sup_{n \geqslant 1} |x_n| \geqslant 1$ and $\|Sx\| < \varepsilon$.*

Suppose that a matrix A with the properties in question exists. Then, in view of (i) and the nonvanishing of a_{nn}, $B = A^{-1}$ exists, and, by (iii), B is conservative. Furthermore, by (ii) and (iv), B meets the conditions on S in Berg's theorem. Hence there is a convergent sequence x such that $\|x\| \geqslant 1$ and $\|A\|\|Bx\| < 1$ where $\|A\| = \sup_{\|x\| \leqslant 1} \|Ax\| < \infty$ by (ii). Observe that $1 \leqslant \|x\| = \|ABx\| \leqslant \|A\|\|Bx\| < 1$, which is a contradiction.

Also solved by the proposer.

A Homogeneous Polynomial

6415 [1983, 60]. *Proposed by Raphael M. Robinson, University of California, Berkeley.*

If $P(x_1, x_2, \ldots, x_n)$ is a polynomial with integer coefficients, find m and a homogeneous polynomial $Q(x_1, x_2, \ldots, x_m)$ with integer coefficients, so that $P = 0$ has a solution in rationals if and only if $Q = 0$ has a solution in integers not all zero.

Solution by Andrew Adler, University of British Columbia. Construct a homogeneous polynomial R by setting

$$R(x_1, x_2, \ldots, x_n, u) = u^d P(x_1/u, x_2/u, \ldots, x_n/u)$$

where d is the degree of P. Let

$$S(x_1,\ldots,x_n,r_1,\ldots,r_4,y)=\sum_{i=1}^{n}x_i^2+\left(\sum_{i=1}^{4}r_i^2\right)-y^2,$$

and let $T(y,z,u)=y^2-2z^2-u^2$. Then $P=0$ has a solution in rationals if and only if $R=0$ has a solution in integers with $u\neq 0$. Suppose $R=0$ has a solution in integers $(x_1,\ldots,x_n,u)$ with $u\neq 0$. Then the Pellian equation $y^2-2z^2=u^2$ has a solution in integers with $y^2\geqslant\Sigma_{i=1}^{n}x_i^2$, and so, by Lagrange's theorem, there are integers $r_1,\ldots,r_4$ such that $S=0$. Conversely, if the system $R=0$, $S=0$, $T=0$ has a solution with $y=0$, then all variables must be 0; if the system has a solution with $y\neq 0$, then $u\neq 0$. It follows that the homogeneous polynomial $Q=R^4+S^{2d}+T^{2d}$ (in $m=n+7$ variables) has the desired property.

Also solved by David G. Cantor and the proposer.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
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The Prehistory of the Theory of Distributions. By Jesper Lützen (Studies in the History of Mathematics and Physical Sciences 7). Springer-Verlag, New York/Heidelberg/Berlin, 1982. viii + 232 pp.

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The theory of distributions has now become an essential tool for modern research in many parts of analysis, in particular in the theory of partial differential equations and in harmonic analysis (both commutative and non-commutative). In many universities it is taught at a level corresponding to first-year graduate or senior undergraduate courses; the essential concepts and results can certainly occupy no more than a one-semester course, and in my opinion they are much easier to grasp and to apply than the Lebesgue integral, with its dangerous subtleties about null sets. Just as a measure μ can be thought of as the linear functional $f\to\int f\,d\mu$, a distribution can be thought of as a continuous linear functional on an appropriate space of functions.

One then may well wonder why it took more than 30 years for distribution theory to be born, after the theory of integration had reached maturity. The book by J. Lützen is centered on that question and on related ones. Even with a scanty knowledge of history, one is struck that the concept of an integral progressed linearly, so to speak, from Cauchy to Lebesgue, every analyst, during that period, trying to go beyond his predecessors in a completely conscious way [**5**]. However, the various ideas which finally converged to the theory of distributions appeared in a kind of random pattern, each being developed in isolation from the others. Such a phenomenon may perhaps be attributed to a more general one, the isolation of the mathematical disciplines from one another which lasted till the middle of the twentieth century; it certainly was due in part to the lack of communication between their practitioners. Very likely a similar situation would be much less probable at present, with the multiplicity of symposia, colloquia, summer schools, seminars, etc., which are now offered to mathematicians.

The author obviously has diligently researched the mathematical literature to obtain a large amount of information on his topic, but I feel that it cannot be said he has been entirely successful in his undertaking. In my opinion, many papers which he describes are irrelevant to the origin of

distribution theory, and, on the other hand, a number of important ones have failed to attract his attention or have not been given the emphasis they deserved. Generally speaking, he has not tried to put some order in the rather chaotic "prehistory of distributions." What is lacking in the book is a clearer picture, uncluttered by side issues, of the main trends and strivings which finally gave birth to the new concepts of Sobolev and L. Schwartz.

In rather vague terms, I think one can summarize these trends as follows: one starts with a family of very "regular" functions (usually with respect to differential properties), which satisfy certain (generally integral) relations, or on which certain operations are possible; and then one discovers that an *a priori larger* family of functions satisfies the *same* relations, or can be subjected to similar operations. Many questions then may naturally be asked: Is this new family really different from the first? If it is, what are the relations between the two families, and can one give a precise description of the new one? It is only in the last stage of the "prehistory" that what may be called a revolutionary point of view will emerge, with the idea that the "new family" might consist of objects *other* than functions.

There are two areas of analysis where this general abstract pattern may be concretely followed in various papers, most of them written with no knowledge of similar ones: weak solutions of partial differential equations, and Fourier transforms.

I. Weak Solutions. The more or less vague notion of "generalized solution" or "weak solution" is essentially linked to integral formulas, of which the simplest is "integration by parts," and the most elaborate Stokes' general formula. The first rather general formula is due to Lagrange: if

$$L(y) \equiv y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y \tag{1}$$

is a linear differential operator on a function y defined in an open interval (a, b) of $\mathbf{R}$, he showed that there is an operator $M(y)$ of order n such that, for any two C^n functions y, z in (a, b)

$$zL(y) - yM(z) = \frac{d}{dx}(B(y, z)), \tag{2}$$

where B is a differential operator of order $n - 1$ which is bilinear in y and z. From this differential formula, one gets, when in addition the function z vanishes in a neighborhood of a and in a neighborhood of b (or, in other words, if z has a *compact carrier*), the integral formula

$$\int_a^b zL(y)\,dx = \int_a^b yM(z)\,dx. \tag{3}$$

It is easy to generalize to any number of variables: if Ω is an open set in $\mathbf{R}^m$, and $P: f \mapsto \Sigma_{|\alpha|\leqslant n} a_\alpha D^\alpha f$ a linear differential operator(*) of order n with C^∞ coefficients a_α defined in Ω, then, if one writes

$$\langle f, g\rangle = \int_\Omega f(x)g(x)\,dx \tag{4}$$

there is a unique linear differential operator ${}^tP$ of order n, called the *adjoint* of P, such that, for any pair of C^∞ functions f, g in Ω, one of which at least has compact carrier, one has

$$\langle P \cdot f, g\rangle = \langle f, {}^tP \cdot g\rangle. \tag{5}$$

If f is a C^n solution of $P \cdot f = 0$, one therefore has $\langle f, {}^tP \cdot g\rangle = 0$ for *all* C^∞ functions g in Ω with compact carrier; a natural question is to ask if conversely, an integrable function f satisfying that

(*)In this now commonly accepted notation, we recall that $\alpha = (\alpha_1, \ldots, \alpha_m)$ is an m-tuple of nonnegative integers, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_m$, and

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_m^{\alpha_m}}.$$

condition (such functions are called weak or generalized solutions) is necessarily a C^n solution of $Pf = 0$. To see that this is not always the case, let $Pf = \partial^2 f/\partial x \partial y$. Using integration by parts, it is easy to verify that ${}^tP = P$. Now if u and v are integrable functions of one variable, not continuous, and $f(x, y) = u(x) + v(y)$, then a simple use of Fubini's Theorem shows that f is a weak solution of $Pf = 0$. Of course this f is not even differentiable.

It would be absurd to think that Lagrange may have had the concept of "weak solution," since he only dealt with analytic functions; for him, the interesting thing in formula (2) was that if z satisfies $M(z) = 0$, the integration of the nth order equation $L(y) = 0$ is reduced to the integration of an equation of order at most $n - 1$. The first attack on the problem of determining if weak solutions are genuine solutions seems to have been made by P. Du Bois-Reymond [**4**] in 1879: in connection with the foundations of the calculus of variations, he showed that a weak solution of the equation $du/dx = 0$, i.e., an integrable function $u(x)$ such that $\int_a^b u(x)g'(x)\,dx = 0$ for *every* C^∞ function g with compact carrier in (a, b), is necessarily a constant (this is relegated by Lützen to a note to Chapter 2, with misplaced emphasis on the calculus of variations instead of the concept of weak solution). The next occurrence of that problem (which is not at all mentioned by Lützen!) is in one of the most famous papers of H. Poincaré, published in 1894 [**6**]; its main topic is the proof of the existence of eigenvalues λ_n for the problem of the vibrating membrane: if Ω is a bounded open set in $\mathbf{R}^3$ with smooth boundary Σ, one attempts to find values of λ for which the equation $\Delta u + \lambda u = 0$ has nontrivial solutions vanishing on Σ. Poincaré tried to extend his results when the boundary condition is $\partial u/\partial n + hu = 0$ with a constant $h > 0$; to apply his method he needed to solve a generalization of the Dirichlet problem, namely, to find a solution u of $\Delta u = 0$ which is C^1 in the closed set $\bar{\Omega}$ and such that $\partial u/\partial n + hu = g$ on Σ, where g is a continuous function on Σ. He was able to obtain a function u continuous in $\bar{\Omega}$ and C^1 in the interior Ω, but he could not prove the existence of the normal derivative $\partial u/\partial n$ on Σ. However, he showed that for *any* C^2 function v in $\bar{\Omega}$, one has

$$\iiint_\Omega u\Delta v\,d\omega + \iint_\Sigma gv\,d\sigma = \iint_\Sigma \left(\frac{\partial v}{\partial n} + hv\right) u\,d\sigma \tag{6}$$

and he added that "physically" this was equivalent to a genuine solution!

A few years later S. Zaremba was able to prove that Poincaré's weak solution was indeed a genuine one; but before 1940 there was no *general* result on operators P such that, for every C^∞ function f, a weak solution u of $P \cdot u = f$ is necessarily a C^∞ solution (what we now call *hypoelliptic* operators).

A variant of the concept of weak solution was based on Green's formula in $\mathbf{R}^2$ or $\mathbf{R}^3$. For instance, Lützen notes (p. 44) that in 1913 H. Weyl defined the (weak) Laplacian of a C^1 function u in $\mathbf{R}^3$ as a continuous function v such that, for *every* open bounded set Ω with smooth boundary Σ, the relation

$$\iiint_\Omega v\,d\omega = -\iint_\Sigma \frac{\partial u}{\partial n}\,d\sigma \tag{7}$$

holds; Lützen also mentions similar ideas proposed by several mathematicians (Bôcher in 1905 (p. 36), Evans in several papers after 1914 (p. 38–42)); he fails to mention E. Cartan in his book of 1922 [**2**]. Some tried to define these weak solutions by limiting processes, more sophisticated than the usual one, and not necessarily valid everywhere but only almost everywhere (in the sense of Lebesgue); for instance (p. 38) Evans suggested defining the Laplacian Δu in $\mathbf{R}^2$ as the limit of $\int_\Gamma (\partial u/\partial n)\,ds$, when Γ is a simple closed curve which tends to a fixed point lying in the domain it bounds. We now see in retrospect that this direction was a blind alley which could not lead to the concept of distribution: as Sobolev observed in 1938, a continuous function f of bounded variation in $\mathbf{R}$ has an ordinary derivative almost everywhere, but that derivative is *not* equal to the weak derivative of f if f is not absolutely continuous. On the other hand, a function such as the characteristic function of the set of rationals ("Dirichlet's function") may be everywhere discontinuous, and still have a weak derivative. Much more promising was the idea of approaching a

function f in L^2 by a sequence of C^∞ functions whose derivatives tend weakly in L^2 to the weak derivative of f; Lützen notes that this idea was put forward by Wiener in 1926 (p. 51) and by Leray in 1934 (p. 54), the latter using for the approximation the regularizations $f * \rho_n$ by the "approximate identity" (ρ_n) used by H. Weyl in his work of 1927 on Lie groups.

However, a function such as the Heaviside function in $\mathbf{R}$, equal to 1 for $x \geqslant 0$ and to 0 for $x < 0$, has no weak derivative, in spite of its very mild discontinuity; at least this is what mathematicians would say, but physicists thought otherwise, since for them there *was* a "derivative" δ, the Dirac "delta function," and they even considered "derivatives of any order $n + 1$" $\delta^{(n)}$. Of course, there was before 1936 no reasonable mathematical definition of these objects; but it is characteristic that they were never used in *bona fide* computations except *under the integral sign*, giving formulas such as

$$\int \delta^{(n)}(x - a) f(x)\, dx = (-1)^n f^{(n)}(a) \tag{8}$$

for a C^n function f. In other words, it seems that before 1936 nobody could think of the linear form $f \mapsto f^{(n)}(a)$ except as patterned after the linear form $f \mapsto \int_{\mathbf{R}} \phi(x) f(x)\, dx$ for a genuine function ϕ. This is exactly the same obstacle which for so long blocked the correct concept of the dual of a finite dimensional vector space, by forcibly identifying, by choice of a basis, a linear form on the space with a vector in the space. There was of course the famous theorem of F. Riesz (1909) identifying the continuous linear forms on the space $\mathscr{C}(I)$ of continuous functions in the interval I with Stieltjes integrals $f \mapsto \int_I f(x)\, d\alpha(x)$; but before 1935 neither Riesz nor anyone else ever said (in spite of the notation) that the derivative of the function α of bounded variation was a *measure*, probably because for mathematicians at that time a measure was defined on certain *subsets* and *not* on functions; and mathematicians certainly considered the idea of a derivative of a measure as unthinkable, except for crazy engineers or physicists!

All this shows how totally unprepared, and without the slightest precursor, was the bold move by which, in a few pages [7], Sobolev in 1936 defined the fundamental notions of the theory of distributions: the topology on the space of test functions, and the weak topology on the space of distributions, the identification of locally integrable functions with particular distributions, the product of a distribution by a C^∞ function, the regularization of distributions, and the most remarkable of all, the derivatives of any order of any distribution. But, bent as he was on the solution of a particular problem, it seems that Sobolev did not at that time realize the power of the ideas he had introduced; at any rate, he made no effort at propagandizing them. For example, in his second pathbreaking 1938 paper on functional analysis, introducing the Sobolev spaces [8], he only used the concept of a *function* which is the weak derivative of another function. Between 1936 and 1940 I have not been able to find a single paper, by Sobolev himself or by other mathematicians, in his own country or elsewhere, which made any mention of Sobolev's "functionals"(*).

These circumstances, and the scientific isolation of France between 1940 and 1945, due to war and occupation by the German army, leave no doubt that L. Schwartz's originality was equal to that of Sobolev when he rediscovered independently the latter's "functionals"; but, in contrast with his predecessor, he very soon became conscious of the wide scope of applications opened to these new tools, and devoted the next 15 years of his career to proving their value by his own papers and those of his pupils.

(*)It is a curious coincidence that the idea of computing with objects *other* than functions in analysis appeared almost at the same time as Sobolev's paper, in connection with other theories: De Rham's "currents" and L. C. Young's "generalized varieties" could be considered as generalizations of the linear forms $\omega \mapsto \int_V \omega$ on the vector space of differential p-forms in some $\mathbf{R}^n$ (V being a smooth p-dimensional manifold); but the publication dates of the corresponding papers show that they could not possibly have influenced Sobolev. One should note here that the clearest paper of L. C. Young introducing "generalized varieties" is the paper [11] of 1937 and not the ones mentioned by Lützen.

II. Fourier Transforms. Soon after the theory of the Fourier integral had been established on firm bases (end of the nineteenth century), analysts began to feel dissatisfaction that the Fourier transform $\mathscr{F}f(t) = \int f(x)e^{-ixt}\,dx$ could only be defined for functions f in $L^1(\mathbf{R})$. In 1910 Plancherel showed how the definition could be extended to functions of $L^2(\mathbf{R})$ by a simple limit process, and thus defined an isometry of $L^2(\mathbf{R})$ onto itself. But in 1911, H. Weyl was led by his researches on linear differential equations to investigate the possibility of defining the Fourier transform of a function f such that $f(x)/(1 + |x|)$ is integrable, and which satisfies an additional regularity condition; he found that this was still possible, but the Fourier transform would not any more be a function, but a bounded *Stieltjes measure* [**10**, vol. 1, pp. 359–360]. On the other hand, in 1920, P. Daniell independently defined the Fourier transform of such a measure, which later because a useful tool in the hands of probabilists [**5**](* *). Weyl's ideas were developed in 1926 by Hahn and Wiener, and it is to be noted that Hahn found out that it was sometimes necessary to introduce, not only Stieltjes measures, but also Cauchy's principal values of integrals, to define his Fourier transforms. Then in 1927 Bochner extended the notion of Fourier transform to all measurable functions f on $\mathbf{R}$ such that $f(x)/(1 + |x|^k)$ is integrable for some integer $k > 0$ (depending on f). Bochner's definition amounts to a particular "tempered distribution" in present day language, but it is doubtful if, when later Bochner reviewed Schwartz's book [**1**], he realized that his process was very far from defining *all* tempered distributions on $\mathbf{R}$, even those having a C^∞ function as Fourier transform (e.g., the Fourier transform of the function $(\sin e^x)e^x$). Furthermore, as Schwartz pointed out, Bochner never tried to develop a systematic study of his operators, in contrast with all the notions later introduced by Sobolev for his "functionals."

Sobolev does not seem to have ever thought of Fourier transforms in connection with his "functionals," and probably was unaware of Bochner's papers. Schwartz's ideas on that topic may have come from a remark made by A. Weil in 1940 ([**9**, p. 118]) (and which is not mentioned by Lützen): when discussing the definition of Fourier transforms of functions in the spaces L^p (studied earlier by W. H. Young and Hausdorff in particular), he considered that when two functions $\phi(x)$, $\Phi(x)$ are such that

$$\int \Phi(x) \cdot f(x)\,dx = \int \phi(x) \cdot \mathscr{F}f(x)\,dx \tag{9}$$

for all functions f such that *both* f and its Fourier transform $\mathscr{F}f$ belong to $L^1 \cap L^\infty$, it is legitimate to consider that Φ is the Fourier transform of ϕ. Schwartz had the idea of replacing these functions f, which are not easy to describe, by the "declining functions" in $\mathbf{R}^n$, which are C^∞ and such that the product of *any* derivative of f by *any* polynomial is integrable. The space $\mathscr{S}(\mathbf{R}^n)$ (now called the Schwartz space) of these functions is given a Fréchet space topology by the seminorms

$$q_{s,m}(f) = \sup_{|\alpha| \leqslant s,\, x \in R^n} (1 + |x|)^m |D^\alpha f(x)|. \tag{10}$$

The usual Fourier operator $f \mapsto \mathscr{F}f$ is then a bicontinuous bijection of $\mathscr{S}(\mathbf{R}^n)$ onto itself, and it is easily seen that the continuous linear forms on $\mathscr{S}(\mathbf{R}^n)$ are special distributions, which Schwartz called tempered distributions. The Fourier transform $\mathscr{F}T$ of such a distribution is then defined by the condition

$$\langle \mathscr{F}T, f \rangle = \langle T, \mathscr{F}f \rangle \tag{11}$$

for every $f \in \mathscr{S}(\mathbf{R}^n)$ (i.e., it is the *transpose* of the Fourier transform in $\mathscr{S}(\mathbf{R}^n)$).

With this happy move, the "prehistory" of distribution theory came to a close, and the stage was set for the giant strides in analysis which it allowed in the next 20 years.

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(* *) Neither Weyl's nor Daniell's paper is mentioned by Lützen.

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Mathematical Structures for Computer Science. By Judith L. Gersting. W. H. Freeman, San Francisco, 1982. x + 426 pp. $22.50.

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In the 1968 version of the Association for Computing Machinery curriculum recommendations —ACM 68--the course in discrete structures is one of four designated as a "basic Computer Science course." Ten years later in ACM 78 it is mentioned in a list of mathematics courses that are "included for completeness." In an abbreviated course program for small colleges, discrete structures has disappeared completely [1].

What happened? Many have ideas and some have written about them [2], [3]. Here is my conjecture. In ACM 68 the suggested approach to teaching this course includes the following phrase:

> The theoretical material should be introduced in a mathematically precise manner...

I suspect that the course was taught by mathematicians who followed this advice with the result (predictable for those of us who roared out of graduate school in the sixties and began teaching Bourbaki to undergraduates) that the students stayed away in droves. I very much doubt that many faculty took successfully the rest of this advice:

> ...with all concepts and results being amply motivated and being illustrated from Computer Science.

I am somewhat confirmed in this guess by the following curious statement that appears in ACM 78 referring to mathematical concepts and techniques deemed "vitally important for a computer scientist":

> Unfortunately, the kind and amount of material needed from these areas for Computer Science only can be obtained, if at all, from the regular courses offered by departments of Mathematics for their own majors.

"Unfortunately," indeed! And when you realize that this report has no less than 84 signatures, the statement represents quite an indictment of teachers of Discrete Structures in the last decade.

I wish it were the purpose of this essay to plead Not Guilty. I cannot. I can only try to talk about what needs to be done to rectify the situation. I believe that the rise of computer science presents an historic opportunity for mathematics education, but I am far from confident that our

profession will succeed in taking advantage of it.

Let me try to explain the opportunity. Contemporary undergraduate mathematics in this country is mainly based on the applications of mathematics to the physical sciences. Calculus, differential equations, numerical analysis, and matrices represent the major portion of our curriculum. It is a little lopsided. Geometry, algebra, topology and logic are not thriving. Maybe this is as it should be for many people believe that the nature of our subject should be determined by its applications. Others disagree—at least with the resulting curriculum [4]. The argument rages [5]. My point is that in any case a revision of the curriculum is called for. Computer science is now a major area for applications of mathematics and we should give the mathematics which is so applied a larger place in our teaching. This means algebra, logic and, to come to the point of this book review, discrete structures.

How is curriculum change brought about? Many things must be done, but eventually the changes must be embodied in textbooks and these textbooks must be adopted. Specifically, there should be textbooks written in discrete structures that exploit the myriad of connections between computer science and finite mathematics. For example, the existence of user-defined data types in programming languages like PASCAL can be used to explain domains of functions. A simple programming exercise in implementing complex arithmetic should permit students to study complex numbers as sets of ordered pairs. The problem of analyzing data flow in a complicated program can serve as the basis for an introduction to graph theory.

Such examples abound and their use makes it easier for students to learn many mathematical concepts. I call for a systematic study of these examples and the development of large portions of our curriculum based on the notion of learning conceptual material through direct computer experiences. I call for nothing less than allowing Papert's Mathworlds [6] to graduate high school and start college.

But this is not what we are getting. What we are getting are texts that pay lip service to introducing computer experience into mathematics education but do it in rather superficial ways, remaining in their hearts standard texts with standard material.

Consider some examples from the present book whose review provided the excuse for this essay. The title is certainly encouraging. The use of 1/4 of each page for a margin presumably increases readability (and also the number of pages which no doubt affects the price—I'd rather buy a tree). But the relevance for computer science appearing in this text is perhaps typified by Exercise 13 on Page 22 which is an example of a fallacious proof that "all computers are built by the same manufacturer." "We will prove," the problem says, "that in any collection of n computers, where m is a positive integer, all of the computers are built by the same manufacturer." Personally, instead of computers, I would have used military weapons and hoped for a class discussion to the effect that although the proof is fallacious, the conclusion may well be true.

A broader example arises in the author's use of simulation to explain homomorphism and isomorphism. Of course! That's what computers do all the time! A program is a simulation, that is, a homomorphism and this notion can be exploited. Instead we get the "simulation" of $[M_2^d(\mathbb{Z}), \cdot\,]$ by $(\mathbb{Z}, \cdot\,)$. What a missed opportunity. The book is full of them.

In fact, except for the title, the margin, the simulation and the manufacturers of computer equipment, this is a text on the standard material from finite mathematics and discrete structures presented in a definition-example-theorem-proof style (although "proof" is invariably called "practice"). Within those confines it is a decent book. But a more serious effort is needed. First, some people need to experiment with uses of computer experiences in teaching. Then they need to write a little about it and various experiences need to be shared. After some time a reasonable text can be written. By then, of course, it will be too late. The book publishers will have flooded the market with texts purporting to contain computer applications. Real analysis of what is a good book will be replaced by lucrative contracts for authors and free rides to the airport (with cocktails) at AMS meetings. Any long-term, considered and possibly collective, effort is likely to

be buried in all the hype. Nevertheless, I hope someone tries.

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The Mathematical Gardner. Edited by David A. Klarner. Wadsworth, Belmont, California, 1981. viii + 382 pp., $24.95.

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For those who love mathematical puzzles related to serious topics in mathematics, this collection is excellent. Written by over thirty contributors in honor of Martin Gardner's 65th birthday, there is a wide variety of topics covered under the six general sections: Games, Geometry, Two-Dimensional Tiling, Three-Dimensional Tiling, Fun and Problems, Numbers and Coding Theory.

Martin Gardner is surely the world's foremost authority on mathematical games and puzzles. His educational impact has been enormous. The role of so-called "recreational" mathematics in mathematical education is an attractive one whose creative possibilities have perhaps not been fully realized. It is a noteworthy fact that many people who claim to have wholly negative attitudes towards mathematics become completely intrigued when presented with a purely mathematical puzzle, providing it is called a "puzzle" rather than "mathematics." [As a perfect case in point, an old colleague of mine phoned me some time ago telling me of his nine-year old son's reaction to a puzzle book I had written. He said in a soft conspiratorial tone: "He loves your book, but when you speak to him, don't let him know that what he is doing is math, because he hates math!"]

I have sometimes wondered what the history of the world would have been like had Euclid written his famous *Elements* in the format of a puzzle book. For example, instead of stating and proving the Pythagorean theorem, he could have drawn a right triangle with squares on the hypotenuse and sides and presented the following puzzle (whose solution would be given at the back of the book): "Suppose these three squares are made of gold plate of uniform thickness and you are offered either the one large square or the two small ones; which would you pick?" I once tried this question on a remedial class in geometry. Interestingly enough, about half the class opted for the one large square and the other half for the two small ones. A lively discussion ensued, and both groups were equally amazed when finally told that it made no difference and immediately wanted to know why! The whole point is that their curiosity was piqued by the Pythagorean theorem presented as a *puzzle*; had I presented it in the normal manner of a theorem and proof, they would have been totally disinterested.

The moral above is important and is one which Martin Gardner has evidently realized full well in his many years of remarkable editorship of the Mathematical Games column of the *Scientific American*. Through these provoking columns and his numerous books, he has intrigued, delighted and instructed vast numbers of people from many walks in life in mathematical and scientific matters. *The Mathematical Gardner* is a fitting tribute to a unique educator of this century.

LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

5 October, 1983

Editor:

I can offer some evidence relating to Professor King's query about the reason for the absence of a Nobel Prize in mathematics (August-September, 1983). While I was an undergraduate at Northwestern in the early 60's, I heard the rumor that Nobel's refusal to endow a prize in mathematics was due to a grudge against Mittag-Leffler. As I heard it then, the two had quarreled over a woman. Later I read that Mittag-Leffler had accumulated a fortune, and somehow annoyed Nobel in the process.

During a three-month visit to the Institut Mittag-Leffler in 1981 I had the opportunity to talk with archivist Barbara Bjornberg, who knows the personal lives of the people around Mittag-Leffler extremely well. She had already heard the rumor and did not believe there was any truth to it. She pointed out to me that Nobel never married and that "Mittag-Leffler's fortune" was actually his wife's dowry. I could not find any evidence to corroborate the rumor. Since the absence of evidence is not the same as evidence of absence, the question really must be investigated from the other end. Who first made the allegations, and on what evidence?

Such evidence as I do have suggests that there is nothing surprising in the absence of a Nobel Prize in mathematics. The question itself seems to conceal a questionable assumption. Why *should* there be a Nobel Prize in mathematics? Nobel may not have considered mathematics important. Or, being a neighbor of Mittag-Leffler, he may have thought that the many prizes and honors Mittag-Leffler had obtained from King Oscar II for mathematicians were already sufficient recognition. These honors were quite extensive; they went to Poincaré, Appell, Bertrand, Hermite, Weierstrass, and Kovalevskaya, among others.

Mittag-Leffler was, at least in his old age, an irascible and eccentric man. Older residents of Djursholm recall that he used to spit on the ceiling during lectures, and that he was a fanatical cyclist, who, in both cycling and lecturing, ran heedlessly over any obstacle in his path. It would not be at all surprising if Nobel disliked him; but until we have positive evidence of a quarrel, it seems to me that we should refrain from spreading rumors of one.

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MISCELLANEA

Triangles are square

129.

"Every triangle consists of n congruent copies of itself" is true if and only if n is a square. (The proof is trivial.)

—Steven H. Cullinane

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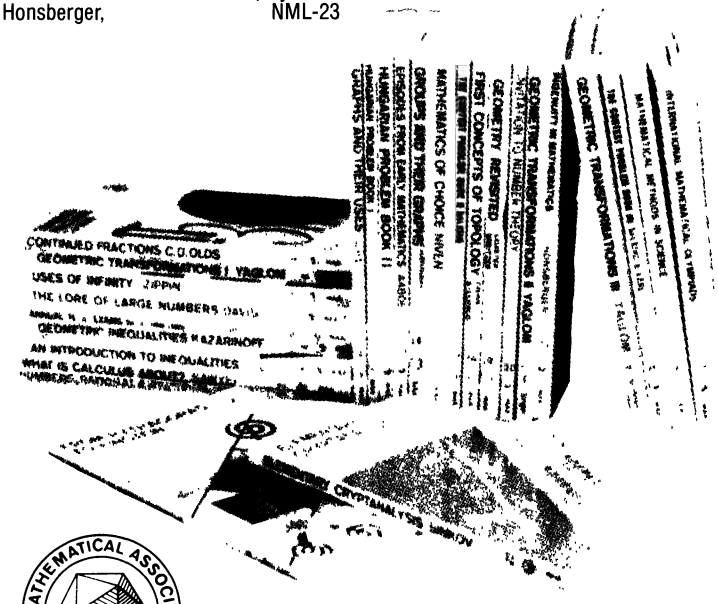
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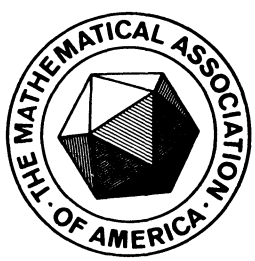
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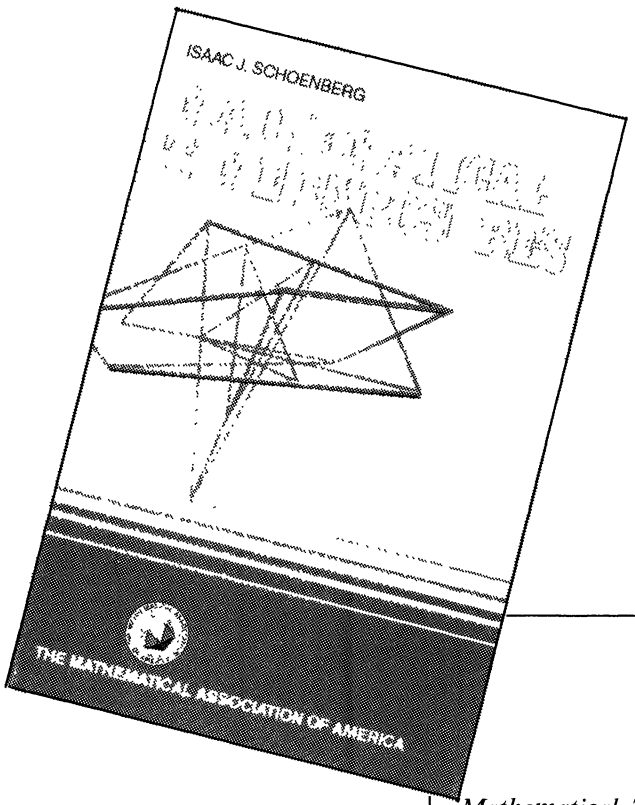
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HOW TO SOLVE A DIOPHANTINE EQUATION

A NUMBER-THEORETIC EXCURSION

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1. Introduction. Some time ago I was asked by Richard T. Bumby to investigate the equation

$$(x^2 - 2)^2 - 2 = 2y^2 \tag{1}$$

in rational integers x and y. In his letter Bumby explained that he had stumbled on a brief proof of the assertion that this diophantine equation has no other solutions than those given by $y = \pm 1$. Without hinting in any way at his own line of reasoning, he asked me to test the hypothesis that his proof is the obvious one in the sense that any "reasonable person" would discover the same proof. Flattered by being considered a reasonable person by implication, I set to work and although I did also find a solution not quite like Bumby's, there is no reason to reject his hypothesis. For, curiously enough, first I found a proof on a par with Bumby's descent argument (see Section 5) and soon after I realized that a more elementary and straightforward solution is possible. Subsequently I learned that equation (1) arose in the context of a search for squares amongst the elements of certain recurrences of order two [**5**].

In the course of my investigations I bumped into several blank walls. That is to say, some natural ways of tackling the problem apparently do not lead to the required solution. However, I also discovered some useful approaches, so that I can offer a few different proofs. What made me write all this down was the realization that quite a few methods for solving diophantine equations can be illustrated non-trivially by this particular equation. Thus, this paper has an expository character; it discusses certain number-theoretic methods which are especially useful for solving diophantine equations.

In the following sections I shall give a systematic exposition of the arguments leading to a solution. Occasionally, a digression is made from the main path for reasons of clarification and illustration. In the final section I shall briefly discuss a selection of references chosen from the most relevant literature.

2. Factorization. First of all it should be noted that the diophantine equation (1) has but finitely many solutions. This can be shown as an application of a famous theorem by A. Thue [**32**] published in 1909, which states that, for any irreducible form $f \in \mathbb{Z}[x, y]$ of degree at least three, the equation

$$f(x, y) = m$$

admits at most finitely many solutions $(x, y) \in \mathbb{Z}^2$ for any rational integer m. At the end of this section it will become clear how this theorem may be used in connection with equation (1).

Roughly speaking, the methods used for solving diophantine equations either are essentially of an analytic nature or use mainly algebraic techniques. The class of analytic methods is based on the theory of diophantine approximation culminating in the so-called Gel'fond-Baker method. This is very high-powered stuff and it seems advisable to consider first something of a more elementary nature. In Section 6 we shall briefly return to this very powerful method.

R. J. Stroeker: I received my mathematical education at the University of Amsterdam. In 1975 I took my doctor's degree there under the guidance of F. Oort. The years 1966–1971 I spent in Manchester (UK), where I taught at the University. During Lent term of 1969 I stayed at Oxford University with B. J. Birch. Since 1971 I have been teaching at Erasmus University, Rotterdam. My mathematical interests are in diophantine analysis, elliptic curves, classical approximation theory, and mathematical economics. I am a fervent bookbinder, I like to play badminton, and I play the cello occasionally.

The most down-to-earth approach is to merely use divisibility techniques such as unique factorization, congruences, quadratic reciprocity and the like. So, to start with, let us look upon our problem from an elementary point of view.

Suppose $(x, y) \in \mathbb{Z}^2$ is a solution of (1). Clearly x is even and this forces y to be odd. Put $x =: 2z$ and rewrite the equation to look like

$$y^2 + 1 = 2(2z^2 - 1)^2. \tag{2}$$

Assuming $y^2 \neq 1$, it follows that $y^2 + 1$ is divisible by an odd prime p which is necessarily of the form $p \equiv 1 \pmod 4$. From (2) it can be seen that each such prime divides $y^2 + 1$ to an even power, so that

$$y^2 + 1 = 2\Pi p^{2\alpha_p} \quad \text{and} \quad 2z^2 - 1 = \Pi p^{\alpha_p} \qquad (\alpha_p \in \mathbb{N}),$$

where the nonempty products are to be taken over all distinct primes $p \equiv 1 \pmod 4$ dividing $y^2 + 1$. Since we do not know anything of the primes p involved beyond the fact that they belong to the infinite set of primes of the form $\equiv 1 \pmod 4$, and because the values of the exponents α_p are also unknown, we soon realize that our line of reasoning most likely leads nowhere. Let us turn our attention in (we hope) a more promising direction.

A widely used technique in diophantine analysis is the technique of factorization. It is based on the application of the *fundamental theorem of arithmetic* (FTA) or its generalization to algebraic number fields. (An algebraic number field is understood to be a finite extension of the field of rationals $\mathbb{Q}$.) This FTA says that every positive integer can be written in one way only as a product of primes, except for the order in which the primes occur in the product. Another way of saying this is that the domain of integers $\mathbb{Z}$ is a *unique factorization domain* (UFD). Unlike $\mathbb{Z}$, the ring of integers O_K of an algebraic number field K often does not have this property. If O_K is a UFD, then K is said to have the unique factorization property. A few examples of such fields are: $\mathbb{Q}$, $\mathbb{Q}(i)$ (the field of Gaussian numbers), $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\exp(2\pi i/5))$. Fields that do not have the unique factorization property are, for instance, $\mathbb{Q}(\sqrt{-5})$, $\mathbb{Q}(\sqrt{10})$, $\mathbb{Q}(\sqrt[3]{7})$ and $\mathbb{Q}(\exp(2\pi i/23))$. For more details on unique factorization it is suggested that the reader consult [**4**], Chapter 3 or [**29**], Chapter 8. In fields that do not have the unique factorization property, unique factorization can be restored in the following sense: replace in our formulation of the FTA the words *positive integer* by *integral ideal* and the word *prime* by *prime ideal*. This leads to Dedekind's celebrated unique factorization theorem of ideals in number fields (cf. [**13**], Theorem I.4.2.).

Factorization of (2) in the ring of Gaussian integers $\mathbb{Z}[i]$ yields

$$y + i = \varepsilon(1 + i)(a + bi)^2$$

for rational integers a and b and $\varepsilon \in \{\pm 1, \pm i\}$. Indeed, $y^2 + 1 = (y + i)(y - i)$ and hence every Gaussian prime divisor of $y + i$ not dividing 2 divides $y + i$ to an even power. Then

$$2z^2 = 1 + a^2 + b^2 \quad \text{with} \quad |a^2 - b^2 \pm 2ab| = 1,$$

so that in particular a and b are relatively prime. Without loss of generality we may choose a and b such that $a^2 - b^2 \pm 2ab = 1$. Hence

$$2z^2 = 1 + a^2 + b^2 = (a^2 - b^2 \pm 2ab) + a^2 + b^2 = 2a(a \pm b)$$

Since a and $a \pm b$ are relatively prime, factorization of $z^2 = a(a \pm b)$ in $\mathbb{Z}$ leads to

$$a = \delta B^2, \quad a \pm b = \delta A^2$$

with $\delta = \pm 1$ and $z = AB$. Substitution of these expressions for a and b into $a^2 - b^2 \pm 2ab = 1$ gives

$$A^4 - 4A^2B^2 + 2B^4 = -1. \tag{3}$$

The equations (2) and (3)—and consequently also (1) and (3)—are equivalent in the sense that the solutions of one lead to all solutions of the other and conversely. But equation (3) has the

advantage of homogeneity, so that Thue's theorem, mentioned at the beginning of this section, can be directly applied. Unfortunately, this theorem does not give the solutions (if they exist) explicitly; the proof of Thue's result has a noneffective character. This leaves us as far from the actual solution of the problem as we were right at the start. Or does equation (3) after all provide a more natural way to look at the problem?

3. Norm Form Equations. Instead of merely asking for rational integers A and B satisfying (3), we may reformulate equation (3) in the following algebraic way.

The polynomial $f(t) := t^4 - 4t^2 + 2$ is irreducible over $\mathbb{Z}$ and the equation $f(t) = 0$ has four real roots, namely θ, $-\theta$, θ' and $-\theta'$, where $\theta := (2 + \sqrt{2})^{1/2}$ and $\theta' := (-1 + \sqrt{2})\theta$. Put $K := \mathbb{Q}(\theta)$, the extension of $\mathbb{Q}$ by θ. Then equation (3) takes the form

$$(A + B\theta)(A - B\theta)(A + B\theta')(A - B\theta') = -1$$

or, written in terms of the norm function,

$$\text{Norm}_{K/\mathbb{Q}}(A + B\theta) = -1. \tag{4}$$

This so-called norm form equation means that $\varepsilon := A + B\theta$ is a unit of O_K of a very special sort. What is so remarkable about it? Let me explain. As it happens, the basis $\{1, \theta, \theta^2, \theta^3\}$ of K as a vector space over $\mathbb{Q}$ is also an integral basis. That is to say, each integer of O_K can be uniquely expressed as a linear combination of these first four powers of θ with rational integral coefficients. The special character of ε lies in the fact that its expression as a linear combination of 1, θ, θ^2 and θ^3 has vanishing coefficients of θ^2 and θ^3.

Now if the structure of the group of units in O_K were known, the extra information on ε provided by its special form could be used to further restrict the possibilities for ε and hence also for the values of A and B. Fortunately, *Dirichlet's unit theorem* gives exact information on the structure of this unit group. To be precise, it says that the group of units in the ring of integers O_L of an algebraic number field L is the direct product of a finite cyclic group of roots of unity and a free abelian group of rank $r + s - 1$, where r is the number of real conjugate fields of L and s is the number of pairs of complex conjugate fields of L (see for instance [**13**], Theorem I.11.19).

In our case, the number field K has no roots of unity other than 1 and -1. Moreover, K is totally real so that $r = 4$ and $s = 0$. This implies that the free abelian group has three generators, ε_1, ε_2, ε_3 say. The set of these so-called *fundamental units* $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ can be effectively computed. A very clear and detailed account of such a process is given in the two papers of Pohst and Zassenhaus [**23**].

As a result of all this, we deduce from the norm form equation (4) the equation

$$A + B\theta = \pm \varepsilon_1^{e_1} \varepsilon_2^{e_2} \varepsilon_3^{e_3}, \tag{5}$$

where the exponents e_i are rational integers. We also get similar expressions for $A - B\theta$, $A + B\theta'$ and $A - B\theta'$ by considering the conjugation maps characterized by $\theta \mapsto -\theta$, $\theta \mapsto \theta'$ and $\theta \mapsto -\theta'$. Elimination of the unknown A and B from (5) and its conjugate equations yields a set of two equations in the three unknown exponents e_1, e_2 and e_3. Unfortunately, this is one unknown too many; Skolem's p-adic method, which is standard in this context, only works if the number of equations equals the number of unknown exponents. A detailed discussion of this method goes far beyond the scope of this paper. The persevering reader is referred to [**4**] or [**17**] (see also [**28**]). We shall only, be it very superficially, describe how the method often works.

Suppose certain expressions E involving the unknown exponents can be shown to satisfy congruences of the form

$$E \equiv 0 \pmod{p^\alpha}$$

for suitable rational primes p and all positive rational integers $\alpha > \alpha_0$. Then necessarily these expressions E must vanish. For instance, if in (5) it could be shown that each of the expressions $E_1 = e_1 - 1$, $E_2 = e_2$, $E_3 = e_3$ vanishes, then $A + B\theta = \pm\varepsilon_1$ which gives essentially one possi-

ble value for A as well as B. One could take $\varepsilon_1 = 1 + \theta$ (check that this is indeed a unit) and then $A = B = \pm 1$ would solve equation (3). Alas, Skolem's method does not apply in this particular case.

We have arrived at yet another dead end, it would seem. Sometimes one may succeed in restoring the conditions necessary for applying Skolem's method successfully, by considering a suitable finite extension of K. However, calculations tend to get very messy (cf. [**18**]), so much so that it often is more reasonable to try something else instead.

4. Recurrences. An obvious way of solving a diophantine equation which has not been mentioned so far, at least not in so many words, is simply that of reducing it to an equation that has already been solved.

Returning to equation (3), we rewrite it this time as

$$(A^2 - 2B^2)^2 - 2B^4 = -1. \tag{6}$$

This equation looks familiar; in [**19**] Ljunggren shows that the only positive integer solutions of the equation

$$y^2 + 1 = 2x^4 \tag{7}$$

are given by $(x, y) = (1,1)$ and $(13,239)$. For equation (6) Ljunggren's result means that the only solutions are given by $A^2 = B^2 = 1$ as a quick calculation confirms. *So we are done and the solution to our original problem is a fact.* However, this way of solving it is not quite satisfactory in the sense that Ljunggren's solution of (7) is very difficult. Couldn't we find a more elementary way of solving (6)? The answer is affirmative, as we shall see shortly.

Factorization of (6) in $\mathbb{Q}(\sqrt{2})$ yields

$$\mathrm{Norm}_{\mathbb{Q}(\sqrt{2})/\mathbb{Q}}(A^2 - 2B^2 + B^2\sqrt{2}) = -1.$$

Since all units of $\mathbb{Q}(\sqrt{2})$, disregarding their sign, are integral powers of $1 + \sqrt{2}$, we deduce that

$$A^2 - 2B^2 + B^2\sqrt{2} = \pm(1 + \sqrt{2})^{2k-1}, \quad k \in \mathbb{Z}. \tag{8}$$

It is easy to see that the $\pm$ sign in (8) may be dropped.

By the binomial theorem it can be seen that the numbers x_k and y_k, defined by the formula

$$x_k + y_k\sqrt{2} = (1 + \sqrt{2})^k, \quad k \in \mathbb{Z} \tag{9}$$

are rational integers. On taking the conjugate equation of (9) into account—this conjugate equation is obtained from (9) on replacing $\sqrt{2}$ by $-\sqrt{2}$ both times—we arrive at the following expressions for x_k and y_k:

$$x_k = \frac{\varepsilon^k + \varepsilon'^k}{\varepsilon + \varepsilon'} \quad \text{and} \quad y_k = \frac{\varepsilon^k - \varepsilon'^k}{\varepsilon - \varepsilon'}, \tag{10}$$

where we have written $\varepsilon := 1 + \sqrt{2}$ and $\varepsilon' := 1 - \sqrt{2}$ for convenience. Using these expressions for x_k and y_k, it is not difficult to verify the formulae:

$$\begin{array}{lll} \text{(i)} & x_0 = 1, x_1 = 1 \quad \text{and} \quad x_{k+1} = 2x_k + x_{k-1} \\ & y_0 = 0, y_1 = 1 \quad \text{and} \quad y_{k+1} = 2y_k + y_{k-1} \\ \text{(ii)} & y_{k+1} = y_k + x_k \quad \text{and} \quad x_{k+1} = x_k + 2y_k \\ \text{(iii)} & x_{-k} = (-1)^k x_k \quad \text{and} \quad y_{-k} = (-1)^{k-1} y_k \end{array}, \quad k \in \mathbb{Z}. \tag{11}$$

From (8) we deduce immediately that $A^2 - 2B^2 = x_{2k-1}$ and $B^2 = y_{2k-1}$ for some $k \in \mathbb{Z}$, $k \geqslant 0$, or by (11) (ii)

$$A^2 = x_{2k}, \; B^2 = y_{2k-1}. \tag{12}$$

This leads to yet another interpretation of equation (3): the solutions of (3) are characterized by

simultaneous squares in the sequences of rational integers $(x_{2n})_n$ and $(y_{2n-1})_n$ defined by (9). Apparently, a possible way to deal with our problem in this form is to use congruence considerations and "in particular" quadratic reciprocity. From (11) (i) we deduce that both x_{2n} and y_{2n-1} are odd for all n. So, from (12) we have

$$x_{2k} = A^2 \equiv 1 \pmod 8,$$

which implies that k is even. Put $k =: 2l$. It can be shown using the expressions (10) that for all n

$$x_{2n} + (-1)^{n+1} = (2y_n)^2.$$

Thus $A^2 - 1 = x_{4l} - 1 = (2y_{2l})^2$ and hence

$$A - 2y_{2l} = A + 2y_{2l} = \pm 1.$$

Clearly this is only possible when $y_{2l} = 0$ and $A^2 = 1$, implying also that $B^2 = 1$ in (3). *This solves equation* (3) *and we have discovered an elementary solution to our problem.* Unexpectedly, this solution turns out to be rather uncomplicated: we merely needed to show that the only square amongst elements of the sequence $(x_n)_{n \geqslant 0}$ with even index is $x_0 = 1$. This simplicity is coincidental. In general it is not so easy to determine all squares in a given linear recurrent sequence of order two. See [**6**], [**7**] and [**14**]. In this particular case Ljunggren's result [**19**] shows that the only squares in the sequence $(x_n)_{n \geqslant 0}$ are $x_0 = 1$, $x_1 = 1$ and those in the sequence $(y_n)_{n \geqslant 0}$ are $y_1 = 1$, $y_7 = 13^2$.

5. Infinite Descent. Although we have settled the problem we set out to solve, our method of proof is not the one Bumby had in mind. In fact, he was thinking of a so-called descent argument, which may be described as follows.

Imagine we are looking for all solutions of problem P in positive integers s. Suppose that, given a solution s of P, it can be proved that a smaller solution s' of P exists. In this way, given an initial positive integer solution, an infinite strictly decreasing sequence of positive integer solutions of P is obtained. However, a nonempty set of positive integers contains a smallest element. We have arrived at a contradiction. Consequently, there exists no such positive integer solution of P.

In the lines to follow we apply this descent argument to our equation. We return to equation (2). Let z be a positive integer satisfying C: $2(2z^2 - 1)^2 - 1 =$ square, subject to $z > 1$.

In the notation of Section 2 we have $z = AB$, where A and B satisfy equation (3). Once again, according to (12), a nonnegative rational integer k exists such that

$$A^2 = x_{2k} \quad \text{and} \quad B^2 = y_{2k-1}.$$

From the formulae (11) we deduce that $k \equiv 0 \pmod 4$. Put $k =: 4m$, then $m \geqslant 1$ because $z > 1$. As before, it can be shown using (10) that for all n

$$x_{2n} + (-1)^n = 2x_n^2.$$

Successive application of this identity with $n = 2m$ and $n = 4m$ yields

$$2(2x_{2m}^2 - 1)^2 - 1 = x_{8m} = A^2 = \text{square},$$

so that x_{2m} also satisfies C (note that $x_{2m} = 1$ implies $z = 1$). Finally, from

$$1 < x_{2m}^2 < x_{8m} y_{8m-1} = A^2 B^2 = z^2$$

it follows that $1 < x_{2m} < z$, which completes our descent argument. Our conclusion must be that $z = 0$ and $z = \pm 1$ are the only solutions of (2). *We have obtained a third solution to our problem, this time based on the classic idea of infinite descent, attributed to Pierre de Fermat.*

6. Diophantine Approximation. Suppose for the moment that all elementary and algebraic ideas considered so far have failed to produce a complete solution. Which other methods can be tried? Are there any more useful approaches to problems of this type? Certainly, the determina-

tion of integer solutions of a given diophantine equation may often be effected by looking for good rational approximations of certain algebraic, irrational numbers. To illustrate this point, consider again equation (8)

$$A^2 - 2B^2 + B^2\sqrt{2} = (1 + \sqrt{2})^{2k-1} \tag{13}$$

and assume $k \in \mathbb{N}$. This assumption does not cause a loss of generality, because an analogous argument can be used when $k \leqslant -1$, $k \in \mathbb{Z}$.

On taking the conjugate equation of (13) and dividing through by B^2, we have

$$2 + \sqrt{2} - \left(\frac{A}{B}\right)^2 = \frac{(1 + \sqrt{2})^{1-2k}}{B^2}.$$

Put $\theta := (2 + \sqrt{2})^{1/2}$, $\varepsilon := 1 + \sqrt{2}$ and assume A and B to be positive. It follows that

$$\theta\left|\theta - \frac{A}{B}\right| < \left|\theta + \frac{A}{B}\right|\left|\theta - \frac{A}{B}\right| = \frac{\varepsilon^{1-2k}}{B^2}$$

and thus

$$\left|\theta - \frac{A}{B}\right| < \frac{\varepsilon^{1-2k}}{\theta B^2} < \frac{1}{2B^2}. \tag{14}$$

In particular, this shows that A/B is a convergent of the regular continued fraction expansion of θ (cf. [**29**], Chapter 7). It also shows that A/B is very close to θ compared with B^{-2} when k is large. However, A/B cannot be too close to θ compared with B^{-4}. To be more precise, N. I. Feldman showed in 1971 [**10**] (see also [**11**]), using the powerful method developed by Gelfond and Baker that, given an algebraic number of degree $n \geqslant 3$ (in our case $n = 4$), effectively computable positive constants c and δ (depending on θ only) may be determined such that

$$\left|\theta - \frac{A}{B}\right| > \frac{1}{cB^{n-\delta}}$$

for any rational integers A, $B > 0$. Combining this with (14), we have

$$B^{2-\delta} > \frac{\theta\varepsilon^{2k-1}}{c}. \tag{15}$$

From (13) it is obvious that $A^2 - 2B^2 \geqslant 1$ since $k \geqslant 1$, and hence $B^2\sqrt{2} < \varepsilon^{2k-1}$. In view of (15), this implies

$$B^{\delta} < \frac{c}{\theta}B^2\varepsilon^{1-2k} < \frac{c}{\theta\sqrt{2}} < \frac{1}{2}c.$$

This inequality means that we have established an effectively computable upper bound B_0 for B. So, at least in principle, all solutions of (13) with $k \geqslant 1$ can be determined by checking the convergents of the regular continued fraction expansion of θ with denominators bounded from above by B_0. However, this constant B_0 may be spoiling the fun by being exceedingly large.

At this stage it may be proper to briefly comment on Baker's method, which has given the study of diophantine equations a tremendous impetus. Its most useful application to the theory of diophantine equations to date is probably given by the following inequality

$$0 < |b_1 \log \alpha_1 + b_2 \log \alpha_2 + \cdots + b_n \log \alpha_n| < e^{-\delta H}, \tag{16}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are given algebraic numbers ($\alpha_i \neq 0$ or 1), $0 < \delta \leqslant 1$ is a given real number and $b_1, b_2, \ldots, b_n$ are unknown rational integers with $H := \max(|b_1|, |b_2|, \ldots, |b_n|)$. If such an inequality holds, then Baker showed that $H < C$, where C is effectively computable in terms of the α's, δ and n. Many polynomial and exponential diophantine equations can be 'reduced' to inequalities of type (16), where the b's correspond to the unknown solutions. Usually, the

constant C is large. This may cause considerable practical problems in the sense that it could make a computer search utterly unfeasible. As an illustration, consider the Catalan equation $x^m - y^n = 1$. Tijdeman [**34**] proved that this equation has only finitely many solutions in integers $x > 1$, $y > 1$, $m > 1$, $n > 1$ and Langevin [**15**] calculated that for a solution (x, y, m, n) the following inequality holds:

$$x^m < \exp\exp\exp(250).$$

It seems that by the existing techniques this hopelessly large upper bound cannot be significantly lowered. This means that Catalan's conjecture (i.e., the only perfect powers differing by 1 are 3^2 and 2^3) is still not resolved.

Possibly the best known example of a successful attempt to solve a diophantine equation by Baker's method, not only in theory but also practically, is that of [**2**]. In order to solve the system of equations

$$3x^2 - 2 = y^2, \quad 8x^2 - 7 = z^2, \tag{17}$$

Baker and Davenport showed by factorization in $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ that to each solution (x, y, z) of (17) a pair (m, n) corresponds with

$$0 < m \log \alpha_1 - n \log \alpha_2 + \log \alpha_3 < 0.11\alpha_1^{-2m},$$

where $\alpha_1 := 2 + \sqrt{3}$, $\alpha_2 := 3 + 2\sqrt{2}$ and $\alpha_3 := 2(1 + \sqrt{3})\sqrt{2}/(\pm 1 + 2\sqrt{2})\sqrt{3}$. They estimated that $m < 10^{487}$ This huge m-region to be checked was reduced to the region $m < 500$ by applying a simple but ingenious lemma from diophantine approximation theory which involved the calculation of the two numbers $\log \alpha_3/\log \alpha_2$ (both signs) to 600 decimal places. Ultimately it was concluded that the only positive solutions of (17) are given by $x = 1$ and $x = 11$. In 1978 Grinstead [**12**] indicated how most of this very large number of m-values to be checked initially could be ruled out quickly by using a simple computer technique based on congruence considerations. It should also be mentioned that in [**14**] an elementary, but complicated, solution to problem (17) is given by means of quadratic reciprocity techniques.

For more examples on the solving of diophantine equations by Baker's method the reader is referred to Ellison [**9**].

We have already indicated a few times how the computer can be used to assist in the process of solving diophantine equations. More facts and ways on this particular approach may be found in [**31**] which also contains an extensive list of references.

7. Literature. Although the field of diophantine analysis is very old, relatively few books have been published dealing with diophantine equations exclusively. Best known are the books written by Mordell [**21**] and Skolem [**27**]. Less well known is the booklet by Bašmakova [**3**]. It is rather elementary, but it contains some interesting historical information. Most textbooks on number theory contain a chapter on diophantine equations. We mention only a few: Borevich & Shafarevich [**4**], LeVeque [**16**], Nagell [**22**] and Stark [**29**]. The algebraic background necessary can be found in Janusz [**13**], Ribenboim [**24**] and Stewart & Tall [**30**]. Schmidt's book [**25**] is a very nice book on diophantine approximation. Information on computer methods useful for diophantine problems can be obtained from Grinstead [**12**], Pohst & Zassenhaus [**23**], Stroeker & Tijdeman [**31**] and Williams [**35**]. Articles of an expository nature treating diophantine equations are: Baker [**1**], Shorey et al [**26**], Tijdeman [**33**], Stroeker & Tijdeman [**31**] and London & Finkelstein [**20**].

Finally, from a historical point of view, Dickson's famous history on the theory of numbers deserves to be mentioned [**8**]. It contains information on most results in diophantine analysis prior to 1920.

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A prolific number theorist, known to most students for another reason. (See p. 412.)

notation) to maximize an individual's utility function $U(x, y, z)$ subject to the budget constraint

$$B = P_x X + P_y Y + P_z Z$$

and the coupon constraint

$$R = r_x X + r_y Y + r_z Z.$$

(d) *Nonlinear Constraints.* Suppose our government agency in (b) can obtain price discounts by purchasing larger quantities of R. We can rewrite the budget constraint as

$$B_0 - rK - wL - \rho(R)R = 0.$$

5. Concluding Comments. We provide several references which the interested reader may wish to consult. Statements of sufficient conditions for local maxima (or minima) can be found in [3] and [6]. A very readable account of some elementary but interesting applications in economics can be found in [4]. An interpretation of the Lagrange multiplier and some consequences of the necessary conditions are provided in [7] but no use of sufficient conditions is mentioned there.

The authors of this paper have for several years jointly taught an upper level seminar in mathematical economics as part of an interdisciplinary major. The type of problem described here is considered in the early weeks of the seminar. Because each of us has learned a lot of useful things from the other, we recommend such interdisciplinary endeavors with enthusiasm.

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131. MISCELLANEA

The fraction of Life can be increased in value not so much by increasing your numerator as by lessening your denominator.

—*A Search for America*, by Frederick Philip Grove (McClelland and Stewart, Toronto, 1982), p. 219.

ANSWER TO PHOTO ON PAGE 393

Edmund Landau.

SUMS OF POWERS OF INTEGERS

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1. Introduction. The Bernoulli family produced several mathematicians, the first of whom was Jacob (1654–1705). He was the son of Nicolaus, a merchant in Basle, and it is reported [**1**] that Jacob had the motto 'Invito patre sidera verso' (against my father's will I study the stars). Despite his father's opposition, Jacob devoted his life to the study of mathematics and astronomy. His most famous work is 'Ars Conjectandi,' which was published after his death, in 1713, and which reflected his interest in probability. It was in this book that the famous Bernoulli numbers were introduced in connection with finding sums of powers of integers. Jacob was very enthusiastic about the technique that he had developed and in comparing his calculations with those of Bullialdus (1605–1694) he states [**2**] 'with the help of this table it took me less than half of a quarter of an hour to find that the tenth powers of the first 1000 numbers being added together will yield the sum

$$91,409,924,241,424,243,424,241,924,242,500.'$$

In this article we show that a simple formula will give highly accurate answers to such sums and with the help of a very crude electronic calculator we can obtain, in 9 seconds,

$$\sum_{r=1}^{1000} r^{10} \simeq 9.14104 \times 10^{31}$$

which is in error by 0.0005%!

The approximation used was

$$\sum_{r=1}^{n} r^k \simeq \frac{\left(n+\frac{1}{2}\right)^{k+1}}{k+1}, \tag{1.1}$$

and the idea stems from another observation of Bernoulli's that the sums can be expressed in the form $p([n(n+1)])$ when k is odd and $(2n+1)p([n(n+1)])$ when k is even, where $p(x)$ is a polynomial [**3**]. We refine the approximation (1.1) and extend the results to the case where k is any real number.

2. $\sum_{r=1}^{n} r^k$ (k is a Positive Integer). It is well known that this sum can be expressed as a polynomial in n of degree $k+1$. Bernoulli's observation mentioned earlier suggests that there is some advantage to be gained by considering the equivalent polynomial in $n+\frac{1}{2}$ which is also of degree $k+1$.

We shall use the method of undetermined coefficients to find this polynomial which we shall denote by $S_{k+1}(n+\frac{1}{2})$. Thus

$$\sum_{r=1}^{n} r^k = S_{k+1}\left(n+\frac{1}{2}\right), \tag{2.1}$$

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where

$$S_{k+1}\left(n+\frac{1}{2}\right)=\sum_{t=0}^{k+1}a_t\left(n+\frac{1}{2}\right)^t \tag{2.2}$$

From (2.1) it follows that

$$\sum_{r=1}^{n-1}r^k=S_{k+1}\left(n-\frac{1}{2}\right), \tag{2.3}$$

and so equations (2.1) and (2.3) give

$$S_{k+1}\left(n+\frac{1}{2}\right)-S_{k+1}\left(n-\frac{1}{2}\right)=n^k. \tag{2.4}$$

Now, using $\binom{k}{r}$ to denote the familiar binomial coefficient we have

$$\left(n+\frac{1}{2}\right)^t-\left(n-\frac{1}{2}\right)^t=\sum_{j=1}^{t}\binom{t}{j}\left(\frac{1}{2}\right)^j n^{t-j}\left(1+(-1)^{j+1}\right) \tag{2.5}$$

so that the right-hand side is either an even or an odd polynomial in n. Equation (2.4) can be rewritten

$$\sum_{t=1}^{k+1}a_t\sum_{j=1}^{t}\binom{t}{j}\left(\frac{1}{2}\right)^j n^{t-j}\left(1+(-1)^{j+1}\right)=n^k. \tag{2.6}$$

Notice that a_0 will have to be determined independently. The coefficients a_t for $t=k+1,k,\ldots,1$ can now be found by equating in (2.6) the terms in n^p for $p=k,k-1,\ldots,0$. This gives the following results:

$$\begin{aligned}
&a_{k+1}\binom{k+1}{1} &&=1,\\
&a_k\binom{k}{1} &&=0,\\
&a_{k+1}\binom{k+1}{3}\left(\frac{1}{2}\right)^2+a_{k-1}\binom{k-1}{1} &&=0,\\
&a_k\binom{k}{3}\left(\frac{1}{2}\right)^2+a_{k-2}\binom{k-2}{1} &&=0,
\end{aligned} \tag{2.7}$$

and generally

$$\begin{aligned}
&a_{k+1}\binom{k+1}{p+1}\left(\frac{1}{2}\right)^{p+1}\left(1+(-1)^{p+2}\right)+a_k\binom{k}{p}\left(\frac{1}{2}\right)^p\left(1+(-1)^{p+1}\right)\\
&+a_{k-1}\binom{k-1}{p-1}\left(\frac{1}{2}\right)^{p-1}\left(1+(-1)^p\right)+\cdots+a_{k+1-p}\binom{k+1-p}{1}=0.
\end{aligned}$$

We first observe that the second of these results immediately gives $a_k=0$ and hence $a_{k-2},a_{k-4},\ldots$ are all zero. From the first of these equations we obtain a_{k+1} and then solving for $a_{k-1},a_{k-3},\ldots$ in turn, we have

$$\begin{aligned}
a_{k+1}&=\frac{1}{k+1},\\
a_{k-1}&=-\frac{1}{24}\binom{k}{1},\\
a_{k-3}&=\frac{7}{960}\binom{k}{3},
\end{aligned} \tag{2.8}$$

$$a_{k-5} = \frac{-31}{2^7 \cdot 63}\binom{k}{5},$$

$$a_{k-7} = \frac{127}{2^7 \cdot 240}\binom{k}{7}.$$

Thus, for example, when $k = 3$ we obtain

$$\sum_{r=1}^{n} r^3 = S_4\left(n + \frac{1}{2}\right) = \frac{1}{4}\left(n + \frac{1}{2}\right)^4 - \frac{1}{8}\left(n + \frac{1}{2}\right)^2 + a_0.$$

When $n = 1$, $S_4(\frac{3}{2}) = 1$ and so $a_0 = \frac{1}{64}$.

Therefore,

$$S_4\left(n + \frac{1}{2}\right) = \frac{1}{4}\left\{\left(n + \frac{1}{2}\right)^2 - \frac{1}{4}\right\}^2 \tag{2.9}$$

$$= \frac{1}{4}n^2(n+1)^2,$$

this final form being the one that is well known.

The results above can perhaps be better appreciated from a computational point of view. Writing X for $N + \frac{1}{2}$ and $S(k)$ for $\sum_{r=1}^{n} r^k$, then we have shown that

$$S(k) = a_{k+1}X^{k+1} + a_{k-1}X^{k-1} + a_{k-3}X^{k-3} + \cdots, \tag{2.10}$$

where the coefficients are given by (2.8) and the last term is a_0 if k is odd and a_1X if k is even.

Thus when $k \geqslant 4$, we may rewrite (2.10) in the form

$$S(k) = \frac{X^{k+1}}{k+1}\left\{1 - \frac{\binom{k+1}{2}}{12X^2} + \frac{7\binom{k+1}{4}}{240X^4} - \cdots\right\}. \tag{2.11}$$

From (2.11) it is clear that good results may be obtained from the approximation

$$S(k) \simeq \frac{X^{k+1}}{k+1}. \tag{2.12}$$

We may illustrate these results by considering $\sum_{r=1}^{35} r^5$ and comparing the exact value of 333,263,700 with just two approximations that can be obtained from (2.10) or equivalently (2.11).

The first approximation, expressed in (2.12), gives

$$\sum_{r=1}^{35} r^5 \simeq \frac{35 \cdot 5^6}{6} = 333{,}594{,}489,$$

which is in error by about 0.1%.

The second approximation, using just the first two terms of (2.10) or (2.11) gives

$$\sum_{r=1}^{35} r^5 \simeq \frac{35 \cdot 5^6}{6} - \frac{5}{24}35 \cdot 5^4 = 333{,}263{,}608$$

which is in error by 92 or about 0.0000003%.

The natural extension of this work is to the case where k is not necessarily an integer.

3. The General Case. In this section we generalize the previous result to deal with $\sum_{r=1}^{n} r^k$ for all real k deriving a result that extends (2.11). In particular we show that the simple approximation obtained in (2.12) provides surprisingly accurate estimates when k is positive.

The formulae we will derive for k not a positive integer result from an application of the methods of asymptotic analysis to sums of powers of integers. A discussion of these techniques appears in [**4**]; in particular equation 3.4.6 of [**4**] gives a result for the convergent case in terms of

powers of n and the Riemann Zeta function.

Consider the Taylor series expansion of $f(x) = x^k$ about the integer $x = r$.

$$f(x) = f(r) + (x-r)f'(r) + \frac{(x-r)^2}{2!}f''(r) + \cdots + \frac{(x-r)^q}{q!}f^q(r) + \frac{(x-r)^{q+1}}{(q+1)!}\cdot f^{q+1}(z_r), \tag{3.1}$$

where $z_r = r + \theta_r(x-r)$ and $0 < \theta_r < 1$.

Using the notation

$$\binom{k}{r} = \frac{k(k-1)\cdots(k-r+1)}{r!}, \tag{3.2}$$

for non-integer k and integer r, we may rewrite (3.1) in the following way, where we have chosen q to be even without loss of generality.

$$x^k = r^k + \binom{k}{1}r^{k-1}(x-r) + \binom{k}{2}r^{k-2}(x-r)^2 + \cdots + \binom{k}{q}r^{k-q}(x-r)^q + \binom{k}{q+1}z_r^{k-q-1}(x-r)^{q+1}. \tag{3.3}$$

Integrating from $r - \frac{1}{2}$ to $r + \frac{1}{2}$ then gives

$$\int_{r-(1/2)}^{r+(1/2)} x^k dx = r^k + \binom{k}{2}r^{k-2}\cdot\frac{2}{3}\left(\frac{1}{2}\right)^3 + \binom{k}{4}r^{k-4}\cdot\frac{2}{5}\left(\frac{1}{2}\right)^5 + \cdots. \tag{3.4}$$

Summing equation (3.4) from $r = 1$ to n yields

$$g(k) = \sum_{r=1}^{n} r^k + \binom{k}{2}\frac{1}{3}\left(\frac{1}{2}\right)^2\sum_{r=1}^{n} r^{k-2} + \binom{k}{4}\frac{1}{5}\left(\frac{1}{2}\right)^4\sum_{r=1}^{n} r^{k-4} + \cdots, \tag{3.5}$$

where

$$g(k) = \frac{1}{k+1}\left\{\left(n+\frac{1}{2}\right)^{k+1} - \left(\frac{1}{2}\right)^{k+1}\right\} \quad (k \neq -1)$$

and $g(-1) = \log(2n+1)$. Write $S(k) = \sum_{r=1}^{n} r^k$. Then equation (3.5) may now be written

$$g(k) = S(k) + \binom{k}{2}\frac{1}{3\cdot 2^2}S(k-2) + \binom{k}{4}\frac{1}{5\cdot 2^4}S(k-4) + \cdots + \binom{k}{q}\frac{1}{(q+1)2^q}S(k-q) + R_{q+1}, \tag{3.6}$$

where

$$R_{q+1} = \binom{k}{q+1}\sum_{r=1}^{n}\int_{r-(1/2)}^{r+(1/2)} z_r^{k-q-1}(x-r)^{q+1}dx$$

and z_r is given in (3.1).

Thus for $q > k-1$,

$$|R_{q+1}| < \binom{k}{q+1}\sum_{r=1}^{n}\int_{r-(1/2)}^{r+(1/2)}\left(r-\frac{1}{2}\right)^{k-q-1}|x-r|^{q+1}dx,$$

that is,

$$|R_{q+1}| < \binom{k}{q+1}\sum_{r=1}^{n}\left(r-\frac{1}{2}\right)^{k-q-1}\frac{2\left(\frac{1}{2}\right)^{q+2}}{q+2}, \tag{3.7}$$

and so

$$|R_{q+1}| < \binom{k}{q+1}\frac{n}{q+2}\left(\frac{1}{2}\right)^k$$

From this we can easily deduce that for fixed k and n, $\lim_{q\to\infty} R_{q+1} = 0$ so that the series (3.6) converges.

Putting $k, k-2, k-4, \ldots$ in turn in equation (3.6) gives the set of equations

(3.8)

$$\begin{aligned}
g(k) &= S(k) + \binom{k}{2}\frac{1}{3\cdot 2^2}S(k-2) + \binom{k}{4}\frac{1}{5\cdot 2^4}S(k-4) + \binom{k}{6}\frac{1}{7\cdot 2^6}S(k-6) + \cdots \\
g(k-2) &= S(k-2) + \binom{k-2}{2}\frac{1}{3\cdot 2^2}S(k-4) + \binom{k-2}{4}\frac{1}{5\cdot 2^4}S(k-6) + \cdots \\
g(k-4) &= S(k-4) + \binom{k-4}{2}\frac{1}{3\cdot 2^2}S(k-6) + \cdots \\
g(k-6) &= S(k-6) + \cdots \\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

These equations can be written in matrix form

$$g = Ms, \tag{3.9}$$

where

$$g = \begin{pmatrix} g(k) \\ g(k-2) \\ g(k-4) \\ \vdots \end{pmatrix}, \quad M = \begin{pmatrix} 1 & \binom{k}{2}\frac{1}{3\cdot 2^2} & \binom{k}{4}\frac{1}{5\cdot 2^4} & \binom{k}{6}\frac{1}{7\cdot 2^6} & \cdots \\ 0 & 1 & \binom{k-2}{2}\frac{1}{3\cdot 2^2} & \binom{k-2}{4}\frac{1}{5\cdot 2^4} & \cdots \\ 0 & 0 & 1 & \binom{k-4}{2}\frac{1}{3\cdot 2^2} & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

and

$$s = \begin{pmatrix} S(k) \\ S(k-2) \\ S(k-4) \\ \vdots \end{pmatrix}$$

We initially solve the problem by truncating the infinite matrix M to a finite size M_l with l rows and columns. In a similar way the vector g is truncated to g_l and s_l is defined so that $M_l s_l = g_l$.

It is worth pointing out that the form of the matrix M is such that having found M_l^{-1} for a given l then this M_l^{-1} forms a submatrix of the inverses of higher orders. Taking as a particular case $l = 4$, then we obtain

$$M_4^{-1} = \begin{pmatrix} 1 & -\binom{k}{2}\frac{1}{3\cdot 2^2} & \binom{k}{4}\frac{7}{15\cdot 2^4} & -\binom{k}{6}\frac{31}{21\cdot 2^6} \\ 0 & 1 & -\binom{k-2}{2}\frac{1}{3\cdot 2^2} & \binom{k-2}{4}\frac{7}{15\cdot 2^4} \\ 0 & 0 & 1 & -\binom{k-4}{2}\frac{1}{3\cdot 2^2} \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.10}$$

so that

$$s_4 = M_4^{-1} g_4. \tag{3.11}$$

From this, we can obtain a 4-term approximation to $S(k)$ given by

$$S_4(k) = g(k) - \binom{k}{2}\frac{1}{3 \cdot 2^2} g(k-2) + \binom{k}{4}\frac{7}{15 \cdot 2^4} g(k-4) - \binom{k}{6}\frac{31}{21 \cdot 2^6} g(k-6). \tag{3.12}$$

Equation (3.12) agrees with the previous results (equations (2.8) and (2.10)) obtained when k was a positive integer.

Again the numerical accuracy of the result may be demonstrated by taking the first few terms as in the next example.

Now, $\sum_{r=1}^{35} r^{2.5} = 76138.722369$ and the table below gives the successive approximations for various values of l.

l	Approximation
1	76160.721097
2	76138.725044
3	76138.722204
4	76138.722459

Note that the approximation $35.5^{3.5}/3.5$ which is analogous to (2.12) is in error by only .003%.

The matrix technique used in this section can be applied in the case where k is a positive integer.

M will now be of finite order m and the previous analysis will still apply provided that the restricted order l is less than m.

4. Modifications when k is Negative. In practice, when k is negative, the accuracy of (3.12) is improved by starting the summation at $r = p$. This is equivalent to redefining $S(k)$ by

$$S(k) = \sum_{r=p}^{n} r^k, \tag{4.1}$$

with the corresponding modifications

$$g(k) = \frac{1}{k+1}\left\{\left(n + \frac{1}{2}\right)^{k+1} - \left(p - \frac{1}{2}\right)^{k+1}\right\}, \quad k \neq -1 \tag{4.2}$$

and

$$g(-1) = \log\left(n + \frac{1}{2}\right) - \log\left(p - \frac{1}{2}\right). \tag{4.3}$$

For example, in the case $k = -1$, we have the explicit form

$$\sum_{r=p}^{n} \frac{1}{r} = \left\{\log\left(n + \frac{1}{2}\right) - \log\left(p - \frac{1}{2}\right)\right\} + \frac{1}{24}\left\{\left(n + \frac{1}{2}\right)^{-2} - \left(p - \frac{1}{2}\right)^{-2}\right\} - \frac{7}{960}\left\{\left(n + \frac{1}{2}\right)^{-4} - \left(p - \frac{1}{2}\right)^{-4}\right\} + \frac{31}{63 \cdot 2^7}\left\{\left(n + \frac{1}{2}\right)^{-6} - \left(p - \frac{1}{2}\right)^{-6}\right\} \cdots. \tag{4.4}$$

Thus for $p = 3$ and $n = 50$, we have used the inverse matrix technique described previously to compute a 1-term approximation of 3.0056826 and a 5-term approximation of 2.9992059. This can be compared with the exact result of 2.9992053 to 7 decimal places.

It is also interesting to compare the well-known approximation

$$\sum_{r=1}^{n} \frac{1}{r} \simeq \log n + \gamma, \tag{4.5}$$

where $\gamma = 0.57721566\ldots$ is Euler's constant, with our first approximation

$$\sum_{r=1}^{n} \frac{1}{r} \simeq \sum_{r=1}^{p-1} \frac{1}{r} + \log\left\{\frac{n+\frac{1}{2}}{p-\frac{1}{2}}\right\} \tag{4.6}$$

and our second approximation

$$\sum_{r=1}^{n} \frac{1}{r} \simeq \sum_{r=1}^{p-1} \frac{1}{r} + \log\left\{\frac{n+\frac{1}{2}}{p-\frac{1}{2}}\right\} + \frac{1}{24}\left\{\left(n+\frac{1}{2}\right)^{-2} - \left(p-\frac{1}{2}\right)^{-2}\right\}. \tag{4.7}$$

For $n = 50$, equation (4.5) leads to the approximation 4.48924 compared with the exact value of 4.499205, an error of .22%. The following table compares this with the approximations (4.6) and (4.7) for various values of p.

Approximation	$p = 2$	$p = 3$	$p = 4$
(4.6)	4.51651	4.40468	4.50254
(4.7)	4.49801	4.49903	4.49916

So far we have only considered examples of finite series of the form $\Sigma_{r=1}^{n} r^k$ since for $k \geqslant -1$ the sum to infinity does not exist. However, equations (3.12) and (4.2) can be used to estimate $\Sigma_{r=1}^{\infty} r^k$ when $k < -1$. In this case we may let $n \to \infty$ and obtain

$$\sum_{r=1}^{\infty} r^k = \sum_{r=1}^{p-1} r^k - \frac{1}{k+1}\left(p-\frac{1}{2}\right)^{k+1} + \binom{k}{2} \cdot \frac{1}{12} \cdot \frac{1}{k-1}\left(p-\frac{1}{2}\right)^{k-1} - \binom{k}{4} \frac{7}{240} \cdot \frac{1}{k-3}\left(p-\frac{1}{2}\right)^{k-3} + \binom{k}{6} \frac{31}{1344} \cdot \frac{1}{k-5}\left(p-\frac{1}{2}\right)^{k-5} - \cdots. \tag{4.8}$$

Equation (4.8) can be written

$$\sum_{r=1}^{\infty} r^k = \sum_{r=1}^{p-1} r^k - \frac{1}{k+1}\left(p-\frac{1}{2}\right)^{k+1} + \frac{k}{24}\left(p-\frac{1}{2}\right)^{k-1} - \binom{k}{3} \frac{7}{960}\left(p-\frac{1}{2}\right)^{k-3} + \binom{k}{5} \frac{31}{8064}\left(p-\frac{1}{2}\right)^{k-5} - \cdots. \tag{4.9}$$

We can easily adapt the proof of convergence of (3.6) to the case where n is infinite (and of course $k < -1$). From a modified form of (3.7) we have

$$|R_q| < \frac{1}{2^q(q+1)} \binom{k}{q} \sum_{r=p}^{\infty} \left(r-\frac{1}{2}\right)^{k-q}.$$

Since $k < -1$, the infinite sum is convergent so that

$$\lim_{q \to \infty} R_q = 0.$$

Computations using the terms given in (4.9) have been carried out for the infinite series where $k = -2$ and $k = -3.5$ with various values of p. The results are shown below.

k	$p = 2$	$p = 3$	$p = 4$	Exact
-2	1.644466	1.644928	1.644934	1.644934
-3.5	1.124871	1.126723	1.126733	1.126730

5. Convergence. In the previous sections we have approximated $\Sigma_{l=p}^{n} r^k$ by taking the first few terms of the series

$$\sum_{l=0}^{\infty} m_l g(k-2l), \tag{5.1}$$

where m_l is the lth entry in the first row of M_l^{-1} (see (3.9)–(3.12)) and $g(k)$ is given by (4.2).

In fact (5.1) converges only when k is a positive integer and in this case the series breaks off after finitely many terms.

In all other cases the series (5.1) diverges and we now show that its terms eventually have increasing modulus.

We first note that from (4.2)

$$g(k-2l) = \frac{1}{k-2l+1}\left\{\left(n+\frac{1}{2}\right)^{k-2l+1} - \left(p-\frac{1}{2}\right)^{k-2l+1}\right\},$$

so that for sufficiently large,

$$|g(k-2l)| > \frac{1}{|k-2l+1|} \cdot \frac{1}{p^{2l-k-1}} \qquad \text{for } p \geqslant 1. \tag{5.2}$$

The matrix M_l can be written in the form

(5.3)

$$M_l = \begin{pmatrix} 1 & a(k) & \lambda_1 a(k)a(k-2) & \lambda_2 \prod_{i=0}^{2} a(k-2i) & \cdots & \lambda_{l-2} \prod_{i=0}^{l-2} a(k-2i) \\ 0 & 1 & a(k-2) & \lambda_1 a(k-2)a(k-4) & \cdots & \lambda_{l-3} \prod_{i=1}^{l-2} a(k-2i) \\ 0 & 0 & 1 & a(k-4) & \cdots & \lambda_{l-4} \prod_{i=2}^{l-2} a(k-2i) \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & a(k-2l+4) \\ & & & & & 1 \end{pmatrix}$$

where

$$a(k) = \binom{k}{2}\frac{1}{3 \cdot 2^2} \quad \text{and} \quad \lambda_r = \frac{6^{r+1}}{(2r+3)!}.$$

Consider the equation

$$M_l x = e, \tag{5.4}$$

where e is the lth column of the $l \times l$ identity matrix. The first entry of x is $x_1 = m_l$. Using Cramer's Rule and noting that $\det M_l = 1$, we have

(5.5)

$$m_l = \begin{vmatrix} 0 & a(k) & \lambda_1 a(k)a(k-2) & \lambda_2 \prod_{i=0}^{2} a(k-2i) & \cdots & \lambda_{l-2} \prod_{i=0}^{l-2} a(k-2i) \\ 0 & 1 & a(k-2) & \lambda_1 a(k-2)a(k-4) & \cdots & \lambda_{l-3} \prod_{i=1}^{l-2} a(k-2i) \\ \vdots & & & & & \vdots \\ 0 & 0 & & & & a(k-2l+4) \\ 1 & 0 & & & & 1 \end{vmatrix}$$

$$= (-1)^{l+1}\begin{vmatrix} a(k) & \lambda_1 a(k)a(k-2) & \lambda_2 \prod_{i=0}^{2} a(k-2i) & \dots & \lambda_{l-2}\prod_{i=0}^{l-2} a(k-2i) \\ 1 & a(k-2) & \lambda_1 a(k-2)a(k-4) & \dots & \vdots \\ 0 & 1 & a(k-4) & \dots & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a(k-2l+4) \end{vmatrix}$$

Use induction on l to prove that

$$m_l = (-1)^{l+1} a(k)a(k-2)\cdots a(k-2l+4)\Delta_l \tag{5.6}$$

where

$$\Delta_l = \begin{vmatrix} 1 & \lambda_1 & \lambda_2 & \dots & \lambda_{l-2} \\ 1 & 1 & \lambda_1 & \dots & \lambda_{l-3} \\ 0 & 1 & 1 & & \\ & 0 & & & \vdots \ \ \vdots \\ \vdots & \vdots & & & 1 \ \ \lambda_1 \\ 0 & 0 & \dots & & 1 \ \ 1 \end{vmatrix}. \tag{5.7}$$

The value of Δ_l can be found by successively eliminating the 'ones' below the main diagonal by subtracting a multiple of the previous row. This produces a triangular determinant given by

$$\Delta_l = \begin{vmatrix} 1 & \lambda_1 & \dots & \lambda_{l-2} \\ 0 & f_1 & & \\ & 0 & f_2 & \\ \vdots & \vdots & & \\ 0 & 0 & \dots & f_{l-2} \end{vmatrix} \tag{5.8}$$

whose value is $f_1 f_2 \cdots f_{l-2}$. Increasing l does not change the previous values of f_l so that for example

$$\Delta_{l+1} = f_1 f_2 \cdots f_{l-2} f_{l-1}. \tag{5.9}$$

A numerical calculation of the sequence $\{f_i\}$ shows that we have a monotonic decreasing sequence tending rapidly to a limit. The table below gives various values of f_i.

i	f_i
1	0.7
2	0.63265306
13	0.60792713
14	0.60792711
15	0.60792710
16	0.60792710

Defining this limit to be $1/\beta$, we see that $\Delta_l \to 0$ as $l \to \infty$. However, from the monoticity we can conclude that

$$\Delta_l > \left(\frac{1}{\beta}\right)^{l-1} \tag{5.10}$$

Now $\Pi_{i=0}^{l-2} a(k-2i)$ can be written as

$$\frac{k(k-1)\cdots(k-2l+3)}{24^{l-1}}. \tag{5.11}$$

Thus combining the results of this section, we have shown that the general term of (5.1) satisfies

$$|m_l g(k-2l)| > \frac{|k(k-1)\cdots(k-2l+3)|}{(24\beta)^{l-1}} \left| \frac{p^{k-2l+1}}{k-2l+1} \right|. \tag{5.12}$$

Since p and k are fixed, (5.11) shows that eventually the terms of the series (5.1) increase in magnitude. This shows that these approximations will always get worse for sufficiently large l. The reason why the estimates given in this paper are good can be seen by regarding equations (3.8) truncated in the form

$$M_l \hat{S}_l = g_l + \varepsilon_l, \tag{5.13}$$

where $\hat{S}_l$ is the first l values of the exact solution and $-\varepsilon_l$ is the vector of the truncation errors in the first l equations (see (3.7)). The exact expression for $S(k)$ is then a finite sum of terms in the form

$$m_t(g(k-2t)+\eta_t), \tag{5.14}$$

where η_t are the elements of ε_l. Taking the particular case $t=3$, we have

$$m_3 = \frac{7k(k-1)(k-2)(k-3)}{4!\cdot 15\cdot 2^4}. \tag{5.15}$$

Since η_3 is small and for positive k the sums are of the order $(n+\frac{1}{2})^{k+1}/(k+1)$, then for large n the relative error

$$|m_3\eta_3|/S(k) \tag{5.16}$$

is small. For negative k (particularly $k<-1$) the approximations are formed by using $p>1$. Since η_3 decreases with p, then it is always possible to reduce the size of $|m_3\eta_3|$. These arguments hold for any t with sufficiently large n or p so that it is always possible to obtain good estimates from the initial terms of (5.1). This gives a heuristic argument why we have been able to obtain surprisingly good approximations to $S(k)$ from a series of terms (5.1) that actually diverges!

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These forms were like the cunning tables used by mathematicians, which may be entered from top, bottom, right, and left, which entrances consist of scores of lines and dozens of columns, and from which may be drawn, without reasoning or thinking, thousands of different conclusions, all unchallengeably precise and true.

—*Martin Eden*, by Jack London,
Macmillan, New York, 1973, p. 226.

LAGRANGE MULTIPLIER PROBLEMS IN ECONOMICS

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1. Introduction. Several surprises are in store for the mathematics student who looks for the first time at nontrivial constrained optimization problems in economics. The usual constrained problem in a mathematics course has only one or two critical points and the selection of the absolute maximum is clear from the geometric nature of the problem. Mathematics texts often ignore sufficient conditions (involving bordered Hessian determinants) for relative extrema and provide no interpretation of the Lagrange multiplier λ, leaving the student with the impression that λ has no significance beyond providing an extra variable which magically transforms the constrained problem into an unconstrained higher dimensional problem.

Our purpose here is to examine carefully one example in order to highlight the following features that are typical in the formulation of problems in economics:

(i) Functions are not usually explicitly given, but are assumed to have characteristic qualitative properties. Thus the problems have an air of "theory" rather than "computation." Yet the problems are meaningful and often generate important insights into economic behavior.

(ii) Normally the economist is not interested in solving for a constrained optimum; rather his starting assumption is that an optimum is achieved and seeks to base predictions of behavioral responses on the assumption that optimizing will continue. For example, assuming a firm minimizes the cost of producing a given output, one wishes to know how changes in input prices will affect the firm's behavior. Thus the problem is not "find the minimum," but "assuming the minimum is attained, what consequences can be deduced?" An interesting result is that the economist is primarily interested in necessary conditions for an optimum and wishes profoundly that sufficient conditions were really necessary.

(iii) The Lagrange multiplier λ generally has significance in economic problems. It can usually be interpreted as the rate of change of the optimal value relative to some parameter.

(iv) The implicit function theorem plays a pivotal role in the problem analysis, both theoretically and computationally, since the analysis requires solving a system of nonlinear equations for the endogenous (dependent) variables and computing partial derivatives of these variables with respect to the exogenous (independent) variables.

2. An Illustrative Example. We analyze in detail an example in utility maximization. We begin with the problem description.

Assume an individual obtains utility (i.e., satisfaction) from the consumption of two goods x and y which are purchased in the marketplace in quantities X and Y, respectively, and from a certain quantity L of leisure time l. The theory of consumer choice can then be characterized mathematically as a constrained optimization problem in which the individual chooses, in an optimal way, the amount of leisure time l (and implicitly money income) and the quantities of x

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and y consumed. We shall assume that the individual spends all his income on the purchases of such goods. If this offends readers who are avid savers, they may assume that one of the consumption goods represents savings. To keep the problem manageable, however, we proceed on the assumption of a two-good world. Let P_x and P_y be the price per unit of x and y, respectively. Let T represent the total time available and L the amount of leisure time chosen by the individual per period (e.g., per week, month, or year). Work time can then be defined as $T - L$. If the market-determined wage rate is w, the individual's income is then $w(T - L)$. Since income equals consumption expenditures, the budget constraint becomes:

$$w(T - L) - P_x X - P_y Y = 0. \tag{1}$$

We assume that the utility U obtained from a particular choice of X, Y, and L is given by a utility function (unique to the individual)

$$U = u(X, Y, L). \tag{2}$$

Thus the consumer's problem is to choose X, Y, L to maximize U subject to the constraint (1). Furthermore it is reasonable to assume that

$$u_X > 0, \quad u_Y > 0, \quad u_L > 0, \tag{3}$$

$$u_{XX} < 0, \quad u_{YY} < 0, \quad u_{LL} < 0, \tag{4}$$

$$u_{XL} > 0, \quad u_{YL} > 0, \tag{5}$$

$$\text{either } u_{XY} > 0 \quad \text{or } u_{XY} < 0 \quad \text{or } u_{XY} = 0, \tag{6}$$

where subscripts denote, as usual, partial derivatives. The first order derivatives in (3) are called marginal utilities. The inequalities in (3) state that utility increases with higher levels of consumption of X and Y and with more leisure time per period. The inequalities in (4) state the "law of diminishing marginal utility." That is, while utility increases with consumption of x, y, and l, it increases at a diminishing rate. The third beer one drinks within an hour does not quench one's thirst as much as the second beer. The inequalities in (5) state that the satisfaction one derives from consuming larger amounts of x (or y) is enhanced by the availability of more leisure time. It takes time to "enjoy things." The sign of u_{XY} in (6) depends on whether x and y are substitutes (pepsi and coke), complements (tennis rackets and tennis lessons), or unrelated (mathematics texts and jellybeans). If the goods are substitutes, $u_{XY} < 0$; if complements, $u_{XY} > 0$; and if unrelated, $u_{XY} = 0$. The inequalities in (3) through (6) are posited relationships from economic theory. [See problem feature (i).]

Applying the method of Lagrange, we introduce the multiplier λ and form the Lagrangian

$$v(X, Y, L, \lambda) = u(X, Y, L) + \lambda\big(w(T - L) - P_x X - P_y Y\big). \tag{7}$$

Assuming the consumer maximizes utility, the optimal quantities X^*, Y^*, L^* and the multiplier λ^* necessarily satisfy the first-order conditions:

$$v_\lambda = w(T - L) - P_x X - P_y Y = 0, \tag{8a}$$

$$v_X = u_X - \lambda P_x \qquad = 0, \tag{8b}$$

$$v_Y = u_Y - \lambda P_y \qquad = 0, \tag{8c}$$

$$v_L = u_L - \lambda w \qquad = 0. \tag{8d}$$

Observe that because of (3) each of the equations (8b–d) imply that the optimal value $\lambda^* > 0$. [See problem feature (ii).]

In the model, the individual chooses X, Y, and L. The prices P_x, P_y, and the wage rate w are given by market conditions beyond the individual's influence or control. T is given by nature. The economist refers to X, Y, and L as endogenous variables and to P_x, P_y, and w as exogenous variables. The central question for the economist is to analyze the impact on the endogenous

variables of changes in the exogenous variables, assuming the individual responds to such changes by making a different set of choices in order to re-maximize utility in the changed context. Thus, if P_x should increase, with P_y and w constant, we might intuitively expect the consumer to adjust by decreasing X and increasing Y or decreasing L in order to earn more income to finance purchases of the more expensive x. Mathematically, we want to solve the four equations (8) for X, Y, L, and λ in terms of P_x, P_y, and w and compute the twelve partial derivatives: $\partial X/\partial P_x$, $\partial X/\partial P_y$, $\partial X/\partial w$, etc. These partial derivatives are referred to as the comparative statics of the model. The model's usefulness is determined by how accurately it predicts adjustments in consumer behavior.

The problem at hand consists of four equations (8a–d) in seven variables. Since we shall assume that the left side of each equation is continuously differentiable and that solutions exist, then by the implicit function theorem, X^*, Y^*, L^*, λ^* will each be continuously differentiable functions of P_x, P_y, and w, if the Jacobian matrix

$$H = \begin{bmatrix} 0 & -P_x & -P_y & -w \\ -P_x & u_{XX} & u_{XY} & u_{XL} \\ -P_y & u_{YX} & u_{YY} & u_{YL} \\ -w & u_{LX} & u_{LY} & u_{LL} \end{bmatrix} \tag{9}$$

is nonsingular at the optimum point (X^*, Y^*, L^*).

For the moment, let us assume that H is nonsingular. Solving (8) for X, Y, L, λ in terms of P_x, P_y, w and substituting the results into (2), we can express the optimized utility U^* in terms of P_x, P_y, w: $U^* = \tilde{u}(P_x, P_y, w)$. Using the chain rule, we obtain

$$\frac{\partial U^*}{\partial w} = u_X \frac{\partial X}{\partial w} + u_Y \frac{\partial Y}{\partial w} + u_L \frac{\partial L}{\partial w}; \tag{10}$$

then from equations (8) in (10), we obtain

$$\frac{\partial U^*}{\partial w} = \lambda \left[P_x \frac{\partial X}{\partial w} + P_y \frac{\partial Y}{\partial w} + w \frac{\partial L}{\partial w} \right]. \tag{11}$$

From (8a), we have the identity

$$wT = P_x X + P_y Y + wL,$$

and differentiation yields

$$T = P_x \frac{\partial X}{\partial w} + P_y \frac{\partial Y}{\partial w} + w \frac{\partial L}{\partial w} + L,$$

which allows us to rewrite (11) as

$$\lambda = \frac{\partial U^*}{\partial w} \frac{1}{T - L}. \tag{12a}$$

Thus the Lagrange multiplier λ may be interpreted as the marginal utility of the wage rate per hour of work. Letting

$$B = w(T - L)$$

be the money income of the individual, we can transform (12a) into an expression of more interest in economics. Recalling that the optimum choice of L depends on P_x, P_y, w, we may substitute to express B in terms of P_x, P_y, w:

$$B = f(P_x, P_y, w).$$

We wish to solve this last equation for w in terms of P_x, P_y, B and substitute the result to express U^* in terms of P_x, P_y, B:

$$U^* = \tilde{u}(P_x, P_y, w) = \tilde{\tilde{u}}(P_x, P_y, B).$$

Then to compute $\partial U^*/\partial B = \tilde{u}_B(P_x, P_y, B)$, we of course hold P_x, P_y constant and use the one-dimensional chain rule to obtain

$$\frac{\partial U^*}{\partial B} = \frac{\partial U^*}{\partial w}\frac{\partial w}{\partial B} = \tilde{u}_w(P_x, P_y, w)\frac{\partial w}{\partial B}.$$

Proceeding to the details, we put

$$h(P_x, P_y, w, B) \equiv f(P_x, P_y, w) - B.$$

We may apply the implicit function theorem to solve $h(P_x, P_y, w, B) = 0$ for w if

$$h_w(P_x, P_y, w, B) = f_w(P_x, P_y, w) \neq 0,$$

after which

$$\frac{\partial w}{\partial B} = \frac{-h_B(P_x, P_y, w, B)}{h_w(P_x, P_y, w, B)} = \frac{1}{f_w(P_x, P_y, w)}$$

Since $f(P_x, P_y, w) = w(T - L)$, it follows that

$$\begin{aligned} f_w(P_x, P_y, w) &= T - L + w\frac{\partial}{\partial w}(T - L) \\ &= (T - L)\left[1 + \frac{w}{T - L}\frac{\partial}{\partial w}(T - L)\right] = (T - L)(1 + \varepsilon_S), \end{aligned}$$

where $\varepsilon_S \equiv \dfrac{w}{T - L}\dfrac{\partial}{\partial w}(T - L)$ is called the individual's elasticity of supply of labor with respect to wage rate; ε_S may be interpreted as the ratio of a percentage change in work time to a percentage change in wage rate. Thus if $\varepsilon_S \neq -1$, then $f_w(P_x, P_y, w) \neq 0$, and we have

$$\frac{\partial U^*}{\partial B} = \frac{1}{(T - L)(1 + \varepsilon_S)}\frac{\partial U^*}{\partial w},$$

from which

$$\lambda = (1 + \varepsilon_S)\frac{\partial U^*}{\partial B}. \tag{12b}$$

Thus, in this model, λ is proportional to the marginal utility of income. The proportionality constant is one plus the elasticity of labor supply. Note that if there is no supply response to a change in the wage rate, i.e., $\varepsilon_S = 0$, the Lagrange multiplier equals the marginal utility of income. We shall return to this finding later in the paper. [See problem feature (iii).]

Returning to the Jacobian matrix (9), since P_x, P_y, w are all positive, we see from (4) and (5) that regardless of what assumption is made about (6), one cannot guarantee that (9) is nonsingular. If one does the tedious work of expanding and analyzing the determinant of (9), one finds that some terms are positive and some terms are negative. However, widespread economic folklore has held for some time that the negative terms will in practice dominate and that the determinant of H may be safely assumed negative. We shall return to a critique of this assumption in the next section.

We are now at the point where the economist deeply wishes that the sufficient conditions be necessary. The Jacobian matrix in (9) is actually a bordered Hessian. According to the mathematical theory of constrained optimization problems, the analogue of the second derivative test is: The sufficient conditions for a solution X^*, Y^*, L^*, λ^* of (8) to yield a relative maximum are (a) the determinant of (9) be negative and (b) the determinant of the 3×3 Hessian matrix

$$\begin{bmatrix} 0 & -P_x & -P_y \\ -P_x & u_{XX} & u_{XY} \\ -P_y & u_{YX} & u_{YY} \end{bmatrix}$$

be positive at the point X^*, Y^*, L^*, λ^*.

Assuming that these sufficient conditions actually occur, then not only is the determinant of (9) nonzero, so that the implicit function theorem can be invoked legitimately, but further results of economic interest can be derived. To illustrate, we shall derive several of the twelve comparative statics mentioned above using the implicit function theorem. [See problem feature (iv).]

Let F be the vector-valued function defined for points $(\lambda, X, Y, L, P_x, P_y, w) \in \mathbb{R}^7$ and taking values in $\mathbb{R}^4$ whose components are given by the left sides of equations (8). By the implicit function theorem, the equation

$$F(\lambda, X, Y, L, P_x, P_y, w) = 0 \tag{8}$$

may be solved in the form

$$\begin{bmatrix} \lambda \\ X \\ Y \\ L \end{bmatrix} = G(P_x, P_y, w). \tag{13}$$

Furthermore, the Jacobian matrix for G is given by

$$\begin{bmatrix} \frac{\partial\lambda}{\partial P_x} & \frac{\partial\lambda}{\partial P_y} & \frac{\partial\lambda}{\partial w} \\ \frac{\partial X}{\partial P_x} & \frac{\partial X}{\partial P_y} & \frac{\partial X}{\partial w} \\ \frac{\partial Y}{\partial P_x} & \frac{\partial Y}{\partial P_y} & \frac{\partial Y}{\partial w} \\ \frac{\partial L}{\partial P_x} & \frac{\partial L}{\partial P_y} & \frac{\partial L}{\partial w} \end{bmatrix} = -H^{-1} \begin{bmatrix} -X & -Y & T-L \\ -\lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & -\lambda \end{bmatrix},$$

where the ith row in the last matrix on the right is obtained by differentiating the ith left side in (8) with respect to P_x, then P_y, and then w. Letting C_{ij} be the cofactor (signed minor determinant) of the element in the ith row and jth column of H, and then inverting H using the method of cofactors gives

$$H^{-1} = \frac{1}{\det H} C^T,$$

where $C = (C_{ij})$. Now we are in a position to derive comparative static results. For example, how does the level of consumption of X change when its price increases? We find that

$$\frac{\partial X}{\partial P_x} = \frac{-1}{\det H}[-XC_{12} - \lambda C_{22}]. \tag{14}$$

Computing C_{12} and C_{22}, we obtain,

$$C_{22} = -P_y^2 u_{LL} - w^2 u_{YY} + 2wP_y u_{LY}$$

and

$$C_{12} = P_x\left[u_{YY}u_{LL} - (u_{LY})^2\right] - P_y[u_{XY}u_{LL} - u_{LY}u_{XL}] + w[u_{XY}u_{YL} - u_{YY}u_{XL}].$$

Because of (4) and (5), $C_{22} > 0$. Since $u_{XY} \geqslant 0$ (x and y are complements or unrelated), the second and third terms of C_{12} are unambiguously positive. If we assume also that

$$\det\begin{bmatrix} u_{LL} & u_{LY} \\ u_{YL} & u_{YY} \end{bmatrix} = u_{LL}u_{YY} - (u_{LY})^2 \geqslant 0,$$

as economists generally do (we examine this assumption in Section 3), then $C_{12} > 0$. Thus

$\partial X/\partial P_x < 0$, which is intuitively correct and is certainly in accord with empirical evidence.

Next, we might ask: How does the amount of leisure time chosen change when the wage rate increases? We find that

$$\frac{\partial L}{\partial w} = \frac{-1}{\det H}\left[-\lambda C_{44} + (T - L)C_{14}\right], \tag{15}$$

where

$$C_{44} = -\left(P_x^2 u_{XX} + P_y^2 u_{YY}\right) + 2P_x P_y u_{XY}$$

and

$$C_{14} = P_x(u_{YX}u_{LY} - u_{LX}u_{YY}) - P_y(u_{XX}u_{LY} - u_{LX}u_{XY}) + w\left(u_{XX}u_{YY} - (u_{XY})^2\right).$$

Continuing to assume that $u_{XY} \geqslant 0$, we infer from the assumptions (4) and (5) that C_{44} is unambiguously positive. Furthermore, the first and second terms in C_{14} are positive. Assuming again with the economists that

$$\det\begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} = u_{XX}u_{YY} - (u_{XY})^2 \geqslant 0,$$

we see that C_{14} is positive. Therefore the sign of $\partial L/\partial w$ is ambiguous. However, this is itself a very interesting result.

Economists refer to the first term in the brackets on the right side of (15), $-\lambda C_{44}$, as the *substitution effect* of a wage increase. When wages increase there is a stronger incentive (reward) to work longer hours, i.e., to choose less leisure time. The second term in the brackets on the right side of (15), $(T - L)C_{14}$, is called the *income effect* by which is meant that a wage hike increases income and allows an individual to opt for more leisure in order to have the time to enjoy the goods he consumes. For example, a 10% wage hike might lead an individual to choose 5% fewer work hours plus a 5% higher income. In general a wage hike gives rise to both income and substitution effects which have opposite influences on an individual choosing a new optimal amount of leisure per period.

However, (15) can be given an even more precise economic interpretation. Using (12a), which defines λ in economic terms, we can rewrite (15),

$$\frac{\partial L}{\partial w} = \frac{-1}{\det H}\frac{1}{(T - L)}\left[-u_w C_{44} + (T - L)^2 C_{14}\right]. \tag{16}$$

From (16), we can see qualitatively that, if u_w is small *or* if an individual works long hours ($T - L$ is large), the income effect dominates and $\partial L/\partial w > 0$. Conversely, if u_w is large *or* if he works few hours ($T - L$ is small), the substitution effect dominates and $\partial L/\partial w < 0$. Put another way, relatively high income, whether because of a high wage rate and/or long hours of work, tends to lead a person to opt for more leisure when w rises.

Finally, with a little more manipulation, we can give a specific interpretation to a zero elasticity of labor supply. Equation (16) can be rewritten

$$\frac{-\partial(T - L)}{\partial w} = \frac{-1}{\det H}\frac{1}{(T - L)}\left[-u_w C_{44} + (T - L)^2 C_{14}\right]$$

or

$$\varepsilon_S = \frac{w}{(T - L)}\frac{\partial(T - L)}{\partial w} = \frac{1}{\det H}\frac{w}{(T - L)^2}\left[-u_w C_{44} + (T - L)^2 C_{14}\right]. \tag{17}$$

But (17) states that when $T \neq L$, $\varepsilon_S = 0$ if and only if the income effect $[(T - L)^2 C_{14}]$ just equals the substitution effect $[-u_w C_{44}]$ of a wage change.

These comparative static results for $\partial X/\partial P_x$ and $\partial L/\partial w$ are intuitively appealing and they

are important. For example, one implication of this analysis is that an income tax cut designed to stimulate work effort, by raising after tax wage rates ("supply side economics"), may generate little response because the tax cut will give rise to both substitution *and* income effects.

3. Discussion of Assumptions. The main reason for the assumptions made by applied mathematicians is that they allow a conclusion to be drawn. In practice, the ends justify any (reasonable) means. The real reason that physicists neglect friction, air resistance, etc., is to make nonlinear problems linear or very difficult problems solvable in closed form. Of course, any scientist with integrity will then examine the validity of his assumptions.

Although the differential equations textbooks of the past twenty-five years are significantly superior to their predecessors, one of these older books by Agnew [**1**] is relevant to the present discussion and, if the reader can locate a copy, the incisive humorous comments there about assumptions in applied mathematics provide insight. Agnew's famous discussion of the "snow-plow problem" (Section 2.8) contains the opinion "much of the progress in science is due to men who have the courage to make assumptions, the good sense to make reasonable assumptions, and the ability to draw correct conclusions from the assumptions." In Section 3.4 on atmospheric pressures, he says "we make progress by application of a time honored method which often brings valuable results in applied mathematics. This method consists in making an assumption known to be absolutely and unequivocally false." But we digress.

We first examine the determinant of H (see (9)) in detail. Expanding the determinant by minors across the first row, then expanding each of the resulting 3×3 determinants down the first column, and finally simplifying, we get

$$
\begin{aligned}
\det H = & -P_x^2 \det\begin{bmatrix} u_{YY} & u_{YL} \\ u_{LY} & u_{LL} \end{bmatrix} - P_y^2 \det\begin{bmatrix} u_{XX} & u_{XL} \\ u_{LX} & u_{LL} \end{bmatrix} \\
& - w^2 \det\begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} + 2P_xP_y \det\begin{bmatrix} u_{XY} & u_{XL} \\ u_{LY} & u_{LL} \end{bmatrix} \\
& - 2P_x w \det\begin{bmatrix} u_{XY} & u_{XL} \\ u_{YY} & u_{YL} \end{bmatrix} + 2P_y w \det\begin{bmatrix} u_{XX} & u_{XY} \\ u_{LX} & u_{LY} \end{bmatrix}.
\end{aligned} \tag{18}
$$

Observe that the first three determinants on the right side of (18) are of similar type and two of them were assumed non-negative in our discussions of (14) and (15).

Let us call two goods x and y weakly dependent (with respect to the utility function u) if

$$
\det\begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} \geqslant 0.
$$

As always, X (or Y) denotes a quantity of good x (or y). If we assume that each pair of the three goods x, y, l are weakly dependent and x and y are complements or unrelated ($u_{XY} \geqslant 0$), it is an immediate consequence of (4) and (5) that $\det H < 0$. However, if $u_{XY} < 0$, then (18) contains positive terms which threaten to change the sign of $\det H$, or make $\det H = 0$. However, we emphasize that our comparative static results in Section 2 are firmly established if each pair of x, y, l are weakly dependent and x, y are complements or unrelated.

Let us examine the concept of weak dependence more closely. Widespread economic folklore has held that for realistic economic situations, the inequalities

$$
|u_{XY}| \leqslant |u_{XX}|, |u_{XY}| \leqslant |u_{YY}| \tag{19}
$$

would surely hold. One rationalization of such inequalities runs as follows. "Consider the extreme cases. If x and y are unrelated, $u_{XY} = 0$ and the inequalities are trivial. If x and y are perfect substitutes (for all practical purposes, indistinguishable), then $u_{XY} = u_{XX} = u_{YY}$ and the inequalities are equations. Other cases lie between these two extremes."

Of course, this argument does not stand up under close examination. In fact, the inequalities in (19) are not generally true and are not really required. The required inequality is

$$\det\begin{bmatrix} u_{XX} & u_{XY} \\ u_{YX} & u_{YY} \end{bmatrix} = u_{XX}u_{YY} - (u_{XY})^2 \geq 0. \tag{20}$$

Note that (19) implies (20) but the converse is not true.

To illustrate the difficulty with (19), suppose that $u_{XY} = 0.1$, $u_{XX} = -1.0$ at some fixed point. If we change Y units to a new unit Y' where $Y = 12Y'$ (for example, Y units are inches, Y' units are feet), then

$$u_{XY'} = u_{XY}\frac{dY}{dY'} = 12u_{XY} = 1.2$$

and thus $|u_{XY'}| \leq |u_{XX}|$ fails using the new units.

It is easy to verify that the sign of a determinant like the one in (20) or $\det H$ in (9) is not affected by changing units of measurement. For example, if X units were yards and $X' = 3X$ measured in feet, the effect of changing X units to X' units on (9) would be to multiply the second column and the second row of H by 3, thus changing $\det H$ by a factor of $1/9$, leaving the sign of $\det H$ unchanged.

If x, y are substitutes ($u_{XY} < 0$), but u_{XY} is sufficiently close to 0, $\det H < 0$ will continue to hold. Comparative static conclusions change in this case, as the reader may verify.

4. Other Representative Problems from Economics. The approach we have just illustrated is useful in addressing a variety of problems in economics. We suggest several other types of problems below, which may be explored by the reader or students in independent study projects. One would begin by applying the Lagrange method to see which features of the previous problem reappear, perhaps with modification.

(a) *Cost Minimization.* Suppose a firm is under contract to produce and deliver (for a fixed price) Q_0 units of output during the next year. It employs capital (K), labor (L), and other inputs (R) in its production process. If the firm seeks to maximize profits while meeting the terms of the contract, its production decision can be characterized as a constrained cost minimization problem in which the firm chooses the least cost combination of K, L, and R necessary to produce Q_0. Its objective is then to minimize the cost function

$$C(K, L, R) = rK + wL + \rho R$$

subject to the output constraint

$$Q_0 = f(K, L, R)$$

when r, w, and ρ are the cost per unit of K, L, and R, respectively, and f is the production function for the firm. Some obvious assumptions need to be formulated regarding the qualitative behavior of f, similar to those we made earlier on the utility function. In this problem, the Lagrange multiplier turns out to be the marginal cost of production $\partial C/\partial Q_0$, that is, the increase in total costs when one more unit of output is produced.

(b) *Output Maximization.* Suppose a government agency is given an annual budget B_0 and is charged with the responsibility of providing as much service to the community as possible. If the agency uses K, L, and R to produce services, then its objective is to maximize the output function

$$Q = g(K, L, R)$$

subject to the budget constraint

$$B_0 = rK + wL + \rho R.$$

(The meaning of the various parameters should be clear.) In this problem, the Lagrange multiplier is $\partial Q/\partial B_0$, the inverse of marginal cost. This problem is thus dual to the cost minimization problem.

(c) *Multiple Constraints.* During wartime, consumers often must pay for goods with money and the surrender of ration coupons. The mathematical problem is then (using an obvious

notation) to maximize an individual's utility function $U(x, y, z)$ subject to the budget constraint

$$B = P_x X + P_y Y + P_z Z$$

and the coupon constraint

$$R = r_x X + r_y Y + r_z Z.$$

(d) *Nonlinear Constraints.* Suppose our government agency in (b) can obtain price discounts by purchasing larger quantities of R. We can rewrite the budget constraint as

$$B_0 - rK - wL - \rho(R)R = 0.$$

5. Concluding Comments. We provide several references which the interested reader may wish to consult. Statements of sufficient conditions for local maxima (or minima) can be found in [3] and [6]. A very readable account of some elementary but interesting applications in economics can be found in [4]. An interpretation of the Lagrange multiplier and some consequences of the necessary conditions are provided in [7] but no use of sufficient conditions is mentioned there.

The authors of this paper have for several years jointly taught an upper level seminar in mathematical economics as part of an interdisciplinary major. The type of problem described here is considered in the early weeks of the seminar. Because each of us has learned a lot of useful things from the other, we recommend such interdisciplinary endeavors with enthusiasm.

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131. MISCELLANEA

The fraction of Life can be increased in value not so much by increasing your numerator as by lessening your denominator.

—*A Search for America*, by Frederick Philip Grove (McClelland and Stewart, Toronto, 1982), p. 219.

ANSWER TO PHOTO ON PAGE 393

Edmund Landau.

PROGRESS REPORTS

EDITED BY THOMAS BANCHOFF AND RICHARD MILLMAN

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

MAPPING THEOREMS IN COMPLEX ANALYSIS

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Mathematicians seek to classify the objects they study; the first step in this program is to determine when two objects are essentially the same. Within the realm of complex analysis, this leads to the biholomorphic mapping problem for domains. A domain is an open and connected set in a complex vector space of finite dimension. Two domains are biholomorphically equivalent if there is an analytic function with an analytic inverse that maps one domain onto the other. These domains then have the same function theory; the analytic function (called a biholomorphic map) amounts merely to a coordinate change.

Let us recall the elegant state of affairs for the complex numbers themselves. Any domain in the complex plane that has no holes, and is not the whole plane, can be mapped biholomorphically onto the unit disk. This is the Riemann mapping theorem. Domains without holes are topologically the same; it is remarkable that they are analytically the same. Of course, the whole plane must be excluded, because a bounded function, analytic in the entire plane, reduces to a constant. The Riemann mapping theorem finds applications in electrostatics, fluid dynamics, and any subject where solving the Laplace equation arises. The reason is easy to understand. Solving the Laplace equation with given boundary values (the Dirichlet problem) is easy to do in the disk. The boundary of the disk is the unit circle, and the methods of Fourier series readily yield a solution. By applying the theorem, one then obtains a solution for any domain as above. Furthermore, it is easy to show that, for a smoothly bounded domain, the map that takes it to the unit disk extends smoothly to the boundary. This means that the mapping from the boundary to the unit circle is smooth, and has a smooth inverse. Hence the boundaries are differentiably equivalent.

Things are much more difficult for domains in the space of several complex variables. Consider the class of domains D_p, where $1 \leq p \leq \infty$, defined as follows:

$$D_p = \{(z,w) : |z|^{2p} + |w|^2 < 1\}.$$

Here z and w are complex numbers. When p equals one, D_p is the unit ball, and as p tends to infinity, D_p tends to the unit polydisk (the product of two disks). This family of domains is topologically equivalent, but analytically inequivalent. More precisely, there is no biholomorphic map from D_p to D_q unless p equals q. The Riemann mapping theorem fails miserably. How can one begin to classify domains?

Around the turn of the century, Poincaré considered the problem of associating "invariants" to the boundaries of domains. These invariants involve differential geometric constructions. However, one would need to know that they would be unaltered by making a biholomorphic map. Hence the question is: When does a biholomorphic map extend smoothly to the boundary?

Charles Fefferman astounded the mathematical world in 1974 by showing that indeed, for a wide class of domains with smooth boundaries, biholomorphic maps extend smoothly to the boundaries. Here is a description of the ideas in his proof. Associated to any bounded domain is its Bergman kernel function. One can reproduce the values of a square integrable analytic function by integration against this kernel. Thus the kernel function behaves like a delta function. It gives rise to a metric that is wildly degenerate at the boundary. Fefferman analyzed the behavior of the geodesics in this metric, and tamed their behavior by using ideas from the n-body problem. He faced the problem of approximation of the domains under consideration (called strongly pseudoconvex domains) by the unit ball, for which the Bergman kernel and metric have elementary explicit formulas. The approximation procedure involved the machinery of the Cauchy-Riemann equations in several variables. Also he invented a calculus of singular integral operators needed in his estimates.

Fefferman's work has inspired a generation of workers in the field of several complex variables. Many mathematicians studied various aspects of his work. In 1979, Bell and Ligocka gave a much simpler proof. Nearly all of the original ideas remain however. For example, Bell took the important step of replacing the Bergman kernel function with the Bergman projection operator. This has led to simplifications and extensions of the original result. It now applies to a class of weakly pseudoconvex boundaries for which the boundaries are not too flat. Furthermore, the theorem has an analog for analytic maps that are finite to one, rather than invertible. The class of domains D_p mentioned above are weakly pseudoconvex if p is larger than one, but can be mapped in a p to one fashion onto the ball, if p is an integer. Mappings such as these are now known to extend smoothly to the boundaries.

The exact class of smoothly bounded domains for which biholomorphic maps extend is unknown. As yet, no one has found a counterexample. These questions are at the center of a very active area. The geometry of the boundary can be very complicated, and the estimates needed in the theory of partial differential equations are quite delicate. Answers here will yield many important results in the theory of analytic functions of several complex variables. Thus, even though there is no hope for a Riemann mapping theorem in several complex variables, the results on boundary smoothness and the ensuing spin-offs are among the most exciting events in complex analysis for a long time.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

PURSUIT AND EVASION IN A CIRCLE

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The pursuer's initial position is the center $p_0 = (0,0)$ of the unit circle $C = \{u \in \mathbb{R}^2 : |u| \leqslant 1\}$ and his maximal velocity is 1. The evader's initial position is a point $q_0 \in C$, and his maximal velocity is a positive number $v \leqslant 1$. The aim of the pursuer is to catch the evader as soon as possible and the aim of the evader is to run free as long as possible. How to make this precise? Here is one way:

The evader can choose any $q_1 \in C$, then the pursuer can choose any $p_1 \in C$ with $|p_1 - p_0| \leqslant |q_1 - q_0|/v$ and again the evader chooses any $q_2 \in C$ and the pursuer any $p_2 \in C$ with $|p_2 - p_1| \leqslant |q_2 - q_1|/v$, etc. The game ends as soon as there is an n such that p_n is in the segment $q_n q_{n-1}$ and

$$(*) \qquad |p_n - p_{n-1}| = |p_n - q_{n-1}|/v.$$

In this case the pursuer pays to the evader the value T defined by

$$T = \sum_{i=0}^{n-1} |p_i - p_{i+1}|.$$

If no such n exists but $\lim_{i\to\infty} |p_i - q_i| = 0$, then the pursuer pays

$$T = \sum_{i=0}^{\infty} |p_i - p_{i+1}|.$$

Finally if neither of the above applies, then the pursuer pays to the evader the value

$$T = \frac{1}{v} \sum_{i=0}^{\infty} |q_i - q_{i+1}|.$$

The following four facts about this game are shown in [**4**].

(i) The game has a value, more exactly, in the case $v < 1$, there exists a real number T_0 such that the pursuer has a strategy which secures $T \leqslant T_0$ for every play of the evader, and for every $\varepsilon > 0$ there exists a strategy for the evader which secures $T \geqslant T_0 - \varepsilon$ for every play of the pursuer. And, in the case $v = 1$, the evader has a strategy which secures $T = \infty$ (the case $v = 1$ is due to Besicovitch, see [**2**]).

(ii) There exists a natural "dual" formalization of this game (see [**3**], §3) in which, for every $\varepsilon > 0$, the pursuer can secure $T \leqslant T_0 + \varepsilon$ and the evader can secure $T \geqslant T_0$ and this value T_0 coincides with that of (i). (Still other characterizations of T_0 are proved in [**4**].)

(iii) Let $T_0 = T_0(v, |q_0|)$ be regarded as a function of v and $|q_0|$. Then, for $v < 1$,

$$|T_0(v, x_1) - T_0(v, x_2)| \leqslant \frac{|x_2 - x_1|}{1 - v},$$

$$T_0(v,x) = \frac{x}{1-v} \quad \text{for } x \leqslant 1-v,$$

$$T_0(v,x) \leqslant \left(\frac{x(2-x)}{1-v^2}\right)^{1/2} \quad \text{for } x \geqslant 1-v.$$

(iv) If $q_0 \neq p_0$ and $v \to 1$, then $T_0 \to \infty$. If $0 < v_1 \leqslant v_2 \leqslant 1$, then

$$T_0(v_1,x) \leqslant T_0(v_2,x) \leqslant \frac{1-v_1}{1-v_2} T_0(v_1,x).$$

But the following natural questions are still open.

Problems. (a) Does $T_0(v,x)$ increase as x increases from 0 to 1?

(b) Let S be the following strategy for the pursuer: If a choice $p_n \in \overline{q_n q_{n-1}}$ satisfying (*) is possible, make this choice. Otherwise, choose p_n on the same radius of C as q_n and as close to q_n as permitted by the rules of the game. Is S an optimal strategy for the pursuer? In other words, does it secure the inequality $T \leqslant T_0(v,x)$?

(c) Is $T_0(v,x)$ a differentiable function?

After the foregoing was accepted for publication, a related question:

WHAT ARE THE SIZE AND SHAPE OF THE LARGEST REGION THAT CAN BE GUARDED AGAINST A FASTER INVADER?

was raised by

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More precisely, an invader, I, whose maximal speed is $v > 1$, wishes to enter a region $G \subseteq \mathbb{R}^2$ A defender, D, with destruction radius $r > 0$ and maximal speed 1 tries to keep I off G. The playing space is the whole plane $\mathbb{R}^2$. We say that D can **guard** G if there are initial positions p_I and p_D in $\mathbb{R}^2$ with $\operatorname{dist}(p_I, p_D) \geqslant r$ such that D has a winning strategy.

For example, D can guard the circle $K = \{z \in \mathbb{R}^2 : |z| < rv/(v-1)\}$ because the angular velocity of I moving on the perimeter of K at the speed v is equal to that of D moving along the circle $|z| = r/(v-1)$ with speed 1.

Problem. Prove that K is the domain of largest area which can be guarded by D.

This game is related to one called "guarding the target" which is described by Isaacs [1]. There are several ways to make a mathematically precise definition of the game. See, for example, the earlier part of this article, or Mycielski's earlier paper [3].

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NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.
For instructions about submitting Notes for publication in this department see the inside front cover.

THE NUMBER OF FURTHEST NEIGHBOUR PAIRS OF A FINITE PLANAR SET

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Geometric problems involving a finite set of points in the plane have intrigued mathematicians for many years. As far back as 1893, J. J. Sylvester [**4**] posed the following problem: If S is a set of points in the plane, not all collinear, must there exist a line that contains exactly two points of S? This problem remained unsolved for forty years until it was rediscovered by P. Erdős in 1933 and solved a few days later by T. Gallai. This problem and a myriad of other famous problems in plane geometry are discussed in the fascinating paper by V. Klee [**2**].

In this note, we are concerned with results about maximum distances between pairs of points in the plane. An early problem of this type was posed by Hopf and Pannwitz [**1**] in 1934: What is the maximum number of pairs of points that can realize the diameter, the maximum distance between points of a set? The answer to this question appears in [**3**] and is the same as the number of points in the set. This is achieved by placing one point at the center of a circle of unit radius and the remaining points on a part of the boundary of the circle so that the two most distant points also are at distance 1 apart. We shall state and solve a variation of this problem.

Consider a set N of n points in the Euclidean plane indexed by the integers $1, 2, \ldots, n$. A point j is called a *furthest neighbour* of a point i if $d(i, j) = \max_{1 \leqslant k \leqslant n} d(i, k)$, where d is the Euclidean distance function. Each point has at least one, and possibly several, furthest neighbours. Let n_i denote the number of furthest neighbours of point i. Finally let

$$M(n, N) = \sum_{i=1}^{n} n_i, \quad \text{and} \quad M(n) = \max_{\substack{\text{All } N \text{ s.t.} \\ |N|=n}} M(n, N)$$

so that $M(n)$ represents the maximum number of ordered furthest neighbour pairs. Our result is that

$$M(n) \leqslant 3n - 3, \tag{1}$$

and that this inequality is achievable for all even values of $n \geqslant 4$. It is unknown whether this is achievable for sets of points with odd cardinality.

This problem was motivated by the corresponding problem in computational geometry, which is to find an efficient algorithm for finding all furthest neighbour pairs. Algorithms of time complexity $O(n \log n)$ have been found for this problem [**6**]. A further variation was suggested by Toussaint [**5**]. Two points i and j are called *symmetric furthest neighbours* if i is a furthest neighbour of j and vice versa. Since a pair of points realizing the diameter of a set is a symmetric furthest neighbour pair, it is clear that there can also be at least n such pairs. The method of [**1**], [**3**] can be used to show that n is in fact the maximum number of symmetric furthest neighbour pairs. In the case where all distances are distinct, it is obvious that only one pair can realize the diameter. In this case, it is shown in [**5**] that the maximum number of symmetric furthest neighbour pairs is the greatest integer in $n/2$, $\lfloor n/2 \rfloor$.

We begin by stating some notation. For each point i, let $r_i = \max_k d(i, k)$ and let C_i denote the circle of radius r_i centered at i. Let D_i denote the disk consisting of the circle C_i and its interior. Further, let m_i denote the number of points j such that i is a furthest neighbour of j. This differs from the definition of n_i which is the number of furthest neighbours of point i. For

convenience we shall assume the points are numbered so that

$$r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n.$$

Observe that all points lie in D_i, $i = 1, 2, \ldots, n$, and hence in the intersections

$$\bigcap_{i=1}^{k} D_i, \qquad k = 1, 2, \ldots, n.$$

The number of points lying on the boundary of D_i is n_i. The number of circles passing through point i is precisely m_i.

For any three noncollinear points s, t and u, let $g(s, t, u)$ denote the radius of the unique circle through the three points. We use the following result from elementary geometry. *For any three noncollinear points s, t and u, any circle C through s whose disk contains t and u has radius $r \geqslant g(s, t, u)$. Further, $r = g(s, t, u)$ if and only if C passes through t and u also.*

We shall prove (1) by starting with the set N and by successively drawing the circles $C_1, C_2, \ldots, C_n$. Initially we set $m_j = 0$, for $j = 1, \ldots, n$. Each time we draw a circle, we update the numbers m_j. Consider circle C_k. It may pass through some points $j \in N$ with $m_j = 0$. This occurs when j does not lie on the boundary of any circle C_i for $1 \leqslant i < k$. Let f_k be the number of such points. In addition, C_k may pass through some points j with $m_j \geqslant 1$. Let h_k be the number of such points. Clearly, $h_1 = 0$ and $n_k = f_k + h_k$. Therefore we may write

$$M(n) = \sum_{k=1}^{n} n_k = \sum_{k=1}^{n} f_k + \sum_{k=2}^{n} h_k \leqslant n + \sum_{k=2}^{n} h_k. \tag{2}$$

The inequality follows from the fact that each point in N can be counted at most once by some f_k. The following lemma provides a bound on h_k.

LEMMA. *$h_k \leqslant 2$ for $k = 2, 3, \ldots, n$.*

Proof. Suppose $h_k \geqslant 3$ and let s, t and u be three points on C_k that satisfied $m_s \geqslant m_t \geqslant m_u \geqslant 1$ when C_k was drawn. Since $m_s \geqslant 1$, some circle C_i passes through s and contains t and u in its disk D_i. By the proposition $r_i \geqslant g(s, t, u)$. Since C_k passes through s, t and u, $r_k = g(s, t, u)$. But $i < k$, so $r_i \leqslant r_k$ and hence $r_i = r_k$. Since there is only one circle passing through the three points, $C_i = C_k$, a contradiction.

Substitution of this result into (2) yields

$$M(n) \leqslant n + 2(n - 1) = 3n - 2. \tag{3}$$

That this bound is almost best possible is illustrated by the configuration of 6 points in FIG. 1.

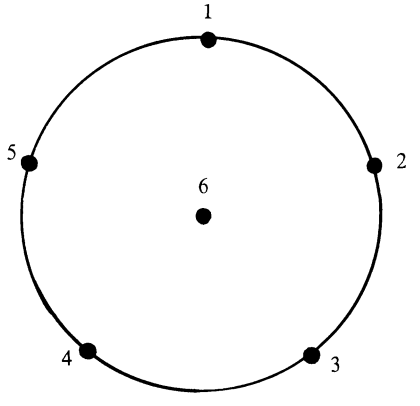


FIG. 1

Point 6 is the center of a circle of radius 1. The other 5 points are placed on the circle to form the vertices of a regular pentagon. Observe that

$$\max_k d(i,k) = \begin{cases} d(i,i+2) \\ d(i,i+3) \end{cases}, \quad i = 1,\ldots,5$$

where indices are taken mod 5. Thus $m_i = 3$, $i = 1,\ldots,5$ and $m_6 = 0$. Generalizing the construction for any even n yields a set of n points for which

$$M(n) = 3n - 3. \tag{4}$$

We shall now eliminate the gap between (3) and (4). First, note that the intersection of the n disks, $D_1,\ldots,D_n$ is a convex set. If N is not a convex set of points, some point $i \in N$ is contained in the interior of this intersection. Hence $m_i = 0$. In this case

$$\sum_{k=1}^{n} f_k \leqslant n - 1,$$

and (1) follows from the arguments that led to (3). Suppose, then, that N is a convex set of points. We shall show that the second circle drawn, C_2, can pass through at most one point that lies on C_1. Hence $h_2 \leqslant 1$, thus proving (1). Assume then that $h_2 \geqslant 2$, implying that C_2 passes through two points i and j that were contained on C_1. Since

$$d(1,i) = d(1,j) \quad \text{and} \quad d(2,i) = d(2,j),$$

both points 1 and 2 lie on the perpendicular bisector of the line segment ij. Furthermore, they lie on the same side of this bisector, since $1 \in D_2$ and $2 \in D_1$ (see FIG. 2). But $r_1 \leqslant r_2$, so point 1 is contained in the triangle formed by points $2, i, j$. This contradicts the assumption that N is convex; hence $h_2 \leqslant 1$ and (1) follows.

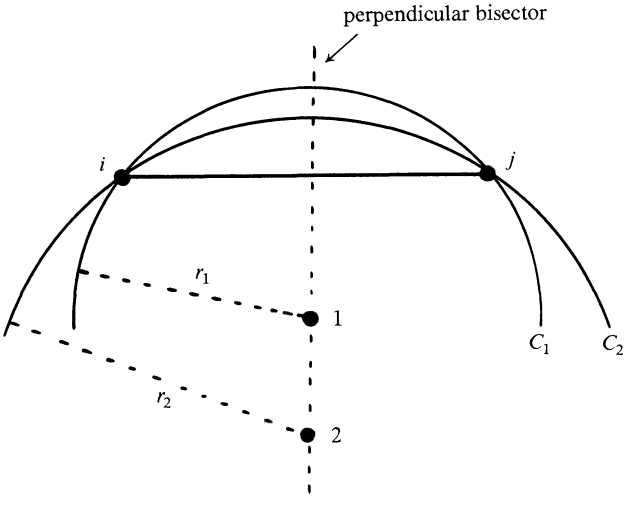


FIG. 2

In the case that n is odd, a modification of the construction gives a set of points for which $M(n) = 3n - 4$. For $n = 3$, the equilateral triangle yields $M(3) = 6$. As mentioned earlier, it is unknown whether (1) holds as an equality for odd values of n greater than 5.

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A COMBINATORIAL PROOF OF THE TRIPLE PRODUCT IDENTITY

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The identity of the title states that

$$\prod_{n=1}^{\infty}(1-q^{2n})(1+tq^{2n-1})(1+t^{-1}q^{2n-1}) = \sum_{n=-\infty}^{\infty} t^n q^{n^2}, \tag{1}$$

where q and t are complex numbers with $|q| < 1$ and $t \neq 0$. Though Gauss was the first to discover (1), this identity was also found by Jacobi in his work on theta functions, where it arises very naturally (Rademacher [**4**, chapter 10] derives (1) in this way). A host of identities may be deduced from (1). Hardy and Wright [**2**, §19.9] give several, and, in particular, substituting $q^{3/2}$ for q and $-q^{1/2}$ for t gives Euler's pentagonal-number theorem. However, Franklin showed, most elegantly, that the pentagonal-number theorem can be proved in a combinatorial manner (see, for example, [**2**, §19.10] or [**3**, §256]), and, inspired by his proof, the purpose of this note is to give a combinatorial proof of (1).

We denote by $p(n)$ the number of partitions of the natural number n, and it is convenient to define $p(0) = 1$ and $p(n) = 0$ for $n < 0$. It was shown by Euler, and can be found in [**2**, §19.3], that

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{(1-q^n)},$$

and it follows that (1) may be rewritten as

$$\prod_{n=1}^{\infty}(1+tq^{2n-1})(1+t^{-1}q^{2n-1}) = \left(\sum_{n=-\infty}^{\infty} t^n q^{n^2}\right)\left(\sum_{n=0}^{\infty} p(n)q^{2n}\right). \tag{2}$$

Comparing the coefficients of $t^k q^N$ on each side of (2) now shows that (1) may be restated as follows:

THEOREM. *For integers k and N, with $N \geqslant 0$, the number of solutions of the equation*

$$N = (a_1 + \cdots + a_{k_1}) + (b_1 + \cdots + b_{k_2}) \tag{3}$$

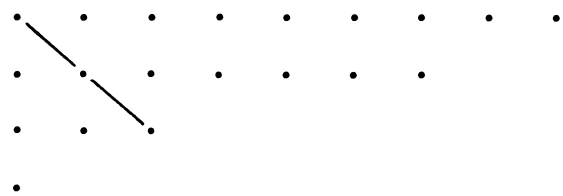
in positive odd integers a_i, b_j with $a_1 > \cdots > a_{k_1}$, $b_1 > \cdots > b_{k_2}$, and with $k_1 - k_2 = k$, is $p(\frac{1}{2}(N-k^2))$.

We assume here that (3) has exactly one solution if $N = k = 0$, and no solution when $N = 0$ and $k \neq 0$. So the theorem is true when $N = k = 0$ (since $p(0) = 1$), and also when $k^2 > N$ (since, as is easily seen, (3) then has no solution, and $p(n) = 0$ for $n < 0$). Note that, if (3) has a solution, $N - k^2$ is even.

Before giving the proof of this theorem, a few remarks on notation are in order. Every partition

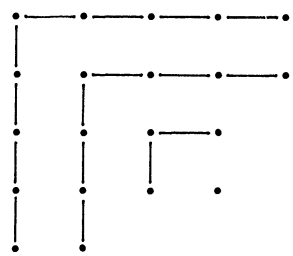
can be represented as an array of points (or other symbols). For example, the partition $9 + 7 + 3 + 1$ is represented by

(4)



This we shall call the *canonical diagram* of the partition (it is usually known as the Ferrers graph). The line of points indicated in (4) we shall call the *diagonal* of the diagram. If the diagram is reflected in its diagonal, the array so obtained is also the canonical diagram of a partition, the partition *conjugate* to that originally represented.

Partitions whose parts are odd and unequal can be represented in another way [**3**, p. 278], under which, for example, $9 + 7 + 3 + 1$ appears as

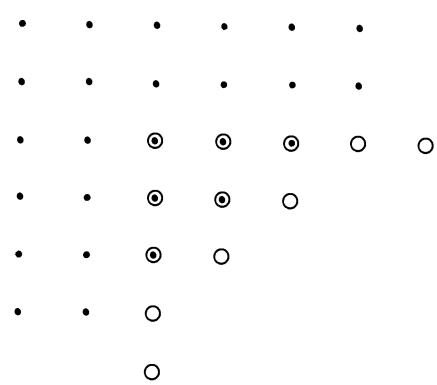


(the parts being represented by the bent lines). This we shall call the *self-conjugate* diagram of the (unequal, odd) partition.

Proof of the Theorem. The proof describes a bijection between the solutions of (3) and the partitions of $\frac{1}{2}(N - k^2)$.

Assume $k \geqslant 0$ and $N > 0$. Given a solution of (3), draw the self-conjugate diagram, A, of the partition $a_1 + \cdots + a_{k_1}$. Superimpose on A the self-conjugate diagram, B, of the partition $b_1 + \cdots + b_{k_2}$, representing the points of B by circles, and with the top left corner of B over the point $k + 1$ places down the diagonal of A. We obtain a composite diagram, call it C, of points and circles. For example, with $N = 38$, $k = 2$ and the solution $38 = (11 + 9 + 5 + 1) + (9 + 3)$ of (3), the diagram C is

(5)



From C we get a new diagram, D, as follows. First remove the top left $k \times k$ square of points,

and then change · to ○ and ○ to · (on or) below the diagonal of C. Note that, after the first step, the symbols remaining on the diagonal of C are all ⊙, while, in carrying out the second step, ⊙ remains ⊙. For our example, diagram D is

(6)

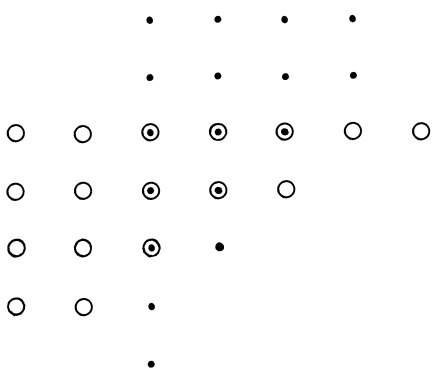
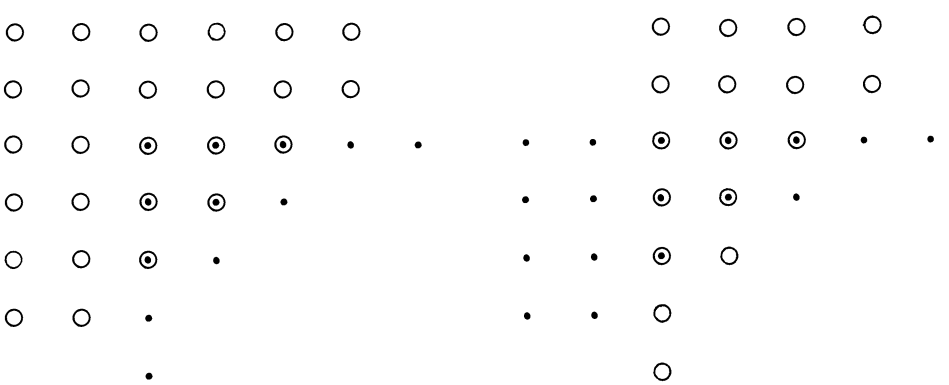


Diagram D is composed of the canonical diagram, drawn with ·, of a partition of $\frac{1}{2}(N - k^2)$ with the canonical diagram of its conjugate, drawn with ○, superimposed. It is this partition that we associate with the solution of (3) with which we set out. So, for the chosen example, the partition $4 + 4 + 3 + 2 + 2 + 1 + 1$ of 17 is associated with the solution $38 = (11 + 9 + 5 + 1) + (9 + 3)$ of (3).

This transformation can clearly be reversed. Given a partition of $\frac{1}{2}(N - k^2)$, draw its canonical diagram, using ·, and superimpose its conjugate, using ○, leaving empty a $k \times k$ square at the top left as portrayed by (6). Change · to ○ and ○ to · (on or) below the diagonal, then fill the $k \times k$ square with ·. (5) reappears, and provides a solution of (3).

The transformation can also be performed when $k < 0$. Here it is, illustrated for the solution $38 = (9 + 3) + (11 + 9 + 5 + 1)$ of (3):



$38 = (9 + 3) + (11 + 9 + 5 + 1)$ $\qquad$ $17 = 7 + 5 + 3 + 2$

The image of $(b_1 + \cdots + b_{k_2}) + (a_1 + \cdots + a_{k_1})$ will be the conjugate of the image of $(a_1 + \cdots + a_{k_1}) + (b_1 + \cdots + b_{k_2})$.

The required bijection has now been described in all cases, so the theorem is proved.

This theorem is not new. It was stated and proved by Sylvester [**5**, pp. 34–36] and his proof is also given by MacMahon [**3**, §323]. In fact Sylvester sets up the same correspondence between solutions of (3) and partitions of $\frac{1}{2}(N - k^2)$ as that described here, but his description of this correspondence is fairly obscure as the diligent reader will discover. Wright [**6**] has also given a combinatorial proof of (1), though the partition theorem he actually proves is different from the theorem proved here. More recently, Garsia and Milne [**1**] have shown that the identity (1) is embodied in their operations CUT and UCUT.

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A FAMILY OF CURVES WITH TWO EQUICHORDAL POINTS ON A SPHERE

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A chord of a plane region R is a segment joining two boundary points of R. A point $p \in R$ is equichordal if all chords through p have the same length. Whether a convex plane region can have two equichordal points is a famous unsolved problem of long standing (see [**1**] for a discussion and references).

It is the aim of this note to show that the corresponding problem on the sphere is easily solvable. We shall describe a one-parameter family of spherically convex regions each of which possesses two (spherically) equichordal points. More precisely, let $S \subset R^3$ be the sphere whose equation is $x^2 + y^2 + z^2 = 1$ and for values of a parameter ϕ between 0 and $\pi/4$ let

$$R_\phi = \{(x, y, z) \in S | z > 0 \quad \text{and} \quad 2x^2 + 2\cos^2\phi\, y^2 \leqslant 1\}.$$

Then(a) R_ϕ does not contain pairs of antipodal points and for any x, $y \in R_\phi$ there is an arc of great circle in R_ϕ which contains x and y, and

(b) if $p = (\sin\phi, 0, \cos\phi)$, and $q = (-\sin\phi, 0, \cos\phi)$, then the intersection with R_ϕ of any great circle through p or q has length $\pi/2$.

The proof of (a) is left to the reader. Let C be the intersection of S with the cylinder whose equation is $2x^2 + 2\cos^2\phi\, y^2 = 1$. Then C is also the intersection of S with the cone

$$x^2 + (2\cos^2\phi - 1)y^2 - z^2 = 0 \tag{1}$$

or with the cylinder given by $2\sin^2\phi\, x^2 - 2\cos^2\phi\, z^2 = \sin^2\phi - \cos^2\phi$.

Let $v = (x, y, z) \in C$. In order to prove (b) it is enough to show that $v' = p - (p \cdot v)v$ satisfies (1), and this is true since after substitution we get

$$\begin{aligned}&\sin^2\phi - 2(p \cdot v)x\sin\phi + (p \cdot v)^2x^2 + (2\cos^2\phi - 1)(p \cdot v)^2y^2\\ &\quad - \cos^2\phi + 2(p \cdot v)z\cos\phi - (p \cdot v)^2z^2\\ &= \sin^2\phi - \cos^2\phi - 2(p \cdot v)(x\sin\phi - z\cos\phi) = \sin^2\phi - \cos^2\phi - 2x^2\sin^2\phi + 2z^2\cos^2\phi = 0.\end{aligned}$$

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THE TAXICAB GROUP

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One of the basic problems in geometric investigations is this: *given a space S endowed with a metric d, describe the group G of isometries of S with respect to the metric d.* For example, if S is the

C E N T E R S E C T I O N
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General, S, P. Computers in Chess: Solving Inexact Search Problems. M.M. Botvinnik. Transl: Arthur A. Brown. Ser. in Symbolic Comput. Springer-Verlag, 1984, xiv + 158 pp, $22. [ISBN: 0-387-90869-2] An update and continuation of the author's classic book Chess, Computers, and Long-Range Planning (1970, TR, January 1971). Since that time, computer programs have attained master ratings (BELLE, Ken Thompson, 1980) however, this has been done by the brute-force method of exhaustive search. Botvinnik insists that further improvements must come about by means of "heuristics" that can be used to prune the tree search, thus more closely imitating a chess master's thought processes. These pages describe a plan in general terms, a major part of which is the construction of a handbook system for recording historical experience, and the elaboration of algorithms for using this experience. LCL

General, S(13-14). Problem Solving with the Programmable Calculator: Puzzles, Games & Simulations with Math & Science Applications. David L. Dunlop, Thomas F. Sigmund. Prentice-Hall, 1983, xi + 227 pp, $10.95 (P). [ISBN: 0-13-721332-8] Collection of 60 activities (32 numerical puzzles, 5 games, 23 simulations of scientific situations), with explanations and fully documented programs. Written specifically for the TI-59, but most programs will also work on the TI-58, and about half on the TI-57. RSK

General, S(15-17), P, L.** Fundamentals of Mathematics. Ed: H. Behnke. Transl: S.H. Gould. MIT Pr, 1983, $40 set (P). Volume I: Foundations of Mathematics, The Real Number System and Algebra, x + 549 pp [ISBN: 0-262-52093-1]; Volume II: Geometry, xi + 685 pp [ISBN: 0-262-52094-X]; Volume III: Analysis, xiii + 541 pp. [ISBN: 0-262-52095-8] Paperback edition of the 1974 hardcover English language translation (TR, April 1975) of the second (1962) German edition. A masterful exposition of classical mathematics, emphasizing "fundamentals" rather than applications. Should be required reading for every beginning graduate student. LAS

Elementary, T(13). College Algebra, Second Edition. Walter Fleming, Dale Varberg. Prentice-Hall, 1984, xiv + 498 pp, $25.95. [ISBN: 0-13-141630-8] Looks like Algebra and Trigonometry by the same authors, without the trigonometry. Does include chapters on systems of equations and probability. BK

Precalculus, T*(13: 1). Precalculus Mathematics. Walter Fleming, Dale Varberg. Prentice-Hall, 1984, xiv + 477 pp, $25.95. [ISBN: 0-13-694851-0] Algebraic, exponential, logarithmic, trigonometric functions followed by optional chapters on complex numbers, polynomials, sequences and induction, and conic sections. Sections are well-motivated and feature unusually good problem sets (including numerous word problems, as well as several requiring a calculator). A careful development done in a direct, easy-to-follow, informal style. LCL

Education, S(14-16). Multicultural Mathematics Materials. Marina C. Krause. NCTM, 1983, iv + 76 pp, $5 (P). [ISBN: 0-87353-206-6] Interesting games and other activites from all parts of the world. Mathematics content is minimal, primarily counting squares in board games. Future elementary teachers will find ideas here, although most ideas would be better classified as social studies. MW

History, P. Armand Borel: Oeuvres Collected Papers. Armand Borel. Springer-Verlag, 1983, $138 set [ISBN: 0-387-12126-9]. Volume I, 1948-1958, viii + 718 pp; Volume II, 1959-1968, iv + 790 pp; Volume III, 1969-1982, iv + 718 pp. The official record (through 1982) of Borel's work which was concentrated in topology, algebraic geometry, and especially algebraic groups. SG

History, P. Wilhelm Magnus, Collected Papers. Ed: Gilbert Baumslag, Bruce Chandler. Springer-Verlag, 1984, xvi + 726 pp, $32. [ISBN: 0-387-90879-X] The collected works of one of the few mathematicians who has done significant work in two completely unrelated fields: combinatorial group theory and diffraction problems and related topics in analysis. Half of the fifty papers are in German. LCL

History, P. Alexander Ostrowski, Collected Mathematical Papers, Volume 1. Alexander Ostrowski. Birkhauser Boston, 1983, 904 pp, $71.95. [ISBN: 3-7643-1506-7] First of six volumes of papers of one of the leading mathematicians of the twentieth century, containing fundamental results ranging from algebra and number theory to numerical analysis. This volume contains papers on determinants, linear algebra and algebraic equations. LAS

History, L. Harish-Chandra, Collected Papers. Ed: V.S. Varadarajan. Springer-Verlag, 1984, $150 set [ISBN: 0-387-90782-3]. Volume I: (1944-1954), lxxvii + 566 pp; Volume II: (1955-1958), v + 539 pp; Volume III: (1959-1968), v + 670 pp; Volume IV: (1970-1983), v + 461 pp. All of Harish-Chandra's papers, arranged chronologically to emphasize the evolution of his own thinking, from theoretical physics to group representations to harmonic analysis on reductive groups. Includes a very brief biography, three surveys of his work, and frontispiece photographs in each volume. LAS

History, S(15-16), P. Contributions à la Théorie du Calcul des Variations. E. Husserl. Papers in Pure & Appl. math., No. 65. Queen's U, 1983, 88 pp, (P). A translation into French of the famous philosopher's thesis and a brief history of the calculus of variations. JD-B

History, S(16-17), P. Georg Cantor: Leben, Werk and Wirkung. Herbert Meschkowski. Bibliographisches Institut, 1983, xvi + 315 pp. [ISBN: 3-411-01653-1] A short account of Cantor's early years, followed by an extensive discussion of his work interspersed with many references to his later life and his relations with other mathematicians. Also letters to and from Cantor, and two chapters on modern results in set theory. JD-B

History, P, L. Mathematics as a Cultural System. Raymond L. Wilder. Found. & Philo. of Sci. & Tech. Ser. Pergamon Pr, 1981, xii + 182 pp, $23. [ISBN: 0-08-025796-8] A loosely-knit tapestry of sociological and historical threads, in partial clarification of the author's Evolution of Mathematical Concepts. No clear pattern emerges from this new perspective--just jumbled images of the familiar classics of mathematics. LAS

History, P. The Seven Liberal Arts in the Middle Ages. Ed: David L. Wagner. Indiana U Pr, 1983, xiii + 282 pp, $25. [ISBN: 0-253-35185-5] An historical study of liberal arts practice and pedagogy in the high middle ages. Four of the 10 individually authored chapters deal with the quadrivium--arithmetic, music, geometry, and astronomy. LAS

Foundations, S*, L*. Post's Machine. V.A. Uspensky. Transl: R. Alavina. Little Math. Lib. MIR Pub, 1983, 88 pp, $2.95 (P). A brief introduction "for school children" to Post's abstract computing machine, a simpler version of the more famous Turing machine. Beginning with utter simplicity (including hints for paper clip and tape models suitable for seven-year-olds), Uspensky carries the reader to the Post-Church thesis on recursive functions, and concludes with a reprint of Post's 1936 original article. LAS

Discrete Mathematics, T(13-14: 1, 2), L. Discrete Mathematical Structures for Computer Science. Bernard Kolman, Robert C. Busby. Prentice-Hall, 1984, ix + 421 pp, $27.95. [ISBN: 0-13-215418-8] Introductory, sophomore level text, suitable for both mathematics and computer science students, with emphasis on topics of interest to further work in computer science. Special emphasis on relationships between relations and digraphs, and among trees, languages, and finite automata. Algebraic topics include order relations, semigroups, groups and coding. RM

Number Theory, P. Computational Methods in Number Theory, Part II. Ed: H.W. Lenstra, Jr., R. Tijdeman. Math. Centre Tracts, No. 155. Math Centrum, 1982, v + 204 pp, Dfl. 27,50 (P). [ISBN: 90-6196-249-8] Seven papers on computational number theory covering such topics as Galois group calculations, class numbers, units, factorization, continued fractions, diophantine equations, and the zeta function. SG

Number Theory, P. Séminaire de Théorie des Nombres, Paris 1981-82: Séminaire Delange-Pisot-Poitou. Ed: Marie-José Bertin. Progress in Math., V. 38. Birkhauser Boston, 1983, vi + 348 pp, $27.50. [ISBN: 0-8176-3155-0] A collection of 25 papers on a variety of topics in number theory, algebraic geometry, and Diophantine geometry. SG

Number Theory, P. Lecture Notes in Mathematics-1030: Selberg's Zeta-, L-, and Eisensteinseries. Ulrich Christian. Springer-Verlag, 1983, xii + 196 pp, $10.50 (P). [ISBN: 0-387-12701-1] The author establishes analytic continuation and functional equations for the so-called Siegel-Eisenstein series and for three classes of functions associated with Selberg, namely Zeta functions, L-series, and Eisenstein series. SG

Linear Algebra, T(16: 1), S. First Course in Linear Algebra. P.B. Bhattacharya, S.K. Jain, S.R. Nagpaul. Halsted Pr, 1983, xi + 280 pp, $19.95. [ISBN: 0-470-27442-5] Written at a more sophisticated level than the typical elementary linear algebra texts, this begins with matrices and determinants and continues with the theory of vector spaces, canonical forms of matrices, Euclidean spaces, tensor algebras and generalized inverses of matrices. JNC

Linear Algebra, T(14: 1), S. Linear Algebra. V.V. Voyevodin. Transl: Vladimir Shokurov. MIR Pub, 1983, 392 pp. Intended to present linear algebra from the point of view of computational mathematics, but curious for this stated goal in both included topics (groups, rings, fields, Jordan canonical form) and omitted topics (Gaussian elimination, well-conditioned matrices, iterative methods). Short problem sets by American standards. AWR

Group Theory, P. Representation Theory of Reductive Groups. Ed: P.C. Trombi. Progress in Math., V. 40. Birkhauser Boston, 1983, xi + 290 pp, $24.95. [ISBN: 0-8176-3135-6] Proceedings of an April 1982 conference held at the University of Utah. LAS

Algebra, P. The Covariant Classification of Two-Dimensional Smooth Commutative Formal Groups Over an Algebraically Closed Field of Positive Characteristic. H.A.W.M. Knepppers. Math. Centre Tracts, No. 157. Math Centrum, 1983, 74 pp, Dfl. 9,90 (P). [ISBN: 90-6196-252-8] The author gives the covariant classification of two-dimensional formal groups (as described in the title), compares this with the contravariant classification of Manin, and gives an application to curves of genus two over fields of characteristic two. SG

Algebra, P. Lecture Notes in Mathematics-996: Invariant Theory. Ed: F. Gheradelli. Springer-Verlag, 1983, v + 159 pp, $9.50 (P). [ISBN: 0-387-12319-9] A collection of six papers presented at a conference in Italy in 1982. Topics covered: complete symmetric varieties; geometric invariant theory; representations of quivers; invariants of Z/pZ; symmetric and flag manifolds; semistable bundles. SG

Algebra, P. Groups of Divisibility. Jiří Močkoř. Math. & Its Appl. D Reidel Pub, 1983, 184 pp, $39.50. [ISBN: 90-277-1539-4] The author brings together the basic ideas and results of the theory of divisibility from ring theory, number theory, and the theory of ordered groups. The author concentrates on the group of divisibility of an integral domain and on various "realization" theorems. SG

Algebra, S(18), P. Lecture Notes in Mathematics-1029: Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin. Ed: M.-P. Malliavin. Springer-Verlag, 1983, v + 339 pp, $14.60 (P). [ISBN: 0-387-12699-6]

Algebra, T(14-15). Elements of Modern Algebra. Jimmie and Linda Gilbert. Prindle, Weber & Schmidt, 1984, x + 294 pp. [ISBN: 0-87150-458-8] Intended for a one-semester course. A bit easy--not enough theoretical problems and too many exercises which are straightforward calculations. BK

Calculus, S(13-14), L. A Brief Course of Higher Mathematics. V.A. Kudryavtsev, B.P. Demidovich. Transl: Leonid Levant. MIR Pub, 1984, 693 pp. Identical with first printing but on cheaper paper--which adds 5/8" thickness to the book. (First Printing, TR, April 1982; Extended Review, October 1982.) JK

Calculus, T(13: 2). Calculus with Applications to Business and Life Sciences, Second Edition. Abe Mizrahi, Michael Sullivan. Wiley, 1984, xii + 440 pp, $28.95. [ISBN: 0-471-05484-4] Any changes from the First Edition (TR, June-July 1976) are relatively minor, mostly rewriting of isolated sections and reorganization of topics. JS

Calculus, S(13). Engineering Mathematics: Programmes and Problems, Second Edition. K.A. Stroud. Springer-Verlag, 1982, xix + 894 pp, $16.50 (P). [ISBN: 0-387-91218-5] Programmed learning approach to those aspects of calculus that are used by engineers. Would be a good book to use for review, as it does not explain the concepts very well. BK

Calculus, T(13-14: 1-3). Calculus and Analytic Geometry, Third Edition. Al Shenk. Scott Foresman, 1984, xvi + 1271 pp, $36.95. [ISBN: 0-673-16582-5] A number of changes have been incorporated into this new edition, including study problems (with solutions), review exercises, and the transplanting of much of the theory of Taylor polyomials and infinite series from Chapters 17-18 to Chapters 10-11, thus preceding the multi-variable calculus. (First Edition, TR, October 1977.) JS

Complex Analysis, P. Lecture Notes in Mathematics-1013: Complex Analysis--Fifth Romanian-Finnish Seminar, Part 1. Ed: C. Andreian Cazacu, et al. Springer-Verlag, 1983, xx + 393 pp, $19 (P) [ISBN: 0-387-12682-1]; Lecture Notes in Mathematics-1014: Complex Analysis--Fifth Romanian-Finnish Seminar, Part 2, xx + 334 pp, $17 (P). [ISBN: 0-387-12683-X] Proceedings of the 1981 Bucharest seminar. 53 papers in four categories: quasiconformal mappings, function theory of one complex variable, several complex variables, and potential theory. Includes two brief essays on work of R. Nevanlinna and S. Stoilow. PZ

Complex Analysis, T(15-16: 1), S, L. Complex Variables and Applications, Fourth Edition. Ruel V. Churchill, James Ward Brown. McGraw-Hill, 1984, x + 339 pp, $23.60. [ISBN: 0-07-010873-0] Covers rudimentary theory (analytic functions, integrals, series, residues and poles) in first six chapters. Last six apply theory to conformal mapping and its physical applications, Poisson-type integrals, further function theory. Changes from Third Edition (TR, November 1974; Extended Review, January 1976): more examples, figures; earlier integration; new treatment of antiderivatives, residues. PZ

Complex Analysis, S(18), P. Complex Analysis: Methods, Trends, and Applications. Ed: Eberhard Lanckau, Wolfgang Tutschke. Akademie-Verlag, 1983, 398 pp, 78 M. Surveys classical and modern methods, generalizations, and applications of complex analysis. 22 chapters by different authors; each introduces general theory of a topic or problem with special attention to modern methods and applications to other fields. Sample chapters: value distribution theory, cohomological methods, ordinary differential equations in the complex domain. Translation slightly quaint, but readable. PZ

Differential Equations, P. Lectures on Partial Differential Equations. G.B. Folland. Springer-Verlag, 1983, 160 pp, $6.40 (P). [ISBN: 0-387-12280-X] Emphasizes applications of Fourier analysis to partial differential equations. Presumes little knowledge of these fields but does presume measure theory and functional analysis. Topics include partial differential equations with constant coefficients, L2 Sobolev spaces, pseudo-differential operators, and Lipschitz estimators. SG

Differential Equations, P. Lecture Notes in Mathematics-1032: Ordinary Differential Equations and Operators. Ed: W.N. Everitt, R.T. Lewis. Springer-Verlag, 1983, xv + 521 pp, $23.20 (P). [ISBN: 0-387-12702-X] Proceedings of a symposium at the University of Dundee, Scotland. Some topics: nonlinear second order boundary value problems; determinantal operators; von Neumann factorization of some self-adjoint extensions; inclusion theorems for solutions; Liouville-Green asymptotic theory for second order equations; quadratic integral inequalities; matrix Riccati inequality and oscillation of second order systems, etc. CU

Differential Equations, P. Systems of Microdifferential Equations. Masaki Kashiwara. Progress in Math., V. 34. Birkhauser Boston, 1983, xv + 159 pp, $16.95. [ISBN: 0-8176-3138-0] Microdifferential operators are sections of the sheaf of rings of pseudodifferential operators on the cotangent bundle of a complex manifold. They are defined and studied microlocally, i.e., locally in the cotangent bundle. Book consists of translated lecture notes from a graduate course given at the University of Paris in 1976-77, with a preface by J.-L. Brykinski. Beware of typos. PZ

Differential Equations, P. Lecture Notes in Mathematics-1015: Equations différentielles et systèmes de Pfaff dans le champ complexe-II. Ed: R. Gérard, J.-P. Ramis. Springer-Verlag, 1983, 411 pp, $18.30 (P). [ISBN: 0-387-12684-8] Contains results from the latter years of the seminar held at Strasbourg. JAS

Differential Equations, P. Lecture Notes in Mathematics-989: Volterra-Stieltjes Integral Equations and Generalized Ordinary Differential Expressions. Angelo B. Mingarelli. Springer-Verlag, 1983, xiv + 318 pp, $15.50 (P). [ISBN: 0-387-12294-X] The aim of these notes is to develop a qualitative and spectral theory of Volterra-Stieltjes integral equations with applications to real ordinary and difference equations of the second order. LCL

Differential Equations, P. Trends in Theory and Practice of Nonlinear Differential Equations. Ed: V. Lakshmikantham. Lect. Notes in Pure & Appl. Math., V. 90. Dekker, 1984, xviii + 562 pp, $59.75 (P). [ISBN: 0-8247-7130-3] Proceedings of a conference held at the University of Texas at Arlington, June 14-18, 1982. Includes a tribute to E.A. Coddington. JAS

Numerical Analysis, S(17-18), P. Computation with Recurrence Relations. Jet Wimp. Applic. Math. Ser. Pitman, 1984, xiii + 310 pp, $50. [ISBN: 0-273-08508-5] An in-depth study of a vast body of computational techniques, together with error analysis, with special attention to the important techniques due to Olver, Clenshaw, and Miller. Three-quarters of the book is devoted to linear algorithms, including practical examples: computations of special functions from mathematical physics, Fourier coefficients, Taylor coefficients, zeros of functions, eigenvalues of differentiable operators. LCL

Functional Analysis, P. Riesz Spaces II. A.C. Zaanen. Math. Lib., V. 30. Elsevier Sci Pub, 1983, xi + 720 pp, $68. [ISBN: 0-444-86626-4] A sequel to Volume I (1971), Volume II contains 10 chapters on aspects of vector lattice theory, including recent results on kernel integral operators, compact operators, Orlicz spaces, and orthomorphisms. Includes an extensive bibliography. PZ

Functional Analysis, S(17). Strict Convexity and Complex Strict Convexity: Theory and Applications. Vasile I. Stratescu. Lect. Notes in Pure. & Appl. Math., V. 89. Dekker, 1984, x + 312 pp, $49.75 (P). [ISBN: 0-8247-1796-1] Presents a comprehensive survey of properties of a Banach space related to strict convexity. After a short introductory chapter containing basic results of linear functional analysis, there follows a chapter devoted to characterizations of strictly convex spaces and applications and a chapter characterizing those Banach spaces for which the maximum modulus holds for analytic functions with values in the space. AWR

Functional Analysis, P?? Matrix Analysis for Applied Sciences, Volume 1. Ivo Marek, Karel Zitný. Teubner-Texte zur Math., B. 60. BG Teubner, 1983, 196 pp, 19M (P). More of a handbook than a useable text, this is largely a summary of selected theorems from functional analysis up through analysis of the spectrum and the Riesz calculus. Unmotivated and badly in need of editing (for example, "nonsubtituable," "sofisticated"); the curious reader would be better off with one of the standard functional analysis texts. JS

Analysis. Global Analysis--Analysis on Manifolds: Dedicated to Marston Morse (1892-1976). Ed: Themistocles M. Rassias. Teubner-Texte zur Math., B. 57. BG Teubner, 1983, 376 pp, 39 M (P). A collection of 27 papers dating from approximately 1982, published in honor of Marsten Morse. JAS

Analysis, P. Lecture Notes in Mathematics-1031: Dynamics and Processes. Ed: Ph. Blanchard, L. Streit. Springer-Verlag, 1983, 213 pp, $10.50 (P). [ISBN: 0-387-12705-4] Proceedings of the third session of the Bielefeld Encounters in Physics and Mathematics, November 30-December 4, 1981. JAS

Analysis, P. Lecture Notes in Mathematics-1028: Séminaire d'Analyse: P. Lelong-P. Dolbeault-H. Skoda. Ed: P. Lelong, P. Dolbeault, H. Skoda. Springer-Verlag, 1983, viii + 328 pp, $15 (P). [ISBN: 0-387-12731-3] Presentations from the years 1981 through 1983 covering complex analysis,

Kaehler geometry, vector bundles, and differential systems. JAS

Analysis, S*(15-17), L*. Catastrophe Theory. V.I. Arnold. Transl: R.K. Thomas. Springer-Verlag, 1984, 79 pp, $8.50 (P). [ISBN: 0-387-12859-X] A masterful, elementary exposition of catastrophe theory, from Whitney's singularity theory to Dynkin diagrams. In the best tradition of popular exposition by Russian mathematicians, this brief volume places an enormous amount of mathematics within the grasp of bright undergraduates. LAS

Analysis, P. Operator Algebras and Group Representations. Ed: Gr. Arsene, et al. Pitman Pub, 1984, $49.95 each. Volume I, Mono. & Stud. in Math., No. 17, xx + 277 pp [ISBN: 0-273-08604-9]; Volume II, Mono. & Stud. in Math., No. 18, xx + 250 pp. [ISBN: 0-273-08605-7] Proceedings of a September 1980 international conference held in Neptune, Romania. LAS

Algebraic Geometry, P. Classification of Algebraic and Analytic Manifolds. Ed: Kenji Ueno. Progress in Math., V. 39. Birkhauser Boston, 1983, viii + 630 pp, $39.95. [ISBN: 0-8176-3137-2] Proceedings of a July 1982 conference held in Katata, Japan. 15 papers plus a sizeable list of open problems. LAS

Differential Geometry, P. Differential Geometry: Proceedings, Special Year, Maryland 1981-82. Ed: Robert Brooks, Alfred Gray, Bruce L. Reinhart. Progress in Math., V. 32. Birkhauser Boston, 1983, vii + 254 pp, $19.95. [ISBN: 3-7643-3134-8] A collection of 13 papers covering a broad range of topics including polyhedra, conformal geometry, foliations, Brownian motion, harmonic functions, and hyperbolic systems of partial differential equations. SG

Differential Geometry, P. Seminar on Minimal Submanifolds. Ed: Enrico Bombieri. Princeton U Pr, 1983, vii + 358 pp, $45; $15 (P). [ISBN: 0-691-08324-X; 0-691-08319-3] A collection of 20 papers on closed geodesics and minimal surfaces, presented in 1979-80 at the Institute for Advanced Study. The first long chapter is a set of survey lectures, introducing the theory and classical problems of minimal submanifolds. PZ

Geometry, T(17: 1, 2), S, P. Géométries affine, projective et euclidienne. Claude Tisseron. Hermann, 1983, xii + 386 pp, 112 F (P). [ISBN: 2-7056-1408-7] A book which is specifically designed to prepare students to take examinations on the subjects of affine, projective and Euclidean (metric) geometry. An excellent source of non-trivial examples and exercises. CEC

Geometry, S(17-18), P. From Affine to Euclidean Geometry: An Axiomatic Approach. Wanda Szmielew. D Reidel Pub, 1983, xiii + 194 pp, $34.95. [ISBN: 90-277-1243-3] Wanda Szmielew died in 1976 after gaining recognition as a specialist in foundations of mathematics with particular emphasis on geometry. This volume, prepared for publication and translated by Maria Moszyńska, consists of the results of her last research on "the best" axiomatization of affine and Euclidean geometry. JNC

Geometry, S*(13-16), P*, L*. Structural Equation Geometry: The Inherent Properties of Curves & Coordinate Systems. J. Lee Kavanau. Science Software Systems, 1983, xxviii + 506 pp, $16.95 (P). [ISBN: 0-937292-02-8] A "much-simplified" version of Chapters I-VII, IX and X of the author's work Symmetry, An Analytical Treatment, (TR, February 1981). Over 150 problems of various levels of difficulty have been added. Aim is "to supply raw materials and suggest new approaches for the revision and rejuvenation of analytic geometry at all levels of instruction." JK

Algebraic Topology, P. Lecture Notes in Mathematics-1009: Controlled Simple Homotopy Theory and Applications. T.A. Chapman. Springer-Verlag, 1983, 94 pp, $8 (P). [ISBN: 0-387-12338-5] A homotopy in a space Y is called ϵ-controlled with respect to a given map p from Y to a metric space B if the p-image of the path of each point during the homotopy has diameter less than ϵ in B. This book develops simple homotopy theory for controlled homotopies. JAS

Dynamical Systems, P. Random Perturbations of Dynamical Systems. M.I. Freidlin, A.D. Wentzell. Transl: Joseph Szücs. Grund. der math. Wissenschaften, B. 260. Springer-Verlag, 1984, viii + 326 pp, $56. [ISBN: 0-387-90858-7] Two quotations from the Foreward suggest both the content and the problems to be encountered with the translation. "One of the important schemes leading to the study of various limit theorems for random processes is dynamical systems subject to the effect of random perturbations. ...The results obtained in this article relate to one-dimensional and partly two-dimensional dynamical systems and perturbations leading to diffusion processes." AWR

Probability, T(14: 1), L. Probability: Modeling Uncertainty. Donald R. Barr, Peter W. Zehna. Addison-Wesley, 1983, ix + 405 pp, $27.95. [ISBN: 0-201-10798-8] A postcalculus text covering random variables, expectation, joint distributions, change-of-variables, moment generating functions, elementary stochastic processes and sampling distributions. MT

Probability, S. Probability by Calculator: Solving Probability Problems with the Programmable Calculator. Peter W. Zehna. Prentice-Hall, 1982, x + 181 pp, $9.95 (P). [ISBN: 0-13-711523-7] Basic probability, from counting problems through random variables. The author's stated goal is not so much to give students a complete theoretical knowledge of probability as to make students intelligent users of results. Toward this end, the book introduces programs, designed for a TI-59 programmable calculator, which give results to be used intelligently. MT

Statistics, T(18: 2). Multivariate Statistics: A Vector Space Approach. Morris L. Eaton. Wiley, 1983, xvi + 512 pp, $34.95. [ISBN: 0-471-02776-6] Treats multivariate statistical theory using

vector space and invariant methods instead of the more traditional distribution theory/likelihood approach. Prerequisites are matrix and vector algebra, measure and integration theory, and a first year graduate course in mathematical statistics. MT

Statistics, T(18: 1), P. Lecture Notes in Statistics-21: Threshold Models in Non-linear Time Series Analysis. Howell Tong. Springer-Verlag, 1983, x + 323 pp, $22.80 (P). [ISBN: 0-387-90918-4] Introduces new methodology of non-linear threshold models. First describes the form and characteristics of threshold models, then discusses identification of models, and concludes with some case studies. Assumes an understanding of the basic theory of time series analysis. MT

Statistics, P. Recent Advances in Statistics: Papers in Honor of Herman Chernoff on His Sixtieth Birthday. Ed: M. Haseeb Rizvi, Jagdish S. Rustagi, David Siegmund. Academic Pr, 1983, xiii + 611 pp, $37.50. [ISBN: 0-12-589320-5] Contains 27 papers divided into five categories: sequential analysis, optimization, nonparametrics, statistical graphics, and other topics. The contributors were selected because of their associations with Herman Chernoff or because of their activity in areas of statistics where Chernoff has made his major contributions. MT

Statistics, T(17: 2). Statistical Methods for Forecasting. Bovas Abraham, Johannes Ledolter. Wiley, 1983, xv + 445 pp, $34.95. [ISBN: 0-471-86764-0] Statistical methods and models used in short-term forecasting, intended to provide an intermediate-level discussion of topics used for time series analysis, including regression, exponential smoothing, and ARIMA models. The intended audience includes advanced undergraduate and beginning graduate students in statistics, business, engineering, and the social sciences. MT

Statistics, S(17), P. Planning and Analysis of Observational Studies. William G. Cochran. Wiley, 1983, xiii + 145 pp, $21.95. [ISBN: 0-471-88719-6] Published posthumously, this book addresses problems which arise in observational studies. More a reference than a text, it focuses on the behavior of statistical procedures when their underlying assumptions are not satisfied. MT

Statistics, S(13). STATMASTER: Exploring and Computing Statistics Workbook. C. Michael Levy, William J. Froming, Marcia Belcher. Little, Brown & Co, 1983, 209 pp, $8.95 (P). [ISBN: 0-316-52237-6] A workbook which, along with a machine-dependent STATMASTER disk, is intended to aid a student's understanding of the application of statistical techniques, including computation of means and variances, Chi-square, standard normal, and t statistics, and correlation and analysis of variance. MT

Statistics, S(17). Introduction to Robust and Quasi-Robust Statistical Methods. William J.J. Rey. Universitext. Springer-Verlag, 1983, ix + 236 pp, $14 (P). [ISBN: 0-387-12866-2] A discussion of several robust statistical techniques (those not very sensitive to departures from their assumptions), including the jackknife and bootstrap methods, M estimators, L estimators, and quantile estimators. MT

Statistics, T*(14-17: 1, 2), L. Design and Analysis of Experiments, Second Edition. Douglas C. Montgomery. Wiley, 1984, xvi + 538 pp, $37.95. [ISBN: 0-471-86812-4] Revision of the author's 1976 text. Changes include a stronger emphasis on using residual analysis to check for model adequacy, reorganization of old material, and new material on handling unbalanced data, fractional factorials, multiple comparison procedures and sample size determination. Assumes only a first course in statistical methods. RSK

Statistics, P. The User's Guide to Multidimensional Scaling. A.P.M. Coxon. Heinemann Educ Books, 1982, xiv + 271 pp, $28. [ISBN: 0-435-82251-9] Multidimensional scaling (MDS) refers to ways of representing information contained in a set of data by a set of points in space in such a way that geometrical relationships reflect empirical relationships in the data. This book provides a conceptual and applications-oriented exposition of MDS, and is designed chiefly for use with the MDS(X) library of computer programs. RSK

Statistics, P*. Key Texts in Multidimensional Scaling. Ed: P.M. Davies, A.P.M. Coxon. Heinemann Educ Books, 1982, xx + 349 pp, $20. [ISBN: 0-435-82253-5] Collection of seventeen articles, most published in the 1970's, viewed as crucial in the development of multidimensional scaling. A general resource book, it is particularly appropriate to use in conjunction with The User's Guide to Multidimensional Scaling (see above). RSK

Statistics, P*. Classification and Regression Trees. Leo Breiman, et al. Statistics/Probability Ser. Wadsworth Pub, 1984, x + 358 pp, $21.95; $13.95 (P). [ISBN: 0-534-98053-8; 0-534-98054-6] First eight chapters give an expository discussion of the use of binary trees as a nonparametric data analysis method. Last four chapters provide the theoretical framework. Good set of references. RSK

Computer Literacy, S(13-15). The Illustrated Computer Dictionary, Revised Edition. Donald D. Spencer. Charles E Merrill, 1983, x + 209 pp, $8.95 (P). [ISBN: 0-675-20075-X] A concise elementary glossary of about 4000 terms, typically with one-sentence definitions. LAS

Computer Literacy, T?(13: 1). An Introduction to Computers: Developing Computer Literacy. Donald D. Spencer. Charles E Merrill, 1983, xiv + 655 pp, $19.95 (P). [ISBN: 0-675-20030-X] A sweeping superficial survey of computer uses in society. Primarily an introduction to vocabulary, not to concepts. In clarity and comprehensiveness, but not in ideas, this is one of the best examples of the vast, depressing, vacuous genre of computer literacy texts. LAS

Computer Programming, S. Your IBM PC: A Guide to the IBM Personal Computer. Lyle J. Graham. Osborne/McGraw-Hill, 1983, xi + 580 pp, $16.95 (P) [ISBN: 0-931988-85-3]; Your IBM PC: A Guide to the IBM PC (DOS 2.0) and XT. Lyle J. Graham, Tim Field, 1984, viii + 600 pp, $17.95 (P). [ISBN: 0-88134-120-7] These two very similar books serve as thorough but introductory user's manuals. Considerable time is spent on BASIC, essentially none on assembly language or any of the popular high-level languages. Beware: even though the introduction to the DOS 2.0 version refers to CP/M-86, there seems to be nothing about it in the text. JAS

Computer Programming. Programming the 8086/8088. James W. Coffron. Sybex, 1983, xiii + 311 pp, $14.95 (P). [ISBN: 0-89588-120-9] Following an introduction that explains "What is programming?", the book moves swiftly into the architecture and addressing modes of the 8086/8088. The middle third is a standard alphabetic presentation of the instruction set mnemonics followed by a few short program segment examples which are not keyed to any particular operating system or assembler. Two chapters on interrupts and I/O are followed by some discussion of the IBM implementation of MS-DOS. Unfortunately, this section ignores the MS-DOS system functions and encourages the direct use of system interrupts. The book may be useful for a software entrepreneur but it encourages bad programming technique. JAS

Computer Programming. 8088 Assembler Language Programming: The IBM PC. David C. Willen, Jeffrey I. Krantz. Howard W Sams, 1983, 235 pp, $15.95 (P). [ISBN: 0-672-22024-5] A fairly gentle introduction to the 8088 architecture and instruction set. Contains sections on IBM PC hardware. Unfortunately the section on using the PC-DOS routines substitutes hardware interrupts for system function calls, thereby encouraging bad programming. JAS

Computer Programming, T?. Programming the IBM Personal Computer: Pascal. Neill Graham. Holt, Rinehart & Winston, 1983, xi + 275 pp, $18.95 (P). [ISBN: 0-03-061982-3] Yet another Pascal programming manual. This one has as its raison d'être the special features of the usual (Microsoft) Pascal for the IBM PC. The presentation is readable, leisurely, and quite complete. JAS

Computer Programming, T. Using Computers: Programming and Problem Solving. Gerald H. Elgarten, Alfred S. Posamentier. Addison-Wesley, 1984, 441 pp, $21.33. [ISBN: 0-201-20205-0] A text which would be appropriate for teaching high school students the principles of programming in the BASIC language. A very slow, very low-level introduction to programming that assumes virtually no mathematical sophistication at all. It might even be appropriate for elementary or junior high school students as well. MS

Computer Programming, T(13: 1, 2). Problem Solving Using Pascal: Algorithm Development and Programming Concepts. Romualdas Skvarcius. Prindle, Weber & Schmidt, 1984, xii + 673 pp, (P). [ISBN: 0-87150-440-5] An introductory programming text designed for a one- or two-semester introduction to computer programming. The text teaches Pascal as a block-structured programming language. No previous background in programming or computer science is assumed. In addition to the syntax of Pascal, the text introduces the student to the concepts of algorithms, algorithmic design and development, and pseudo-code. MS

Software Systems, L. The Software Catalog: Microcomputers, Fall 1983. Elsevier Sci Pub, 1983, xiii + 1020 pp, $69 (P). [ISBN: 0-444-00776-8] Nearly 10,000 programs listed by vendor, computer, operating system, language, and microprocessor (there is another volume for minicomputers). This volume is produced from the International Software Database which is currently available online from the DIALOG Information Retrieval Service, phone 800-227-1927. JAS

Software Systems, S(14-15). The IBM PC-DOS Handbook. Richard Allen King. Sybex, 1983, xx + 296 pp, $16.95 (P). [ISBN: 0-89588-103-9] Really a book about MS-DOS, including specifics of the IBM implementation with considerable detail on the system functions, diskette logical structure, and the system utility programs. Useful index and appendices; including MS-DOS 2.0 directory structures. JAS

Software Systems, S, L. The Unix System Guidebook: An Introductory Guide for Serious Users. Peter P. Silvester. Springer-Verlag, 1984, xi + 207 pp, $14.50 (P). [ISBN: 0-387-90906-0] A brief, well-written yet superficial introduction illustrating UNIX commands without giving enough detail for effective use. Covers shell commands, editors, languages, and the kernel. Best used as a preface to other more complete documents. LAS

Software Systems, P. Computer Algebra: Symbolic and Algebraic Computation, Second Edition. Ed: B. Buchberger, G.E. Collins, R. Loos. Springer-Verlag, 1983, vii + 283 pp, $24.50 (P). [ISBN: 0-387-81776-X] A concise survey of computer algebra theory and systems, with extensive references. Corrected reprint of 1982 edition (TR, June-July 1983). LAS

Computer Science, S(17), P, L. Fuzzy Sets, Natural Language Computations, and Risk Analysis. Kurt J. Schmucker. Computer Sci Pr, 1984, xv + 192 pp, $32.95. [ISBN: 0-914894-83-8] Presents a new approach to the analysis of risks to which a computer system may be subject, such as faulty software, malfunction, pranks, terrorism and sabotage. The approach is based on fuzzy set theory. An entirely non-numeric approach which appears to give a 20% increase in accuracy of risk analysis. CEC

Computer Science, P. Lecture Notes in Computer Science-161: DIANA, An Intermediate Language for Ada, Revised Version. Ed: G. Goos, et al. Springer-Verlag, 1983, vii + 201 pp, $11 (P). [ISBN: 0-

387-12695-3] Introduction to and reference manual for DIANA (Descriptive Intermediate Attributed Notation for ADA), designed to retain the structure of the program and serve as an interface between different parts of a compiler. RM

Systems Theory, P. Reproducing Kernel Hilbert Spaces: Applications in Statistical Signal Processing. Ed: Howard L. Weinert. Benchmark Papers in Elect. Engin. & Comp. Sci., V. 25. Hutchinson Ross Pub, 1982, xiii + 654 pp, $61. [ISBN: 0-87933-434-7] In the Benchmark Series intended to provide in each volume a practical point of entry into a given area of research, this volume differs by focusing not on an area but on a methodology applicable to different areas: detection, estimation, and approximation. A reproducing kernel Hilbert space is a Hilbert function space containing a kernel which reproduces via an inner product every function in the space. Thirty papers explain, then employ this principle. AWR

Applications, P. Lecture Notes in Mathematics-1035: The Mathematics and Physics of Disordered Media: Percolation, Random Walk, Modeling, and Simulation. Ed: B.D. Hughes, B.W. Ninham. Springer-Verlag, 1983, vii + 431 pp, $18.30 (P). [ISBN: 0-387-12707-0] Proceedings of a February 1983 interdisciplinary workshop at the Institute for Mathematics and Its Applications in Minneapolis, Minnesota, featuring a major (100 page) survey by B.D. Hughes and S. Prager on "Random processes and random systems." LAS

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-200: The Resource Sector in an Open Economy. Ed: Horst Siebert. Springer-Verlag, 1984, ix + 161 pp, $11.20 (P). [ISBN: 0-387-12700-3] Some economic problems of a resource-exporting country are addressed in papers presented in a seminar "Allocation Policy in a Market Economy," and collected in this volume. AWR

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-221: An Inter-Industry Translog Model of Prices and Technical Change for the West German Economy. Shinichiro Nakamura. Springer-Verlag, 1984, xiv + 290 pp, $18.70 (P). [ISBN: 0-387-12709-7] Results of a research project in a disaggregated econometric forecasting model where prices are explained as dual variables of the underlying production model. LAS

Applications (Engineering), P. Electronic Systems Effectiveness and Life Cycle Costing. Ed: J.K. Skwirzynski. NATO ASI Ser. F., No. 3. Springer-Verlag, 1983, xvii + 732 pp, $51.50. [ISBN: 0-387-12287-7] Complete proceedings of a July 1982 NATO Advanced Study Institute held at Norwich, U.K. Papers range from mathematical background to practical reliability assessments. LAS

Applications (Engineering), P. Annual Review of Fluid Mechanics, Volume 16, 1984. Ed: Milton Van Dyke, J.V. Wehausen, John L. Lumley. Annual Reviews, 1984, 444 pp, $28. [ISBN: 0-8243-0716-X] Fifteen survey papers offering well-documented expositions of topics in fluid dynamics. LAS

Applications (Inequalities), P. General Inequalities 3. Ed: E.F. Beckenbach, W. Walter. ISNM 64. Birkhauser Boston, 1983, xxi + 563 pp, $54.95. [ISBN: 3-7643-1539-3] The proceedings of the Third International Conference on General Inequalities held at Oberwolfach, Spring 1981. The thirty-six research papers span a broad range of topics: series and integral inequalities, matrix inequalities, functional inequalities (especially sequence spaces), geometric inequalities, differential inequalities; special emphasis on majorization and optimization techniques. LCL

Applications (Physics), S(18), P. Mathématique et Physique: Séminaire de l'Ecole Normale Supérieure 1979-1982. Ed: Louis Boutet de Monvel, Adrien Douady, Jean-Louis Verdier. Progress in Math., V. 37. Birkhauser Boston, 1983, x + 438 pp, $37.95. [ISBN: 0-8176-3154-2]

Applications (Physics), T(15-16: 1), S, L. A First Course in the Mathematical Foundations of Thermodynamics. David R. Owen. Undergrad. Texts in Math. Springer-Verlag, 1984, xvii + 134 pp, $26. [ISBN: 0-387-90897-8] Mathematical development of classical and nonclassical thermodynamics; theory and examples. Uses elementary vector calculus in three dimensions, including line integrals and exact differentials. Many helpful figures; few problems. PZ

Applications (Social Science), P. Systems Analysis in Urban Policy-Making and Planning. Ed: Michael Batty, Bruce Hutchinson. NATO Conf. Ser.: Systems Sci., V. 12. Plenum Pr, 1983, xii + 619 pp, $75. [ISBN: 0-306-41118-0] Proceedings of the Advanced Research Institute sponsored by NATO and held at New College, Oxford, September 21-27, 1980. JAS

Reviewers

RJA: Richard J. Allen, St. Olaf; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JDB: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MT: Michael Tveite, St. Olaf; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

Florida Section

The seventeenth annual meeting of the Florida Section was held March 9-10, 1984 at the University of Tampa, in Tampa. There were 169 registrants.

Invited Addresses:

"Intuitive and Counterintuitive Mathematics," by Warren Page, New York City Technical College.
"Discrete Mathematics in the First Two Years," by Stephen Maurer, Swarthmore College and Sloan Foundation.
"On the Addressing Problem in Loop Switching--Or, How to Imbed an Arbitrary Graph in a Squashed Cube," by H.O. Pollak, Central Services Organization for Regional Bell Companies.
"Finiteness Conditions and Torsion," by Mark Teply, University of Florida.
"Length and Area," by J.R. Quine, Florida State University.
"Character Theory and Finite Groups," by Pamela Ferguson, University of Miami.
"Topology, Algebra, and Image Processing," by Gerhard Ritter, University of Florida.
"Rational Representations of Finite Groups," by Dennis Kletzing, Stetson University.
"Mathematics Education in the Soviet Union--A View from Three Cities," by Denisse Thompson, Manatee Junior College.
"Some Interesting Problems from the Calculus," by William R. Rice, St. Petersburg Junior College.
"Aspects of Legislation Affecting Mathematics Education," by Representative Mary Figg, Florida Legislature.

Short Presentations:

* "Patterned Integer Products," by Alan Wayne, Pasco-Hernando Community College.
* "Predicative Mathematics: What, Why, and How?" by Jay Hook, Florida International University.
"Laser Analysis," by Donna Burke, University of Central Florida.
"Statistics--A Survey," by Willie H. Green, National Aeronautics and Space Administration.
"Which Triangles Have Their Areas To Be a Multiple of the Perimeter," by John Goehl and M.S. Jagadish, Barry University.
"Supplemental Instruction for College Algebra," by Lee H. Armstrong, University of Central Florida.
* "Order in Chaos and Disorder in Tranquility," by M.W. Hudgins, Lutz, Florida.
* "Patterns in Repeating Decimal Arithmetic," by Samuel Yates, Delray Beach, Florida.

Special Sessions:

Regional Meetings Session, organized by David Sherry and chaired by James Weaver, University of West Florida.
Special Session on Computing, organized by Gareth Williams, Stetson University:
"Ill Conditioning in Computational Mathematics--Examples on the Apple," by Bruce Edwards, University of Florida.
"Demonstration of Math Software for the Apple," by James Lang, Valencia Community College.
"Menu Driven Math Software," by Gareth Williams, Stetson University.
"The Computer in Undergraduate Mathematics," Panel discussion moderated by Gene Medlin, Stetson University. Panelists: Bill Caldwell, University of North Florida; David Jesse, Manatee Junior College; Wendell Motter, Florida A&M University.
Mathematics Articulation Session, organized by Ernest R. Ross, St. Petersburg Junior College.
Florida Two-Year College Mathematics Association business meeting, organized by B.A. Braun, St. Petersburg Junior College.
Women and Mathematics Session, organized by Betty Lichtenberg, University of South Florida, for the Association of Women in Mathematics. Speakers: Kitty Freeman and Denise Strenglein, University of South Florida.
Special Session on Discrete Mathematics, organized by Bettye Anne Case, Florida State University:
"Graph Theory," by Neil White, University of Florida.
"A Sufficient Condition for a Digraph To Be Hamiltonian," by Tony Shershin, Florida International University.
"Combinatorics," by Joe Mott, Florida State University.
"Logic and Set Theory," by Joe Hoffman, Tallahassee Community College.
Undergraduate Paper Session, sponsored by Pi Mu Epsilon and organized by Fredric J. Zerla, University of South Florida:
"Subset Selection," by David Van Brackle, University of Central Florida.
"On the Relative Frequency of Obtuse Triangles," by Michael Engling, University of Tampa.
"Real Maximal Ideals," by David Mullins, University of South Florida, New College.
"A Software Simulation of Floating Point Computation," by Peter Nylen, Stetson University.
"Transformations of Tensors," by Rockford Rathgeber, University of South Florida.

At the business meeting an award was presented to William R. Rice of St. Petersburg Junior College for outstanding contributions to mathematics and the mathematical community of the Florida Section. MAA memberships were presented to the undergraduates who presented papers at the meeting.

Eastern Pennsylvania and Delaware Section

The annual spring meeting of the Eastern Pennsylvania and Delaware Section was held on March 31, 1984 at Lehigh University. There were 90 persons in attendance.

Invited Addresses:

"Codes That Detect and Correct Errors," by Chester Salwach, Lafayette College.
"Public Key Cryptography," by Andrew Odlyzko, Bell Telephone Laboratories.
"Computer System Performance Evaluation," by Walter Hivner, Hewlett-Packard Corporation.
"Multi-access Communications," by Jack Wolf, University of Massachusetts.

A first prize award in student paper competition was presented to Rebecca J. Plassmann of Bryn Mawr College. She presented her paper "A Proof of the Euler-Poincaré Formula As Needed for Thurston's Work on Manifolds."

Indiana Section

The spring meeting of the Indiana Section was held on April 14, 1984 on the campus of Rose-Hulman Institute of Technology, Terre Haute, Indiana. The meeting was held in conjunction with the Indiana Small College Mathematics Competition, and was attended by approximately 80 people.

Presentations:

"The Dining Philosophers," by Judy Morrel, Butler University.
"Checking Counterexamples to Fermat's Last Theorem," by Constant Goutziers, Rose-Hulman Institute of Technology.
"The Apportionment Problem for the United States House of Representatives," by Jerome Goldstein, Tulane University.

Student Papers:

"Optimization of a Corporate Telephone Distribution," by Baron Gemmer, Rose-Hulman Institute of Technology.
"The Mathematical Derivation of the Determination of Cardiac Output," by Gerald Cleaver, Valparaiso University.

North Central Section

The spring meeting of the North Central Section was held at Macalester College, St. Paul, Minnesota on April 27-28, 1984. There were 119 registrants.

Invited Addresses:

"Bijections and Enumerations," by Dennis White, University of Minnesota, Minneapolis.
"The Mathematical Sciences: A Capital View," by Marcia Sward, Associate Director of the Mathematical Association of America.

Short Presentations:

* "Generalized Geometric Random Variables," by Ronald Rietz, Gustavus Adolphus College.
* "Infinite Classes of Sequence Generated Circles," by Gerald E. Bergum, South Dakota State University.
* "Mathematics Education in China--A Current Reform," by Sen Fan, University of Minnesota, Morris.
"Quixotic Physics," by Paul Fjelstad, St. Olaf College.
"Pick's Theorem Revisited," by Dale Varberg, Hamline University.
* "Computer Graphics and the Mathematics Curriculum: A Time for Implementation," by Allen Holmes, St. Paul Academy.
* "What Is the Dual Space to a Space of Analytic Functions?" by Paul Zorn, St. Olaf College.
* "Writing Across the Mathematics Curriculum," by Jennifer Galovich, St. Olaf College.
* "The Excess Number of Friday the 13th's in the Gregorian Calendar," by George Bridgman, Macalester College.
"Remarks on Some Fixed Point Theorems and Their Extensions," by Duane Anderson, University of Minnesota, Duluth.

Student Paper:

"Another Step Beyond Reality," by Paul Borman, St. Olaf College.

Allegheny Mountain Section

The annual spring meeting of the Allegheny Mountain Section was held on April 13-14, 1984 at Washington and Jefferson College, Washington, Pennsylvania. There were 95 registrants.

Invited Addresses:

"The Exponential Formula: Combinatorics Best Kept Secret," by Herbert S. Wilf, University of Pennsylvania.
* "The Computational Discovery of Mathematical Laws: Experiments in Bin Packing," by Jon Bentley, Bell Labs at Murray Hill, New Jersey.
* "Decision Making in Complex Environments," by Thomas L. Saaty, University of Pittsburgh.
"Some Peculiar Properties of Rationals and Irrationals," by Ivan Niven, University of Oregon.

Short Presentations:

"Teaching Problem Solving to Non-mathematicians: Difficulties and Opportunities," by James B. Reber, Indiana University of Pennsylvania.
"Structure of Multi-block Circulants," by Franz X. Hiergeist, West Virginia University.
"The Lifting Property of Differentiation," by Nicholas Ford, Pennsylvania State University, Fayette Campus.
"Eleven Coefficients in a Series Expansion in the Harmonic Mean," by Mike Mays, West Virginia University.
"Experiences of a Computer Science Retrainee," by Al Rabenstein, Washington and Jefferson College.
"Obtuse Triangle Problem," by Douglas Frank, Indiana University of Pennsylvania.

Panel Discussions:

"Some Discrete and No-So-Discrete Thoughts About the First Two Years of Mathematics," by Stephen Maurer, Sloan Foundation and Swarthmore College, and James Crawford, Lafayette College.
"Discrete Mathematics: When and How Should It Be?" by Mario Benedicty, University of Pittsburgh, and Richard McDermot, Allegheny College.

Student Presentations:

"Computing Ruin Probabilities in Insurance Risk Theory," by Julia Groves, Allegheny College.
"Constructing the Reals Using the Nested Interval Property," by Glen Rhoads, Allegheny College.
"Differential Equation Models of Venereal Diseases," by John Clements, Allegheny College.

Nebraska Section

The sixty-first annual meeting of the Nebraska Section was held April 13-14, 1984 jointly with the ninety-fourth annual meeting of the Nebraska Academy of Sciences at Nebraska Wesleyan University, Lincoln. There were 65 registrants.

Invited Address:

"What is Combinatorial Number theory?" by Ronald Graham, Bell Tel Labs, Murray Hill, New Jersey.

Short Presentations:

"A Moving Boundary Problem in Biological Transport," by Mike Wang, University of Nebraska, Omaha.
"Dagaev-Pell: From Russia in Haste," by Alexander Mahaffey, Jr., University of South Dakota.
"Repeating Decimals," by William G. Leavitt, University of Nebraska, Lincoln.
"An Unsymmetric Fubini Theorem," by Gerald W. Johnson, University of Nebraska, Lincoln.
"A Family of Weakened Continuity Conditions," by Edwin Halfar, University of Nebraska, Lincoln.
"The Good News and Bad News About Singular Differential Equations," by Leon Hall, University of Nebraska, Lincoln.
"Some Simple Applications of Difference Equations," by Allan Peterson, University of Nebraska, Lincoln.
"Some Tiling and Packing Problems," by Earl Kramer, University of Nebraska, Lincoln.
"Tiling a Block with Blocks," by David Klarner, University of Nebraska, Lincoln.
"Report on the 1984 American High School Mathematics Examinations," by Richard L. Barlow, Kearney State College.
"If $1 < \lambda \leq q_{n+1}/q_n$ and $0 \leq a < b \leq 1$, Is There An $x \in [a,b]$ Such That $\{q_n x\}_{n=1}^{\infty}$ Is Not Dense mod(1)?" by Gordon Woodward, University of Nebraska, Lincoln.

Southeastern Section

The sixty-third annual spring meeting of the Southeastern Section was held on April 6-7, 1984 at Tennessee Technological University, Cookeville, Tennessee. There were 223 persons in attendance.

Invited Lectures:

"The Role of Group Inversion in Analyzing Finite Markov Chains," by Carl D. Meyer, Jr., North Carolina State University.
"Applied Mathematics in the 1980's," by Garrett Birkhoff, Harvard University and Georgia Institute of Technology.
"Bits and Pieces From the Classroom," by Leonard Gillman, University of Texas.

Short Presentations:

"An Application of Global Approximations in the Finite Element Method," by E.J. Allen, University of North Carolina, Asheville.
"Orlicz Space Conjectures," by Arnold S. Goldstein, Savannah, Georgia.
"Evaluation of Improper Integrals," by Richard D. Carmichael and Elmer K. Hayashi, Wake Forest University.
"The Set of Ordinates of All Local Maxima of a Real Function is Countable," by J.E. Vaughan, University of North Carolina, Greensboro.
"Mathematical Models in Economics--An Example," by John Baxley, Wake Forest University.
"A Visitor's Impressions of Mathematics Education in the U.S.S.R.," by James G. Ware, University of Tennessee at Chattanooga.
"Antipodean Mathematics," by Beth Hardy, Georgia Southern College.
"A Comparative Study of Mathematics Education Between the Province of Taiwan, Republic of China, and the United States," by Ping-Tung Chang, Augusta College.
"Are the Japanese Really Ahead in Mathematics Education?" by Donald R. Peeples, Wingate College.
"Mathematics Instruction in China: Professors, Students, and the Curriculum," by W. Howard Bare, Berry College.
"On the Linear Diophantine Equation Over the Ring of Integral Matrices," by Jau-shyong Shiue, Gardner-Webb College.
"Number of Generators of Invertible Ideals in Certain Polynomial Rings," by Mary Barone Martin, University of North Carolina, Chapel Hill.
"When Are Regular Elements of Unital Subrings Regular?" by James Kuzmanovich, Wake Forest University.
"What A Tangent Line Is When It Isn't A Limit," by Irl C. Bivens, Davidson College.
"Use of Numerical Analysis in Approximating Solutions to Differential Equations," by Susan Y. Johnson, Spelman College.
"The North Carolina Mathematics Curriculum Study," by Benjamin G. Klein, Davidson College.
"The Influence of a Well Conceived Examination Procedure on the Way Mathematics is Taught and Studied," by Myrtle H. Lewin, Agnes Scott College.
"Hypothesis Testing: A Logically Indefensible, Intellectual Bankrupt Decision-Making Mechanism," by R.W. Hoyer, University of North Carolina, Asheville.
"The Influence of Course Upon Measures of Teaching Effectiveness in Mathematics," by Fred A. Massey, Georgia State University.
"Tag Team Teaching," by Linda Exley and Vincent Smith, DeKalb Community College.
"An Application of Non-Linear Regression," by Lyndell Kerley, East Tennessee State University.
"Exchangeability, Uncorrelation, and the Central Limit Theorem," by Peter Z. Daffer, Vanderbilt University.
"The Solution of an Experimental Design Problem Involving a First Order Auto-regressive Model," by S.F Ebey, University of the South.
"Geometric Representations for Various Means," by Richard P. Savage, Jr., Tennessee Technological University.
"A Simple Nonstandard Way of Looking at Nets and Filters," by Jonathan W. Lewin, Kennesaw College.
"Almost Open Sets and A-functions," by David A. Rose, Francis Marion College.
"Hausdorff Dimension, Trees, and Distribution Functions," by Mark Smiley, Auburn University, Montgomery.
"Elementary Proof of a Formula of Ramanujan," by Robert L. Lamphere, Francis Marion College.
"Some Wonderful Properties of Bernoulli's Table of Combinations," by Sandra C. McLaurin, University of North Carolina, Wilmington.

Special Sessions:

"Using a Computer to Guess at Algebra Theorems," by Billy F. Bryant, Vanderbilt University.
"Carrying Capacity in Density-Dependent Selection," by C.E. Clark, University of Tennessee, Knoxville.
"Triangular Fibonacci and Lucas Numbers," by Charles R. Wall, Trident Technical College.
"Some Computer-assisted Theorem Relating to $(GF(q))_n$," by Joel V. Brawley, Jr., Clemson University.
"Empirical Computer Studies of Combinatorial Structures, by Kevin T. Phelps, Georgia Institute of Technology.
"Using the Computer to Find Visible Paths," by David R. Stone, Georgia Southern College.
"Hamilton, Petersen, and Permutation Graphs--Computing Your Way to Theory," by Richard D. Ringeisen, Clemson University.
"Empirical Studies and Statistical Research," by Claude E. McHenry, Tennessee Technological University.

THE MATHEMATICAL ASSOCIATION OF AMERICA

January 1, 1984

BYLAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

ARTICLE I — NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of the mathematical sciences in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by cooperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II — MEMBERSHIP

1. There shall be two classes of members: individual and institutional.

2. Any person interested in the field of collegiate mathematics shall be eligible for election to individual membership in the Association.

3. Any institution, academic or corporate, interested in the support of collegiate mathematics shall be eligible for election to institutional membership in the Association.

4. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission. In the case of individuals qualifying for student dues, the application shall be endorsed by two individual members of the Association.

ARTICLE III — BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a President-Elect (only during a year immediately prior to the expiration of a President's term), a Past-President (only during a year immediately following the expiration of a President's term), a First Vice-President, a Second Vice-President, an Editor of its publication entitled "THE AMERICAN MATHEMATICAL MONTHLY," a Secretary, and a Treasurer.

2. There shall be a Board of Governors (herein called "the Board") to consist of the officers, the ex-presidents for terms of six years after the expiration of their respective presidential terms, the Editor of each of its two publications entitled TWO-YEAR COLLEGE MATHEMATICS JOURNAL and MATHEMATICS MAGAZINE, the members of the Finance Committee, and additional elected members (herein called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association.

3. There shall be an Executive Committee of the Board consisting of the President, the President-Elect (only during a year immediately preceding the expiration of a President's term), the Past-President (only during a year immediately following the expiration of a President's term), the two Vice-Presidents, the Editor of the AMERICAN MATHEMATICAL MONTHLY, the Secretary, and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or interest to the Association. This Committee shall prepare the agenda for meetings of the Board and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent or as may in any way facilitate the Board's work.

4. At all meetings of the Board of Governors a quorum shall consist of not less than 25 per cent of the membership of the Board, and no business may be validly transacted at a meeting at which less than a quorum is present.

5. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This Committee shall consist of six members including the President, the Past-President (for a term of one year), the Secretary, and the Treasurer.

6. The Board shall hold a meeting each year immediately preceding the annual business meeting of the Association. Other meetings of the Board may be held from time to time at the call of the President or of any six (6) members of the Board.

7. Notice of all meetings of the Board shall be given by the Secretary to each member of the Board at least fifteen (15) days prior to the date set therefor.

8. A member of the Board may waive notice with the same effect as if due notice had been given.

9. The Board may refer a matter to a referendum mail vote of the entire membership and shall make such reference if a referendum is requested, prior to final action by the Board, by three hundred or more members. The taking of a referendum shall act as a stay upon Board action until the votes have been canvassed, and thereafter no action may be taken by the Board except in accordance with a plurality of the votes cast in the referendum.

ARTICLE IV — ELECTIONS, APPOINTMENTS, AND TERMS OF OFFICERS AND MEMBERS OF THE BOARD

1. (a) The membership at large shall elect biennially a President-Elect for a term of one year and a First Vice-President for a term of two years. The President-Elect shall become President for a two-year term at the expiration of the one-year term as President-Elect and shall become Past-President for a one-year term at the expiration of the term as President.

(b) The membership in each Section shall elect triennially a Governor for a term of three years beginning July 1. For these elections at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section. A Governor who has moved his or her place of employment from the Section by which he or she was elected shall be considered to have ended his or her term of office on the Board.

(c) The Board shall elect annually two Governors for terms of three years and at appropriate times by ballot and for terms stated: a Second Vice-President for two years, an Editor of the AMERICAN MATHEMATICAL MONTHLY, an Editor of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL, an Editor of MATHEMATICS MAGAZINE, a Secretary, and a Treasurer each for five years, and members of the Finance Committee (other than the President, the Past-President, the Secretary, and the Treasurer) for four years.

(d) The beginning and end of the term of every officer and member of the Board (except as provided in Section (b) of this Article) shall occur at the adjournment of the annual business meeting. All officers and members of the Board shall hold over until their respective successors have been duly elected or appointed and qualified.

(e) The President shall be ineligible for reelection as President-Elect or as President. The Vice-Presidents, the Editors, and the Governors shall be eligible for reelection only after an interim equal to their respective terms of office except that Governors having served less than a year and a half shall be eligible for reelection for a term of three years.

(f) The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of business of the Association.

(g) Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President, Governors (except Sectional Governors), and members of the Finance Committee. The Board may make additional nominations.

2. For general elections by the membership of the Association there shall be a Nominating Committee appointed annually by the President with the approval of the Board. The general election shall be conducted in two stages, a primary mail voting concluding approximately five months before the date of the annual meeting and a final voting concluding at the time of the annual meeting. For the primary voting the Nominating Committee shall prepare printed ballots with five or more nominees for President-Elect and three or more for each other office to be filled by the members. Blank spaces on the ballot shall be provided for write-in votes. From the results of the primary voting the Nominating Committee shall prepare a printed ballot for the final voting. This ballot shall be mailed to the membership approximately one month before the annual meeting and the voting shall close on the day of the annual business meeting. The final ballot shall present one nominee for President-Elect, to be selected by the Nominating Committee out of the three persons who received the most votes in the primary voting. For each other office the final ballot shall present two names, one being that of the person who received the highest vote in the primary voting.

3. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual business meeting of the Association, shall be Chairman of the Executive Committee and of the Finance Committee; and shall have the usual duties pertaining to the office and such other duties as may from time to time be assigned by the Board of Governors.

4. In the absence of the President, the First Vice-President (or in his or her absence the Second Vice-President) shall have and exercise the powers of the President, except that the Past-President shall preside at meetings of the Finance Committee (or in his or absence the senior member, in terms of service on the Committee, of the two elected members of the Finance Committee). The Board of Governors may assign to the Vice-President such duties as may from time to time be determined.

5. The Secretary shall have the usual duties pertaining to the office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual business meeting and special meetings, and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the terms of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

6. The Treasurer shall have the usual duties pertaining to the office including the collection of dues and the supervision and safekeeping of the funds of the Association.

7. (a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, except when they meet in executive session, but he shall not be *ex officio* a member of these bodies. He shall be especially responsible for implementing and coordinating Section activities.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee.

8. The Association shall, to the extent allowed by law, indemnify any member or former member of the Board of Governors or any other committee of the Association against expenses (including attorneys' fees), judgments, fines and amounts paid in settlement actually and reasonably incurred by such member in connection with the defense of any action, suit, or proceeding in which he or she is made a party by reason of having been made a member and having acted in good faith as a member of the Board of Governors or other committee. The Association may purchase and maintain insurance on behalf of such member to provide for indemnifying him or her against such liabilities.

ARTICLE V—BUSINESS MEETINGS OF THE ASSOCIATION

1. A business meeting of the Association shall be held annually, at such time and place as the Board may direct. Other business meetings of the Association may be called from time to time by the Board or by the President of the Association to be held at such time and place as may appear from the call.

2. Notice of any business meeting of the Association shall be given by the Secretary to each member of the Association at least thirty (30) days prior to the date set for each meeting.

3. Any member of the Association may waive notice with the same effect as if due notice had been given.

4. At all business meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present.

ARTICLE VI—SECTIONS

1. In the interest of more effective promotion of the objectives of the Association on a local level, the United States, Canada and their possessions shall be subdivided by the Board of Governors into non-overlapping geographical areas, and a Section of the Association shall be established in each of these areas. The subdivision into non-overlapping areas may be changed by the Board, upon recommendation by the Committee on Sections (see paragraph 7).

2. Each member of the Association residing in the United States, Canada or their possessions shall belong to one and only one Section. He will belong to the Section in whose geographic area he resides, except that a member who resides in one area and is employed in a different area may elect the Section to which he prefers to belong. Any member may petition the Committee on Sections for reassignment of his membership to another Section.

3. Each Section shall adopt a set of Bylaws which, along with any subsequent changes, must be approved by the Board. The geographic area covered by a Section shall be described in the Bylaws for the Section.

4. If there are members of the Association residing in a geographic area in which no Section has been organized, any ten or more members of this Association residing or employed in this area may petition the Board for authority to organize a Section covering that area.

5. A group of not less than twenty-five members of an existing Section may petition the Board to partition the area and the Section into two or more Sections. The Board shall have the authority to approve or deny this petition. The Board may specify conditions under which such action may be accomplished. It may conduct a poll of some or all members of the Association in the Section to determine the advisability of allowing the proposed partition of the Section. If separate Sections are approved then each new Section must prepare its own set of Bylaws to be approved by the Board.

6. A group of not less than twenty-five members residing or employed in that part of the area of an existing Section which they desire to become part of another existing Section may petition the Board to redefine the geographic boundaries of the Sections affected. The Board shall have the authority to approve or deny this petition. It may conduct a poll of some or all members of the Sections involved to determine the advisability of permitting such action.

7. There shall be a standing Committee on Sections through which the Board shall maintain general supervision over the activities of all Sections. This Committee, in particular, shall study all matters involving creation of Sections or modification of boundaries of Sections and make appropriate recommendations to the Board.

8. The Association shall not be obligated to pay from its treasury any of the expenses of a Section except as the Board may provide.

ARTICLE VII — PUBLICATIONS

1. The Association shall publish one or more journals.

2. The Board shall have full control of the publication and sale of each journal, and of all other publications.

3. There shall be appointed by the Board a body of Associate Editors for each journal.

4. The Board shall from time to time, as the need arises, make special provisions for the management of any other publications.

5. The Board shall fix the price of each journal and of any other publication of the Association.

ARTICLE VIII — DUES

1. The Board shall establish the annual dues and privileges of membership for individual and institutional members. The dues of individual members shall include a subscription to one or more of the Association's journals.

2. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, that member shall be dropped from the list after due notice.

3. New members entering the Association after April 1 of any year may have their dues prorated for the balance of the year, except when they desire to receive the full current volume of a journal.

4. Any individual member who because of age is no longer in active service, who is in good standing at the time of retirement, and who has been a member of the Association for twenty years, may upon notifying the central office of said retirement, be exempt from the payment of dues, while retaining all of the privileges of membership except receipt of journals. Such a member may elect to receive one or more journals at an annual cost of one half of the dues paid by a regular individual member receiving the same journals.

ARTICLE IX — AMENDMENTS TO THE ARTICLES OF ASSOCIATION AND BYLAWS

1. Changes in the Articles of Association or amendments to the Bylaws may be made at any annual business meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-third (2/3) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in each official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association or amendments to these Bylaws shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be recorded in the office of the Recorder of Deeds for Cook County, Illinois.

(see [5], p. 295, and [4], p. 222). Fig 4 points out that the Euclidean metric cannot be replaced by the taxicab metric d_T in this theorem. (The basic reason for this, of course, is that Taxicab geometry obeys all Euclidean axioms *except* that of congruence; see [3] and [6], pp. 116–118.) In fact, the theorem above even fails for certain infinite sets of points. For example, if **a** and **b**, **a**′ and **b**′ are points on lines having different positive slopes and $d_T(\mathbf{a}, \mathbf{b}) = d_T(\mathbf{a}', \mathbf{b}')$, then it is possible to choose infinite sequences of points $\{\mathbf{x}_n\}, \{\mathbf{x}_n'\}$ from the "midpoint" sets $M(\mathbf{a}, \mathbf{b}), M(\mathbf{a}', \mathbf{b}')$, respectively, such that $d_T(\mathbf{x}_i, \mathbf{x}_j) = d_T(\mathbf{x}_i', \mathbf{x}_j')$ for all i, j (Fig. 6). Then, under the correspondence $\mathbf{a} \to \mathbf{a}'$, $\mathbf{b} \to \mathbf{b}'$, $\mathbf{x}_i \to \mathbf{x}_i'$, all taxicab distances are preserved, but there is no taxicab isometry which extends this correspondence. What (if any) is the taxicab metric analogue of the theorem above for Euclidean isometries?

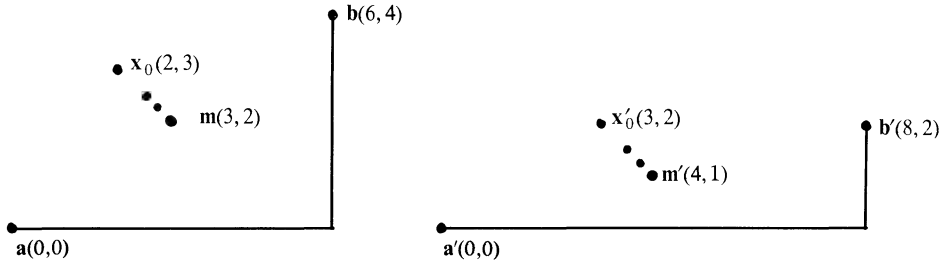


FIG. 6. The points $\mathbf{x}_i$, $\mathbf{x}_i'$ are chosen on the line segments joining $\mathbf{x}_0$ to $\mathbf{m}$ and $\mathbf{x}_0'$ to $\mathbf{m}'$, respectively, so that $d_T(\mathbf{x}_i, \mathbf{x}_{i+1}) = d_T(\mathbf{x}_i', \mathbf{x}_{i+1}') = 1/2^i$.

The ability to pose interesting mathematical problems deserves high recognition. I am grateful to Joe Malkevitch for asking me the question whose answer is given by the theorem.

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EULER'S CONSTANT AND *e*

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Euler's constant γ may be defined by

$$\gamma = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \ln(n + 1).$$

Glaisher [5] gave an account of the early history of this limit beginning with the connection, first established by Euler, between the harmonic series and the natural logarithm.

An inspection of most tables of definite integrals as well as past problems in this MONTHLY

shows that Euler's constant appears in many analytic formulas. Some of these problems (4045, 4046, 4353, E819) were included among The Four Hundred "Best" Problems (1918–1950) in The Otto Dunkel Memorial Problem Book [**1**].

The proof of the existence of the limit in Euler's constant is given in many of the standard calculus texts. Bers [**3**], pp. 512–513, uses a proof based on properties of bounded monotone sequences; a proof given by Apostol [**2**], pp. 405–406, is based on alternating series.

There is current interest in Euler's constant. Recently Johnsonbaugh [**6**] gave a new simple proof of its existence. In the present note we will prove a familiar fact about e to set the stage for another new proof of the existence of Euler's constant.

We follow the customary approach in elementary calculus courses by using the definition $\ln(x) = \int_1^x t^{-1}\,dt$. The base e is then defined to be the number for which

$$\ln(e) = \int_1^e t^{-1}\,dt = 1.$$

When this definition is used, then it remains to prove that $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e$.

We will show how one comes in a natural way to consider $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$, a fact not immediately evident in much of the textbook literature. Edwards [**4**] presented an account of the original work of Euler relative to $\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$.

We begin with an integration by parts to show that

$$\int_{1/(n+1)}^{1/n} \ln(x)\,dx = \frac{1}{n(n+1)}\ln\left[\frac{(n+1)^n}{n^{n+1}}\right] - \frac{1}{n(n+1)}.$$

By the mean value theorem for integrals,

$$\int_{1/(n+1)}^{1/n} \ln(x)\,dx = \left(\frac{1}{n} - \frac{1}{n+1}\right)\ln(c_n),$$

where $1/(n+1) < c_n < 1/n$. The last two equations give

$$\ln\left[\frac{(n+1)^n}{n^{n+1}}\right] - 1 = \ln(c_n),$$

or

$$\ln\left[\frac{(n+1)^n/n^{n+1}}{c_n}\right] = 1.$$

Thus

$$e = \frac{(n+1)^n/n^{n+1}}{c_n},$$

or

$$e = \left(1 + \frac{1}{n}\right)^n\left(\frac{1}{nc_n}\right).$$

By the conditions on c_n we have

$$1 < \frac{1}{nc_n} < 1 + \frac{1}{n}.$$

Let $a_n = 1/nc_n$. Then $1 < a_n < 1 + \frac{1}{n}$ and $e = a_n\left(1 + \frac{1}{n}\right)^n$ for every positive integer n. Now

apply the standard limit theorems from calculus. By the theorem often called the sandwich theorem, $\lim_{n\to\infty} a_n = 1$. Hence $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ exists and equals e. From $e = a_n\left(1+\frac{1}{n}\right)^n$ we have, taking logarithms,

$$1 = \ln(a_n) + n\{\ln(n+1) - \ln(n)\}.$$

Substituting values $1, 2, 3, \ldots$ we have the equations

$$\begin{aligned}
1 &= \ln(a_1) + \ln(2) - \ln(1),\\
\frac{1}{2} &= \ln\left(a_2^{1/2}\right) + \ln(3) - \ln(2),\\
\frac{1}{3} &= \ln\left(a_3^{1/3}\right) + \ln(4) - \ln(3),\\
&\cdots\\
\frac{1}{n} &= \ln\left(a_n^{1/n}\right) + \ln(n+1) - \ln(n).
\end{aligned}$$

Summing, we obtain

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln(n+1) = \ln\left(a_1 a_2^{1/2} a_3^{1/3} \cdots a_n^{1/n}\right).$$

The sequence $\{\ln(a_1 a_2^{1/2} \cdots a_n^{1/n})\}_{n=1}^{\infty}$ is increasing. Moreover we have

$$\begin{aligned}
\ln\left(a_1 a_2^{1/2} \cdots a_n^{1/n}\right) &= \ln(a_1) + \frac{1}{2}\ln(a_2) + \cdots + \frac{1}{n}\ln(a_n)\\
&< \ln\left(1+\frac{1}{1}\right) + \frac{1}{2}\ln\left(1+\frac{1}{2}\right) + \cdots + \frac{1}{n}\ln\left(1+\frac{1}{n}\right).
\end{aligned}$$

Now we use the fact that $1 < \left(1+\frac{1}{n}\right)^n < e$ for all n, since $e = \left(1+\frac{1}{n}\right)^n 1/nc_n$ where $1/nc_n > 1$. Since the function $F(x) = \ln(x)$ is increasing, it now follows that

$$\ln(1) < \ln\left(1+\frac{1}{n}\right)^n < \ln(e), \quad \text{or} \quad 0 < \ln\left(1+\frac{1}{n}\right) < \frac{1}{n}.$$

Using this in our last estimate, we have

$$\ln\left(a_1 a_2^{1/2} \cdots a_n^{1/n}\right) < 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2},$$

so that the sequence $\{\ln(a_1 a_2^{1/2} \cdots a_n^{1/n})\}_{n=1}^{\infty}$ is bounded above $\left(\text{by } \frac{\pi^2}{6}\right)$.

It now follows that

$$\gamma = \lim_{n\to\infty}\left\{\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \ln(n+1)\right\}$$

exists.

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THE TEACHING OF MATHEMATICS

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ARCWISE CONNECTEDNESS AND THE PERSISTENCE OF ERRORS

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It is very important that mathematics students, from the junior-senior level on, develop the habit of critically examining all details, stated and unstated, of any argument they are considering. Many working mathematicians (myself included) are somewhat lax in this, at least with respect to published arguments. After all, if the author states that such-and-such is the case, and the assertion seems plausible, it is probably correct, so why bother with the grubby details? While I can more or less get by with such carelessness, *students cannot*. For example, the most common textbook arguments [**1**], [**2**], [**4**], [**5**], [**7**], [**12**] for the arcwise connectedness theorem of elementary point set topology all contain the same glaring error. Any student reading one of these "proofs" should notice that almost the simplest possible examples of the construction used (for example, the illustrations given in [**4**]) serve to refute the argument. Of course, this *was* overlooked by the authors of at least six textbooks, but they are all working mathematicians and don't have to be as careful as students.

The error, though not subtle, seems to have considerable appeal and it might be well to put it to rest now, lest it becomes established as fact by induction, or through the Method of Infinite Repetition. I propose, therefore, to describe the argument, point out the flaw, and suggest ways of correcting it. The theorem and its proof are simple enough to be understood by anyone having even a rudimentary knowledge of topology.

A compact, connected metric space is called a continuum, and a continuum in which each point has arbitrarily small connected open neighborhoods is said to be *locally connected*. An *arc* is a metric space homeomorphic to the real number interval [0, 1] or, equivalently, a continuum having precisely two non-separating points. The arcwise connectedness theorem asserts that if M is a locally connected continuum (more generally, a connected open subset of such a continuum), then every two points of M can be joined by an arc lying entirely in M. With proper care, the desired arc can be obtained as the intersection of a sequence $K_1 \supset K_2 \supset K_3 \supset \cdots$ of "thinner and thinner" subcontinua of M. Of course, the notion of thinness must be appropriately defined, and this is usually done by means of chains of (small) sets.

A *simple chain* is a finite sequence $\mathscr{C} = (C_1, C_2, \ldots, C_n)$ of nonempty sets such that every two adjacent terms of the sequence intersect but no two non-adjacent terms do so. The sets $C_1, C_2, \ldots, C_n$ are called *links* of $\mathscr{C}$, and if $a \in C_1$ and $b \in C_n$, then $\mathscr{C}$ is said to be a chain *from a to b*. If $\mathscr{D}_1$ and $\mathscr{D}_2$ are simple chains such that the concatenation of the sequence $\mathscr{D}_1$ followed by the sequence $\mathscr{D}_2$ is a simple chain $\mathscr{D}$, we will write $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2$. Note that if $\mathscr{D} = \mathscr{D}_1 + \mathscr{D}_2 + \cdots + \mathscr{D}_k$, then for $i < j$ each link of $\mathscr{D}_i$ precedes each link of $\mathscr{D}_j$ in $\mathscr{D}$.

Now suppose M is a locally connected continuum (or a connected open subset of a locally connected continuum) and $a, b \in M$, $a \neq b$. It is easy to obtain a sequence $\mathscr{C}^1, \mathscr{C}^2, \mathscr{C}^3, \ldots$ of simple chains from a to b, $\mathscr{C}^n = (C_1^n, C_2^n, \ldots, C_{k_n}^n)$, such that for each n,

(1) every link of $\mathscr{C}^n$ is a connected open subset of M of diameter less than $1/n$,
(2) the closure of each link of $\mathscr{C}^{n+1}$ is a subset of some link of $\mathscr{C}^n$,
(3) there exist chains $\mathscr{D}_1^n, \mathscr{D}_2^n, \ldots, \mathscr{D}_{k_n}^n$ such that $\mathscr{C}^{n+1} = \mathscr{D}_1^n + \mathscr{D}_2^n + \cdots + \mathscr{D}_{k_n}^n$ and such that each link of $\mathscr{D}_i^n$ is a subset of C_i^n, $i = 1, \ldots, k_n$.

Condition (3), or something like it, is needed to assure that there is not too much "doubling back" of $\mathscr{C}^{n+1}$ inside $\mathscr{C}^n$. As given here (3) is a formalization of the construction described in [1], [4], [7], [12], but is not explicitly stated in this form in any of the cited arguments. The actual conditions used are:

(3a) If both C_i^{n+1} and C_k^{n+1} are contained in C_r^n, then $C_j^{n+1} \subset C_r^n$ for every j between i and k [4], [7].

(3b) If $C_i^{n+1} \subset C_{r+2}^n$, then $C_j^{n+1} \cap C_r^n = \varnothing$ for all $j \geqslant i$ [5].

(3c) If $C_i^{n+1} \subset C_r^n$ and $C_i^{n+1} \not\subset C_{r-1}^n$, then for each $j \geqslant i$ there is an $s \geqslant r$ such that $C_j^{n+1} \subset C_s^n$ [2].

No explicit condition is stated in [1], [12], but the construction used is the same as in [4], [7]. (Condition (3c) follows easily from (3). Despite assertions to the contrary in [4] and [7], however, condition (3a) does not follow from the construction given, although (3) does. The construction of [5] does not give (3b), as claimed; it does not even give (3), or any other condition adequate for a proof of the theorem).

After stating conditions (1), (2) and whatever is used in place of (3), the six arguments in question differ mainly in notation and are equivalent to the following.

For each n, let K_n denote the union of the closures of the links of $\mathscr{C}^n$ and let $K = \cap_{n=1}^{\infty} K_n$. It is clear that K is a continuum lying in M and containing a and b; to prove that K is an arc it is sufficient to show that each point of $K - \{a, b\}$ is a separating point of K. To this end, suppose $p \in K - \{a, b\}$ and for each n, let A_n (respectively, B_n) denote the union of all links of $\mathscr{C}^n$ which precede (respectively, follow) every link of $\mathscr{C}^n$ containing p, and let $A = K \cap (\cup_{n=1}^{\infty} A_n)$, $B = K \cap (\cup_{n=1}^{\infty} B_n)$. In each argument, it is observed that $a \in A$, $b \in B$, A and B are open in K, and $K - \{p\} = A \cup B$, and then it is asserted that (A, B) is a separation of $K - \{p\}$. As shown in Fig. 1, however, A and B need not be disjoint, so the arguments all fail for the same reason. (Only in [2] is any attempt made to show that $A \cap B = \varnothing$; the argument fails by claiming a contradiction where none exists.)

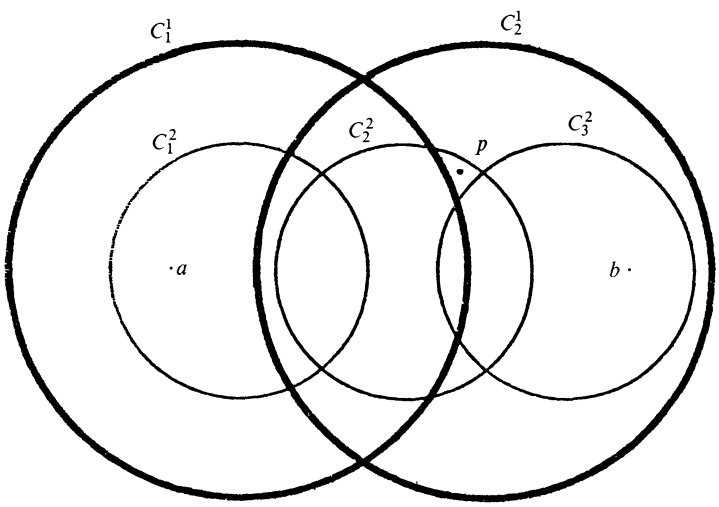


FIG. 1

Although the preceding *argument* is invalid, it is nevertheless true that $K - \{p\}$ is not connected. To show this, it is obviously necessary to redefine the sets A and B in some way.

One way to do this is to let A (B) denote the set of all points of K which belong to A_n (B_n) for

all but a finite number of values of n. Another way (my personal favorite) is to let $A = \cup_{n=1}^{\infty} S_n$, $B = \cup_{n=1}^{\infty} T_n$ where S_n (T_n) is the set of all $x \in K$ such that *every* link of $\mathscr{C}^n$ containing x precedes (follows) every link of $\mathscr{C}^n$ containing p. It is also possible to redefine the sets A_n, B_n in any of several ways, keeping the formal definitions of A and B as initially given.

Each of these modifications requires considerable additional argument and uses properties of simple chains which are nowhere suggested in any of the cited arguments. Details will not be given here since the purpose of this article is not to prove the theorem (correct proofs may be found in [3], [6], [8], [9], [10], [11]), but only to point out that an argument which is invalid in a nontrivial way seems to have gained rather widespread acceptance during the last twenty years or so.

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TANGENT PLANES WITH INFINITESIMALS

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Both pedagogically and aesthetically, the greatest blemish in a vector calculus course is the definition of differentiability and the tangent plane. Using infinitesimals, a very simple and natural definition is possible, and one that is a perfect generalization of the two-dimensional case.

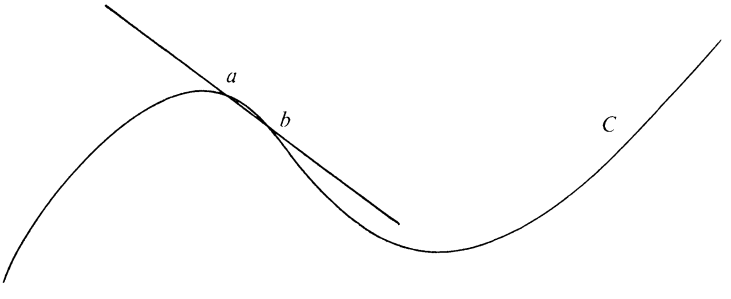


FIG. 1

We recall the nonstandard definition of tangent *line*. A line l passing through point a on curve C is tangent to C if and only if for all points b on C, $a \approx b$ (a infinitely close to b), $a \neq b$, the line determined by a and b is infinitely close to l (Fig. 1). Our measure of infinitely close for lines is the difference in slope.

Algebraically, the derivative of a function f is calculated as the unique real number infinitely close to $(f(a + \iota) - f(a))/\iota$, where ι is any infinitesimal.

In R^3 we replace the line determined by two points with the plane determined by three. We say a plane P passing through a point $\vec{u}$ on a surface S is tangent to S if and only if for all pairs of points $\vec{v}, \vec{w}$ on $S, \vec{v}, \vec{w} \approx \vec{u}$, not all collinear, the plane determined by $\vec{v}, \vec{w}, \vec{u}$ is infinitely close to P (Fig. 2). Our measure of infinitely close for planes is the difference in unit normal vectors.

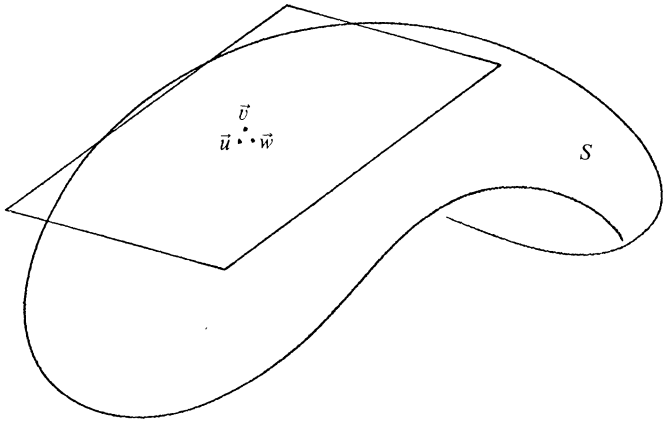


FIG. 2

f: $\mathbb{R}^2 \to \mathbb{R}$ is differentiable at (a, b) if and only if the graph of f has a tangent plane at $\vec{u} = (a, b, f(a, b))$. The equivalence of this to various standard definitions is a routine application of nonstandard techniques.

We can obtain the usual expression for the tangent plane using points

$$\vec{v} = (a + \iota, b, f(a + \iota, b)) \quad \text{and} \quad \vec{w} = (a, b + \iota, f(a, b + \iota))$$

where ι is infinitesimal. The expression $((\vec{v} - \vec{u}) \times (\vec{u} - \vec{w}))/\iota$, normal to the plane of $\vec{u}, \vec{v}, \vec{w}$, simplifies to

$$\left(\frac{f(a + \iota, b) - f(a, b)}{\iota}, \frac{f(a, b + \iota) - f(a, b)}{\iota}, -1 \right)$$

which is infinitely close to $(f_x, f_y, -1)$.

The failure of differentiability in a function can be witnessed easily by producing appropriate points. For example

$$f(x, y) = \begin{cases} \dfrac{xy\sqrt{2}}{\sqrt{x^2 + y^2}} & \text{if } x^2 + y^2 \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

is not differentiable at $(0, 0, 0)$, since the planes through $(0, 0, 0)$, $(0, \iota, 0)$, $(\iota, 0, 0)$ and through $(0, 0, 0)$, $(0, \iota, 0)$, (ι, ι, ι) have unit normals respectively, $(0, 0, 0)$ and $(1/\sqrt{2}, 0, -1/\sqrt{2})$, which are not infinitely close.

Beyond the clear geometric picture of the tangent plane this presents, the definition allows an improved version and shorter proof of the usual theorem in which we require the continuity of only one partial derivative.

THEOREM. *If f: $\mathbb{R}^2 \to \mathbb{R}$ has the property that f_x and f_y exist and at least one is continuous, then f is differentiable.*

Proof. Given $\vec{u} = (a, b, f(a, b))$, it is sufficient to show for any $\vec{v} \approx \vec{u}$, $\vec{v} \neq \vec{u}$ on the graph of f that the angle θ between $\vec{v} - \vec{u}$ and $\vec{n} = (f_x(a, b), f_y(a, b), -1)$ is infinitely close to $\pi/2$ (Fig. 3).

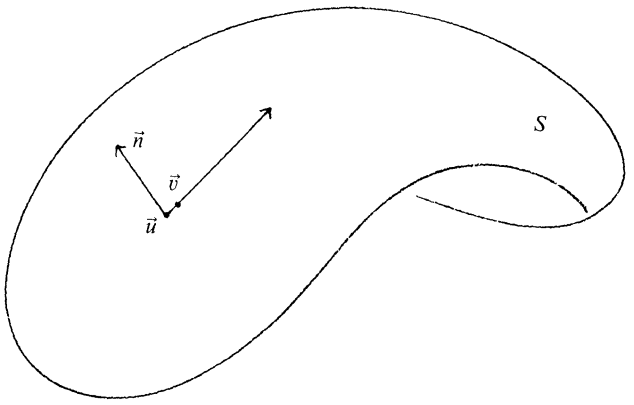


FIG. 3

Suppose f_y is continuous. Let $\vec{v} = (a + \iota_1, b + \iota_2, f(a + \iota_1, b + \iota_2))$, where $\iota_1, \iota_2 \approx 0$ (but are not both equal to 0). Then

$$\cos(\theta) = \frac{\vec{n} \cdot (\vec{v} - \vec{u})}{\|\vec{n}\| \|\vec{v} \cdot \vec{u}\|} = \frac{1}{\|\vec{n}\| \|\vec{v} - \vec{u}\|} \big(f_x(a,b)\iota_1 + f_y(a,b)\iota_2 - f(a + \iota_1, b + \iota_2) = f(a,b)\big).$$

Since $\frac{1}{\|\vec{n}\|}$ is real and $\frac{\iota_1}{\|\vec{v} - \vec{u}\|}, \frac{\iota_2}{\|\vec{v} - \vec{u}\|}$ are finite, the following substitutions cause only an infinitesimal change:

$$f_x(a,b) \approx \frac{f(a + \iota_1, b) - f(a,b)}{\iota_1}$$

and $f_y(a,b) \approx f_y(a + \iota_1, b) \approx \frac{f(a + \iota_1, b + \iota_2) - f(a + \iota_1, b)}{\iota_2}$. [This last follows from the continuity of f_y: by the Mean Value Theorem,

$$\frac{f(a + \iota_1, b + \iota_2) - f(a + \iota_1, b)}{\iota_2} = f_y(a + \iota_1, b')$$

for some b' between b and $b + \iota_2$. This in turn is infinitely close to $f_y(a + \iota_1, b)$.]

These yield:

$$\cos(\theta) \approx \frac{1}{\|\vec{n}\|} \frac{1}{\|\vec{v} - \vec{u}\|}[0] = 0,$$

proving the theorem.

The definitions can be generalized easily to higher dimensions.

132. MISCELLANEA

The advance of our knowledge in this branch of electrical development has been very much retarded by the phantasies of visionary mathematicians who monopolize the columns of our technical literature and fill the mind of the student with false conclusions. I have no sympathy with the pure mathematician who scorns the practical man, scoffs at his experience, directs the universe from his couch, and invents laws to suit his fads.

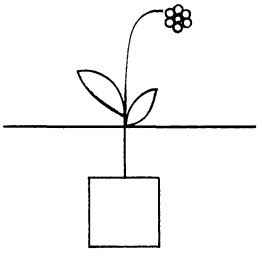
—Sir William Henry Preece, quoted in the article titled "Oliver Heaviside: genius and curmudgeon," IEEE Spectrum 20 (1983) 63–69, by Paul J. Nahin.

REBUSES

According to the American Heritage Dictionary, a *rebus* is a riddle composed of words or syllables depicted by symbols or pictures that suggest the words or syllables they represent. A recent flurry of rebus activity was started by Peter G. Casazza (first at the University of Alabama and later at the University of Missouri-Columbia) who is, in fact, the inventor of the samples not otherwise ascribed below. Can you decipher them? If not, see p. 456.

A few additional mathematical rebuses will appear in this department from time to time. If you can think of others, please send them to the editor. They will be refereed and, if accepted, they will be published, so that they may become parts of your curriculum vitae.

1.



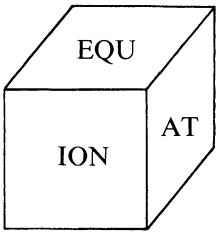
—Brian Sells
D. Ross

2. derivat

3. CA – – OR —Folklore

4. DOD —Jim Griffith

5.



—John Hayward

6. – •−− −−− > −−− −• • —Steve Markvonen

7. $x^{\infty} + y^{\infty} = z^{\infty}$

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS), H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.*

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E 3048. *Proposed by J. L. Brenner, Palo Alto, California; and D. M. Silberger, SUNY at New Paltz, New York, and Universidade Federal de Santa Catarina, Brazil.*

Prove that every element a in the alternating group A_n $(n > 4)$ can be written in the form $a = xxyyxy$ $(x, y \in A_n)$.

E 3049. *Proposed by Jordi Dou, Barcelona, Spain.*

Determine a planar region of area 4 which can be partitioned into four subregions of unit area in such a way that the total length of all bounding arcs is a minimum. For example, a square of area 4 can be partitioned into four unit squares so that the bounding arcs have total length 12, while a circle of area 4 can be partitioned into four sectors of unit area so that the total length of the bounding arcs is $(4\pi + 8)/\sqrt{\pi} \approx 11.6$.

E 3050. *Proposed by Jordi Dou, Barcelona, Spain.*

An optimal tripartition of a plane region is a partition into three subregions of equal area such that the total length of the separating arcs is a minimum. Determine those isosceles triangles

Editor:

Robert C. Thompson's article "Author vs. Referee: A Case History for Middle Level Mathematicians" (this MONTHLY, December 1983, pp. 661–668) gives me an excuse to raise an issue that has long troubled me.

We all know that the referee's name is not usually submitted to the author of a mathematical paper. But Professor Thompson's experience, as well as my own, indicates that authors' names *are* sent to referees (cf. p. 664, item 4, where Thompson's referee writes, "I suggest not publishing the enclosed paper of R. C. Thompson..."). Of course, this practice is not restricted to the MONTHLY but permeates most journals, apparently; Thompson's paper was sent to the *Rocky Mountain Journal of Mathematics*.

Why the lack of symmetry (referee knows author but not vice versa)? More seriously, is not the objectivity of refereeing impaired if the referee knows who the author is? Does this practice not give the paper written by a known author from a large university a different chance of being accepted than the same paper written by a newcomer from a small college? At the very least, should not anonymity be a *right* of the prospective author if he wishes?

Clearly the *editor* must know the names of both referee and author, but shouldn't it stop there? This discipline of ours, which prides itself on objectivity, has run aground on an unnecessary reef. Let the prospective paper stand on its own merits *without* the buttress or hindrance of the author's name and institution! To the retort, "Other disciplines are guilty, too!" I answer simply (1) two wrongs do not one right make in any modular arithmetic, and (2) all other journals do NOT do this (the *Journal of Finance*—official publication of the American Finance Association—is one counterexample).

Has my thinking gone astray? Am I missing something?

Donald P. Minassian
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Butler University
Indianapolis, IN 46208

P.S. Would Thompson's paper have been accepted if his name had been lacking? We'll never know.

ANSWERS TO REBUSES ON PAGES 436–437

1. Square root

2. Partial derivative

3. Cantor with the middle third removed

4. Odd permutation

5. Cubic equation

6. Morse inequality

7. This is definitely Fermat's *last* theorem.

6. – •−− −−− > −−− −• • —Steve Markvonen

7. $x^{\infty} + y^{\infty} = z^{\infty}$

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An optimal tripartition of a plane region is a partition into three subregions of equal area such that the total length of the separating arcs is a minimum. Determine those isosceles triangles

which admit (a) two optimal tripartitions, (b) two optimal tripartitions among those partitions using only straight line segments.

E 3051. *Proposed by P. O'Hara and H. Sherwood, University of Central Florida.*

It is observed in plane analytic geometry that any set S that is symmetric with respect to both the x and y axes is also symmetric with respect to the origin. Does the statement remain valid if the y-axis is replaced by a line through the origin with inclination angle α?

E 3052. *Proposed by Peter Ungar, New York University.*

Show that

$$\sum_{k=1}^{n-1} \frac{(-1)^{k-1}}{k}\binom{n}{k}(n-k)^n = n^n\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right).$$

E 3053. *Proposed by Irl C. Bivens, Davidson College.*

Let m denote a fixed nonzero real number. Find all continuous functions $f\colon R \to R$ such that

$$f\left(2x - \frac{f(x)}{m}\right) = mx.$$

(Compare with E 2893 [1981, 444; 1982, 702].)

SOLUTIONS OF ELEMENTARY PROBLEMS

$$\sum_{m=0}^{\infty} \frac{1}{F(2^m)} = \frac{7-\sqrt{5}}{2}$$

E 2922 [1982, 63]. *Proposed by Roger Cuculière, Paris, France.*

Let $\{F(n)\}$ be the Fibonacci sequence: $F(0) = 0$, $F(1) = 1$, $F(n) = F(n-1) + F(n-2)$. Find

$$\sum_{m=0}^{\infty} \frac{1}{F(2^m)}.$$

Composite Solution. Since

$$F(n) = (a^n - b^n)/\sqrt{5},$$

where $a = (1+\sqrt{5})/2$, $ab = -1$, one obtains

$$\sum_{m=0}^{\infty} \frac{1}{F(2^m)} = 1 + \sqrt{5}\sum_{m=1}^{\infty}\left[\frac{1}{a^{2^m}-1} - \frac{1}{a^{2^{m+1}}-1}\right],$$

which is a telescopic series yielding the sum

$$1 + \frac{\sqrt{5}}{a^2-1} = \frac{7-\sqrt{5}}{2}$$

Solved by 56 readers, including the proposer. As pointed out by several of the solvers, this problem has already appeared in print on numerous occasions: I. J. Good presented a solution in 1974, V. E. Hoggatt and M. Bicknell gave nine additional solutions in 1976 and considered generalizations. More general results were also found by P. S. Bruckman and I. J. Good. For references, see Fibonacci Quart., 12 (1974), p. 346; 14 (1976), pp. 272–76; 14 (1976), pp. 453–55; 14 (1976), pp. 193–96.

$$\lim \sum_{1}^{\infty} \left(\frac{x}{n} \right)^{n}$$

E 2928 [1982, 130]. *Proposed by Doug Wiedemann, Institute for Defense Analysis.*

Find $\lim(\Sigma_1^\infty x^n/n^n)$, $x \to -\infty$.

Solution by Mark Pinsky, Northwestern University, Evanston, IL. For any x,

$$\sum_{n=1}^{\infty} x^n/n^n = x\int_0^1 e^{xu\ln(1/u)}\,du.$$

(This can be seen by expanding the exponential in a power series, uniformly convergent for $0 \leqslant u \leqslant 1$, and integrating term by term.)

If $x < 0$, the integrand is less than 1, except at $u = 0$ and $u = 1$. For any $\delta > 0$, when $x \to -\infty$ integrating by parts yields

$$\int_0^{\delta} e^{xu\ln(1/u)}\,du = o(1/x)$$

$$\int_{\delta}^{1-\delta} e^{xu\ln(1/u)}\,du = O(1/x^2),$$

and

$$\int_{1-\delta}^{1} e^{xu\ln(1/u)}\,du = -1/x + O(1/x^2).$$

Therefore $\Sigma_{n=1}^\infty x^n/n^n \to -1$ when $x \to -\infty$.

Also solved by P. F. Byrd, M. L. Glasser, J. E. Jamison and C. C. Rousseau, O. P. Lossers (The Netherlands), L. E. Mattics, W. A. Newcomb, O. G. Ruehr, M. El-Tohami, and the proposer.

Equations Forcing Commutativity

E 2931 [1982, 131]. *Proposed by Chen-Te Yen, Chung Yuan Christian University, Chung-Li, Taiwan.*

Let n, m be fixed positive integers. Suppose G is a group such that for all x, y in G, $(xy)^n = x^n y^n$, $(xy)^{n+m} = x^{n+m}y^{n+m}$, and $(xy)^{n+2m} = x^{n+2m}y^{n+2m}$. Show that G is abelian if $m = 1$ or 2. What about $m \geqslant 3$?

Solution by I. N. Herstein, University of Chicago, and Rony Teitler, Wolfson College, Oxford, England. The solvers call attention to F. W. Levi, *Notes on group theory*, J. Indian Math. Soc. 8, (1944) 1–9.

THEOREM (Levi). *If $(xy)^n = x^n y^n$ for all x, y in a group G and all n in a set S of integers, then G is forced to be commutative if and only if* $\gcd\{n^2 - n,\ n \in S\}$ *is* 2. (See also this MONTHLY, E2411; 1974, 410.)

Teitler referred also to Ronse, Teitler, *Groupes dans lesquels l'élévation à une puissance entière est un endomorphisme*, Bull. Acad. Roy. Belg. 5e sér., 62 (1976) 539–564.

Teitler proved also the following results, which he believes to be new.

1. If G is a group in which the maps $x \to \phi(x)$, $x \to x\phi(x)$, $x \to x^2\phi(x)$ are endomorphisms of G, then G is abelian.

2. If $x \to \phi(x)$, $x \to x^2\phi(x)$, $x \to x^4\phi(x)$ are endomorphisms, then G is abelian.

Proof of 1. Set $B(x) = x\phi(x)$. From $B(xy) = B(x)B(y)$ it follows that $xy\phi(x)\phi(y) =$

$x\phi(x)y\phi(y)$, so that $y\phi(x) = \phi(x)y$, and $\phi(x)$ must always lie in the center of G.

Define $\Gamma(x) = x^2\phi(x) = xB(x)$. Then $x\phi(x)$ is in the center. Thus x is in the center.

Proof of 2. Set $B(x) = x^2\phi(x)$. Then $B(xy) = B(x)B(y)$ gives $(xy)^2\phi(x)\phi(y) = x^2\phi(x)y^2\phi(y)$, so that

$$(*) \qquad yxy\phi(x) = x\phi(x)y^2.$$

Set $\Gamma(x) = x^4\phi(x) = x^2B(x)$. Then $yxyB(x) = xB(x)y^2$, so that

$$(**) \qquad yxyx^2\phi(x) = xx^2\phi(x)y^2.$$

From (*) (with $y = x$), $x^2\phi(x) = \phi(x)x^2$ for all x in G.

Next invert the left-hand side, right-hand side, of (*) and multiply by (**).

$$\phi(x)^{-1}(yxy)^{-1}(yxy)x^2\phi(x) = y^{-2}\phi(x)^{-1}x^{-1}xx^2\phi(x)y^2.$$

But $\phi(x)$ commutes with x^2, so that

$$x^2 = y^{-2}x^2y^2.$$

Thus squares commute. Now set $x = t^2$ in (*): $yt^2y\phi(t)^2 = t^2\phi(t)^2y^2$, so that

$$yt^2y = t^2y^2, \qquad yt^2 = t^2y.$$

Thus the square t^2 of every element t is in the center of G.(*) can now be simplified to $xy = yx$, so that G is abelian.

F. W. Barnes rediscovered Levi's Theorem. G. P. Wene mentioned H. E. Bell, *The identity* $(xy)^n = x^ny^n$*: does it buy commutativity?*, Math. Mag., 55 (1982) 165–169.

Also solved by A. Bager (Denmark), F. W. Barnes (Kenya), M. Bencze (Romania), S. D. Bronn, P. L. Chabot, C.-N. Lee (student), L. Kuipers (Switzerland), J. J. Martinez, D. McCevitt (student), E. T. Ordman, G. P. Wene, D. Wiedemann, and E. T. Wong.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by January 31, 1985. The solver's full post-office address should be on each sheet.

6463. *Proposed by José M. Bayod, Santander, Spain.*

Let $S \subset R^n$ be a Lebesgue-measurable set with finite measure $\mu(S)$. Assume $f\colon S \to R$ is a real function that can be decomposed in the following way: $f = h \cdot g$, with g real, measurable and bounded (call $a = \inf g(S)$ and $b = \sup g(S)$), and h absolutely continuous on $[a, b]$. Then prove that f is integrable over S, and

$$\int_S f = \mu(S)h(b) - \int_a^b h'(t)\mu\left(g^{-1}[a,t]\right)dt.$$

6464. *Proposed by Roger Cooke, University of Vermont.*

What is the least upper bound of real numbers b such that there exists a continuous real-valued function $f(x)$ satisfying

$$xf(x) - \int_1^x \left(f(t) + t(f(t))^2\right)dt - b(\ln(x)) \to \infty \quad \text{as } x \to \infty?$$

6465. *Proposed by F. S. Cater, Portland State University.*

Prove that for each integer $n \geq 3$, there is a largest number h_n such that there exists a polynomial p of degree at most n which increases on the interval $(-h_n, h_n)$ and satisfies

$p(1) \leqslant p(-1)$. Prove that $0 < h_{2k-1} = h_{2k} < 1$ for $k \geqslant 2$, and that $\lim_{n\to\infty} h_n = 1$. (Compare with Problem 6449.)

6466. *Proposed by I. J. Schoenberg, University of Wisconsin-Madison, from a problem seminar of A. Ostrowski* (1924).

Let F be a compact plane set, not contained in a line. Let

$I = \sup\{\text{area of triangle } ABC\colon \{A, B, C\} \subset F\}$,

$J = \inf\{\text{area of triangle } ABC\colon F \subset \text{closed triangle } ABC\}$.

Show that $4I \geqslant J$ and that equality can occur.

6467. *Proposed by Bu Qi-yue (student), Shanghai Jiao Tong University, People's Republic of China.*

Suppose that $g\colon D \subset R^n \to R^1$ has a second Gateaux derivative at $X_0 \in \operatorname{int} D$. Suppose also that X_0 is a critical point of g and $g''(X_0) > 0$. Must X_0 be a proper local minimizer of g? Answer the same question with "Fréchet" instead of "Gateaux."

6468. *Proposed by Calvin T. Long, Washington State University.*

If a and n are integers with $n > 1$, prove that $\Sigma_{d|n}\phi(d)a^{n/d} \equiv 0 \pmod{n}$. Note that if p is a prime this reduces to Fermat's Little Theorem, $a^p \equiv a \pmod{p}$.

SOLUTIONS OF ADVANCED PROBLEMS

Latin Rectangles

6400 [1982, 602]. *Proposed by Fred Galvin, University of Kansas, and Karel Prikry and J. Ian Richards, University of Minnesota.*

Let $2 \leqslant k \leqslant n$. A $k \times n$ Latin rectangle is a $k \times n$ matrix with entries $1, 2, \ldots, n$, such that no row or column contains a repetition. Let $R(k, n) = R^+(k, n) - R^-(k, n)$ where $R^+(k, n)[R^-(k, n)]$ is the number of $k \times n$ Latin rectangles whose first two rows are of the form

$$\begin{array}{cccc} 1 & 2 & \ldots & n \\ \pi(1) & \pi(2) & \ldots & \pi(n) \end{array}$$

where π is an even [odd] permutation of $\{1, 2, \ldots, n\}$.

(i) Show that $R(2, n) = (-1)^{n-1}(n - 1)$.

(ii) Show that $(-1)^{n-1}R(3, n) \geqslant 0$; the inequality is strict for $n > 4$.

(iii)* Is $(-1)^{n-1}R(k, n) \geqslant 0$ for $k \geqslant 4$? In particular, is $(-1)^{n-1}R(n, n) \geqslant 0$ for all n?

Partial solution by the proposers. Both (i) and (ii) will follow from the theorem proved below. In fact, (i) can be proved directly by a straightforward application of the principle of inclusion and exclusion. We do not know the answer to (iii)*.

Let S_n, A_n, and D_n be the set of all permutations, all even permutations, and all derangements (fixed-point-free permutations) of $\{1, 2, \ldots, n\}$. For $P \subseteq S_n$ let $\#P = |P \cap A_n| - |P \setminus A_n|$, i.e., $\#P$ is the *net* number of permutations in P, counting even permutations positively and odd permutations negatively. For any mapping $\sigma\colon \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, we define

$$d(\sigma) = \#\{\pi \in S_n\colon x \neq \pi(x) \neq \sigma(x) \text{ for all } x\}.$$

By an even (odd) cycle we mean a cycle of even (odd) *length*, i.e., an even cycle is an odd permutation.

THEOREM. *Let n be a positive integer, and consider any mapping $\sigma: \{1,2,\dots,n\} \to \{1,2,\dots,n\}$. If σ contains an even cycle, then $d(\sigma) = 0$; otherwise $d(\sigma) = (-1)^{n-1} \cdot 2^{k-1} \cdot (p + 2q - 2)$ where:*

k is the number of cycles in σ of length > 1;
$p = |\{x: \sigma^r(x) \neq \sigma^{r+1}(x)$ for all $r\}|$;
$q = |\{x: \sigma^r(x) = \sigma^{r+1}(x)$ for some $r \geqslant 0$, and the least such r is even$\}|$.

Before proving the theorem, we show that it implies (i) and (ii). For (i), let σ be the identity mapping on $\{1,2,\dots,n\}$; then $k = p = 0$, $q = n$, and $R(2,n) = d(\sigma) = (-1)^{n-1} \cdot 2^{-1} \cdot (2n-2) = (-1)^{n-1}(n-1)$. For (ii) note that

$$R(3,n) = \sum_{\sigma \in D_n} d(\sigma).$$

It is easy to see from the theorem that $(-1)^{n-1}d(\sigma) \geqslant 0$ for every mapping $\sigma: \{1,2,\dots,n\} \to \{1,2,\dots,n\}$, with equality only if σ contains an even cycle or if σ is a constant mapping. Moreover, if $\sigma \in D_n$ and σ contains no even cycles, then we have $p = n$, $q = 0$ and $(-1)^{n-1}d(\sigma) = 2^{k-1}(n-2)$ where k is the number of cycles in σ. It follows that

$$(-1)^{n-1}R(3,n) = (n-2) \cdot \sum_k f(n,k) \cdot 2^{k-1},$$

where $f(n,k)$ is the number of permutations σ of $\{1,2,\dots,n\}$ consisting of k cycles, all odd and of length $\geqslant 3$. Clearly $f(n,1) > 0$ for odd $n \geqslant 3$ and $f(n,2) > 0$ for even $n \geqslant 6$; hence $(-1)^{n-1}R(3,n) > 0$ for $n \geqslant 3$, $n \neq 4$.

Proof of the theorem. For $X \subseteq \{1,\dots,n\}$ let

$$A_X = \{\pi \in S_n: \pi(x) \in \{x, \sigma(x)\} \text{ for all } x \in X\},$$
$$B_X = A_{\{1,\dots,n\}\setminus X}, \quad a_X = \#A_X, \quad b_X = \#B_X = a_{\{1,\dots,n\}\setminus X}.$$

Note that $a_X = 0$ if $|X| \leqslant n-2$; thus, by the principle of inclusion and exclusion, we have

$$d(\sigma) = \sum_{X \subseteq \{1,\dots,n\}} (-1)^{|X|} a_X = (-1)^{n-1}\left(-b_\varnothing + \sum_{u=1}^{n} b_{\{u\}}\right). \tag{1}$$

There are k sets $X \subseteq \{1,\dots,n\}$ such that $\sigma|X$ is a nontrivial cyclic permutation. Call these sets $X_1,\dots,X_k$, and define $\sigma_i \in S_n$ for $1 \leqslant i \leqslant k$ by putting $\sigma_i(x) = \sigma(x)$ for $x \in X_i$, $\sigma_i(x) = x$ for $x \notin X_i$. It is easy to see that the elements of $B_\varnothing$ are just the products $\sigma_1^{\varepsilon_1}\sigma_2^{\varepsilon_2} \cdots \sigma_k^{\varepsilon_k}$ where each ε_1 is 0 or 1. Thus $B_\varnothing$ has 2^k elements; these are all even permutations if σ contains only odd cycles, otherwise half of them are even and half are odd. Hence

$$b_\varnothing = \begin{cases} 0 & \text{if } \sigma \text{ contains an even cycle,} \\ 2^k & \text{otherwise.} \end{cases} \tag{2}$$

We still have to compute $\sum_{u=1}^{n} b_{\{u\}}$. For $u, v \in \{1,\dots,n\}$, let

$$B_{u,v} = \{\pi \in B_{\{u\}}: \pi(u) = v\} = \{\pi \in S_n: \pi(u) = v \text{ and } \pi(x) \in \{x, \sigma(x)\} \text{ for } x \neq u\}.$$

Let $b_{u,v} = \#B_{u,v}$ and $c_v = \sum_{u=1}^{n} b_{u,v}$. Then

$$\sum_{u=1}^{n} b_{\{u\}} = \sum_{u=1}^{n}\sum_{v=1}^{n} b_{u,v} = \sum_{v=1}^{n}\sum_{u=1}^{n} b_{u,v} = \sum_{v=1}^{n} c_v. \tag{3}$$

Finally, we compute the c_v's. Note that $c_v = \#C_v$ where

$$C_v = \{\pi \in S_n: \pi(x) \in \{x, \sigma(x), v\} \text{ for all } x\}.$$

Note further that, if $\pi \in C_v$ and $\pi(u) = v$, then $u = \sigma^i(v)$ for some $i \geqslant 0$. It follows that the components of σ are invariant under π, for every $\pi \in C_v$. From this it is easy to see that $c_v = 0$ if

σ has an even cycle outside the component of v.

Suppose now that σ has no even cycles, except possibly in the component of v. Let $r \geqslant 0$ be minimal such that $\sigma^r(v) = \sigma^{r+s}(v)$ for some $s > 0$, and let $s > 0$ be minimal with $\sigma^r(v) = \sigma^{r+s}(v)$, i.e., s is the length of the cycle in the component of v, and r is the distance from v to the cycle. Then one can verify that

$$c_v = \begin{cases} 2^k & \text{if } s = 1 \text{ and } r \text{ is even;} \\ 0 & \text{if } s = 1 \text{ and } r \text{ is odd;} \\ 2^{k-1} & \text{if } s \text{ is odd and } s > 1; \\ 0 & \text{if } s \text{ is even.} \end{cases}$$

Hence, for an arbitrary mapping σ, we have

$$\sum_{v=1}^{n} c_v = \begin{cases} 0 & \text{if } \sigma \text{ contains an even cycle;} \\ q \cdot 2^k + p \cdot 2^{k-1} & \text{otherwise.} \end{cases} \tag{4}$$

Combining (1)–(4), we get the theorem.

Triples of Integers with Identical Ratios

6401 [1982, 603]. *Proposed by J. C. Lagarias, Bell Laboratories, Murray Hill, NJ, and D. G. Sturtevant, Massachusetts Institute of Technology.*

Let $A = \{a_1, \ldots, a_n\}$ where $1 \leqslant a_1 < a_2 < \cdots < a_n$ are positive integers. Let r be a given positive rational number, with $r = p/q$ and $(p, q) = 1$. Define

$$d_r(A) = \#\left\{(i, j, k) : i < j < k \text{ and } \frac{a_j - a_i}{a_k - a_j} = r\right\}.$$

Now let A vary and set

$$d_r(n) = \operatorname*{MAX}_A \{d_r(A)\}.$$

Prove or disprove that

$$d_r(n) \sim \frac{n^2}{2(p+q)} \text{ as } n \to \infty.$$

Partial solution by the proposers. We can only solve the case $r = 1$ of the problem.

PROPOSITION 1. *For all r, $d_r(n) \leqslant n^2/4$.*

To see this, count the number $a_j(A)$ of triples (i, j, k) having j fixed, such that

$$\frac{a_j - a_i}{a_k - a_j} = r. \tag{*}$$

Note that if i, j are given, there is at most one k for which (*) is true. Since $i < j$, this gives $a_j(A) \leqslant j - 1$. Similarly given j, k, there is at most one value of i satisfying (*), so $a_j(A) \leqslant n - j - 1$. Hence

$$d_r(A) \leqslant \sum_{j=1}^{n} a_j(A) \leqslant 2 \sum_{j=1}^{[(n-1)/2]} (j - 1) \leqslant \frac{n^2}{4}.$$

PROPOSITION 2. *For the sequence $A_n = \{1, \ldots, n\}$,*

$$d_r(A_n) \sim \frac{n^2}{2(p+q)}$$

as $n \to \infty$, r remaining fixed.

This result together with Proposition 1 shows that $d_1(n) \sim n^2/4$.

To prove Proposition 2, note that the set of triples (i, j, k) with $\frac{j-i}{k-j} = \frac{p}{q}$ is just the set of triples with $j - i = mp, k - j = mq$ for $m = 1, 2 \ldots$. Then $k - i = m(p + q)$ and, consequently, there are $n - m(p + q)$ such triples $(1, 1 + mp, 1 + m(p + q))$, $(2, 2 + mp, 2 + m(p + q)) \ldots$. Finally, it is simple to check that

$$\sum_{m=1}^{[n/(p+q)]} (n - m(p + q)) \sim \frac{n^2}{2(p + q)}$$

as $n \to \infty$, with p, q fixed.

Generation of the Alternating Group by Two Conjugate Four-Groups

6402 [1982, 603]. *Proposed by the Editors.*

Prove that if $n \geqslant 8$, the alternating group $\mathscr{A}_n$ can be generated by two conjugate four-groups.

Solution by the Editors. Use is made of the following theorem due to A. Williamson [*On primitive permutation groups containing a cycle*, Math. Z., 130 (1973), 159–162]: *If a primitive transitive group G of degree n contains a t-cycle, then G contains* $\mathscr{A}_n$ *provided* $t \leqslant (n - t)!$.

CASE 1. $n = 4k, k \geqslant 2$. Take

$$a = (1\,2)(3\,4) \cdots (4k - 1, 4k), \qquad b = (1\,2)(3\,4) \cdots (2k - 1, 2k),$$
$$c = (2\,3)(4\,5) \cdots (4k, 1), \qquad d = (2\,3)(4\,5) \cdots (2k, 2k + 1).$$

Clearly $|\langle a, b\rangle| = 4, |\langle c, d\rangle| = 4$. Also, $H = \langle a, b, c, d\rangle$ is transitive and

$$abcd = (1, 4k, 4k - 2, \ldots, 2k + 2, 2k + 1, 2k + 3, \ldots, 4k - 1),$$

so H is primitive. Hence, by Williamson's theorem, $H = \mathscr{A}_n$.

CASE 2. $n = 4k + 1, k \geqslant 2$. Take

$$a = (1\,2)(3\,4) \cdots (4k - 1, 4k), \qquad b = (1\,2)(3\,4) \cdots (2k - 1, 2k),$$
$$c = (2\,3)(4\,5) \cdots (4k, 4k + 1), \qquad d = (2\,3)(4\,5) \cdots (2k, 2k + 1).$$

Here

$$abcd = (2k + 1, 2k + 3, \ldots, 4k - 1, 4k + 1, 4k, 4k - 2, 4k - 4, \ldots, 2k + 2)$$

and the argument is the same as in Case 1.

CASE 3. $n = 4k + 2, k \geqslant 2$. Take

$$a = (1\,2)(3\,4) \cdots (4k - 1, 4k), \qquad b = (4k - 1, 4k)(4k + 1, 4k + 2),$$
$$c = (2\,3)(4\,5) \cdots (4k, 4k + 1), \qquad d = (4k, 4k + 1)(4k + 2, 1).$$

Here

$$db = (4k, 4k + 2, 1, 4k + 1, 4k - 1)$$

and the argument is the same as in Case 1.

CASE 4. $n = 4k + 3, k \geqslant 2$. Take

$$a = (1\,2)(3\,4) \cdots (4k - 1, 4k), \qquad b = (4k - 1, 4k)(4k + 1, 4k + 2),$$
$$c = (2\,3)(4\,5) \cdots (4k, 4k + 1), \qquad d = (4k, 4k + 1)(4k + 2, 4k + 3).$$

Here

$$db = (4k, 4k + 2, 4k + 3, 4k + 1, 4k - 1)$$

and the argument is the same as in Case 1.

Pathological Graphs

6412 [1982, 788]. *Proposed by F. S. Cater, Portland State University.*

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and let $G = \{(x, f(x)) : x \in \mathbb{R}\}$ be its graph. Prove the following:

(i) If f is not continuous everywhere and G is connected, then $\overline{G}$ contains a line segment which is perpendicular to the x-axis. Thus G is not closed.

(ii) There exists a function f whose graph G is a connected and dense subset of $\mathbb{R}^2$. This f is nowhere continuous.

(iii) There exists a function f which is continuous at every irrational point and discontinuous at every rational point and whose graph G is connected.

Solution by Pier Ivan Pastro, Villorba, Italy.

(i) There exist real numbers $\bar{x}, \varepsilon > 0$ and a real sequence $\{x_n\}$ converging to $\bar{x}$ such that $|f(\bar{x}) - f(x_n)| > \varepsilon$. Without loss of generality we may suppose one of the following cases to hold for $n = 1, 2, \ldots$: $f(x_n) - f(\bar{x}) > \varepsilon$; $f(\bar{x}) - f(x_n) > \varepsilon$. Consider the first case (the proof for the other case is analogous). Let $0 < d < \varepsilon$ and suppose that $P = (\bar{x}, f(x) + d) \notin \overline{G}$. Then there exists an $r > 0$ such that $D \cap G \neq \varnothing$ where D is the closed disk with centre P and radius r. Further, there exists an integer m such that $|x_m - \bar{x}| < r$. Let C be the closed set $\{(\bar{x}, y) : y \geqslant f(\bar{x}) + d\} \cup D \cup \{(x_m, y) : y \leqslant f(\bar{x}) + d\}$. It is easily seen that C divides the plane into two open disjoint sets A, B such that $A \cup B \supset G, A \cap G \neq \varnothing$ and $B \cap G \neq \varnothing$. Hence if $P \notin \overline{G}$, then G is not connected, contrary to the hypothesis. It follows that $\overline{G} \supset \{(\bar{x}, y) : f(\bar{x}) \leqslant y \leqslant \varepsilon\}$ as required, and thus G is not closed.

(ii) From here on we shall denote by $(A, B; C)$ a partition of $\mathbb{R}^2$ with A, B open and C closed, and by $P(X)$ the projection on the x-axis of the set X. Let $(A, B; C)$ be a partition. Then $P(A), P(B)$ are open, and $P(A) \cup P(B) \cup P(C) = \mathbb{R}$. Further, we have:

PROPOSITION I(a). *If $P(A) \cap P(B) = \varnothing$, then there is an $\bar{x} \in \mathbb{R} - (P(A) \cup P(B))$ and so $C \supseteq \{(\bar{x}, y) : y \in \mathbb{R}\}$.*

(b) *If $x \in P(A) \cap P(B)$, then there is a $y \in \mathbb{R}$ such that $(x, y) \in C$ and so $P(C) \supset P(A) \cap P(B)$.*

Part (a) is immediate since $\mathbb{R}$ is connected; and part (b) is a consequence of A, B being disjoint and $\{x\} \times \mathbb{R}$ being connected.

Let H be the set of all partitions $(A, B; C)$ with $P(A) \cap P(B) \neq \varnothing$. Then H has the cardinality of the continuum and hence, in view of Proposition I(b) and the well-ordering principle, given $h = (A, B; C) \in H$ we can choose a distinct $x = x_h$ such that x is in the open set $P(A) \cap P(B) \subset P(C)$. Let $S = \cup_{h \in H} \{x_h\}$ and define $g\colon S \to \mathbb{R}$ by setting, for $x \in S$, $g(x)$ to be any y such that $(x, y) \in C$. By Proposition I(b) there is such a y. We then have:

PROPOSITION II. *The graph of g is dense in $\mathbb{R}^2$.*

This follows from the fact that H contains elements $(A, B; C)$ where A is any open disk, $B = \mathbb{R}^2 \setminus \overline{A}$ and $C = B \setminus A$.

Finally, we have:

THEOREM. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an extension of g. Then the graph of f is connected and dense in $\mathbb{R}^2$.*

That the graph is dense follows from Proposition II, and that it is connected from the definition of g and Proposition I(a) which together show that for every partition $(A, B; C)$ there is an x such that $(x, f(x)) \in C$.

(iii) Let $f_n\colon \mathbb{R} \to \mathbb{R}$ be a function defined by:

$$f_n(x) = 0 \text{ if } x \geqslant r_n; f_n(x) = \sin \frac{1}{x - r_n} \text{ if } x < r_n,$$

where $r_n, n = 1, 2, \ldots$, is an enumeration of the rational numbers. Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function

defined by:

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} f_n(x).$$

We shall prove that the function f:

(1) is continuous at every irrational point;
(2) is right continuous but not continuous at every rational point;
(3) has a connected graph.

(1) Let $\bar{x}$ be an irrational number. Then

$$|f(x) - f(\bar{x})| \leqslant \sum_{j=1}^{n} 2^{-j} |f_j(x) - f_j(\bar{x})| + 2^{-n},$$

and hence, since each f_j is continuous at $\bar{x}$, we have $\limsup_{x \to \bar{x}} |f(x) - f(\bar{x})| \leqslant 2^{-n}$ for $n = 1, 2, \ldots$. It follows that $\lim_{x \to \bar{x}} |f(x) - f(\bar{x})| = 0$, and so f is continuous at $\bar{x}$.

(2) Let q be a rational number. Then there exists an integer m such that $q = r_m$, and $f(x) = 2^{-m} f_m(x) + h(x)$ where

$$h(x) = \sum_{j \neq m} 2^{-j} f_j(x).$$

As in (1), h is continuous at q and so f, like f_m, is right continuous but not continuous at q.

(3) Suppose G, the graph of f, is not connected. Then there exist open sets A, B such that $A \cup B \supset G, A \cap G \neq \varnothing, B \cap G \neq \varnothing$ and $A \cap B \cap G \neq \varnothing$; the sets $P(A \cap G), P(B \cap G)$ partition $\mathbb{R}$. Let $\bar{x}$ be an irrational number and suppose, without loss in generality, that $\bar{x} \in P(A \cap G)$. Since f is continuous at $\bar{x}$, it is easily seen that $P(A \cap G)$ is a neighbourhood of $\bar{x}$. Let $s = \sup\{x : [\bar{x}, x] \subset P(A \cap G)\}$. Then s cannot be an irrational number, for if it were either $P(A \cap G)$ or $P(B \cap G)$ would be a neighbourhood of s, contradicting the definition of s. Suppose s is a rational number, then $\lim_{k \to \infty} f(s - 1/k\pi) = f(s)$; and hence $s \in P(A \cap G)$, for otherwise $(s, f(s)) \in B$ and so $s - 1/k\pi \in P(B \cap G)$ for all k sufficiently large. Since f is right continuous at s, it follows that $[s, s + c] \subset P(A \cap G)$ for some $c > 0$, again contradicting the definition of s. Thus we must have $s = \infty$. This shows that $[\bar{x}, \infty) \subset P(A \cap G)$ for any irrational $\bar{x} \in P(A \cap G)$. Since the same holds for $P(B \cap G)$, it follows that no irrational point belongs to $P(B \cap G)$. Hence $P(A \cap G) = \mathbb{R}$, which is impossible since $P(B \cap G) \neq \varnothing$. Consequently G must be connected.

Also solved by the proposer. Partially solved by Victor Pambuccian (Rumania).

A Second-Order Differential Inequality

6416 [1983, 60]. *Proposed by Sanford S. Miller, SUNY, Brockport, NY.*

Let $p(z) = 1 + p_1 z + \cdots$ be regular in the unit disc U, and let α be a real number such that $\alpha < 1$. Show that if

$$\operatorname{Re}\left[z^2 p''(z) + 3zp'(z) - \alpha p^2(z)\right] > -1$$

for $z \in U$, then $\operatorname{Re} p(z) > 0$.

Solution by the proposer. Note that $\operatorname{Re} p(0) = 1$. Suppose there is a point $z_0 \in U$ such that $\operatorname{Re} p(z) > 0$ for $|z| < |z_0|$, but $\operatorname{Re} p(z_0) = 0$. Then $p(z_0) = ri$ where r is real, and, by a result of Sanford Miller and Petru Mocanu [*Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65(1978) 289–305; Theorem 4], $2z_0 p'(z_0) \leqslant -1 - r^2$ is real and nonpositive and

$$\operatorname{Re}\left[z_0^2 p''(z_0) + z_0 p'(z_0)\right] \leqslant 0.$$

Hence

$$\begin{aligned}\operatorname{Re}\left[z_0^2 p''(z_0) + 3z_0 p'(z_0) - \alpha p^2(z_0)\right] &= \operatorname{Re}\left[z_0^2 p''(z_0) + z_0 p'(z_0)\right] + \operatorname{Re}\left[2z_0 p'(z_0) - \alpha p^2(z_0)\right] \\ &\leqslant 0 - 1 - r^2 + \alpha r^2 \leqslant -1,\end{aligned}$$

which contradicts the hypothesis. Therefore $\operatorname{Re} p(z) > 0$ for all $z \in U$.

Positive Cosine Transforms

6417 [1983, 60]. *Proposed by R. P. Boas, Northwestern University.*

It is a well-known theorem of Pólya that although the cosine transform of a decreasing positive function is not necessarily positive, the cosine transform of a positive convex function is positive. [Math. Z., 2(1918) 352-382 (378) = *Collected Papers, vol.* 2, M.I.T. Press, 1974, p. 192.] Show that there is a positive number $0 < \sigma < \frac{1}{2}$ such that the cosine transform of a positive function ϕ with $t^{\mu}\phi(t)$ decreasing is positive provided that $\mu > \sigma$ but not necessarily when $\mu < \sigma$. (Note that, although $\phi(t)$ necessarily becomes infinite at $t = 0$, $\phi(t)$ might be 0 for $t > t_0$.)

Solution by K. F. Andersen, University of Alberta, Canada. Let

$$F(\mu) = \int_0^{3\pi/2} t^{-\mu} \cos t \, dt, 0 \leqslant \mu < 1.$$

Then F is continuous on $[0,1)$, and, for $0 < \mu < 1$,

$$F'(\mu) = \int_0^{3\pi/2} (-\log t) t^{-\mu} \cos t \, dt.$$

Deleting a portion of the interval $(0, 3\pi/2)$ where the integrand is positive and using the monotone properties of $\log t$ and $t^{-\mu}$, we see that, for $0 < \mu < 1$,

$$\begin{aligned}F'(\mu) &> \left(\int_0^{2/\pi} + \int_1^{\pi/2}\right)(-\log t) t^{-\mu} \cos t \, dt \\ &> \left(\log \frac{\pi}{2}\right)\left(\int_0^{2/\pi} - \int_1^{\pi/2}\right)\cos t \, dt = \left(\log \frac{\pi}{2}\right)\left(\sin \frac{2}{\pi} + \sin 1 - 1\right) > 0.\end{aligned}$$

Thus, F is strictly increasing on $[0,1)$. Further, $F(0) < 0$; and, since $\cos\theta > 1 - 2\theta/\pi$ for $0 < \theta < \pi/2$,

$$\begin{aligned}F(1/2) &= \int_0^{3\pi/2} t^{-1/2} \cos t \, dt \\ &> \int_0^{\pi/2} t^{-1/2}(1 - 2t/\pi) \, dt + \left(\frac{\pi}{2}\right)^{-1/2} \int_{\pi/2}^{3\pi/2} \cos t \, dt \\ &= \left(\frac{\pi}{2}\right)^{1/2}\left(2 - \frac{2}{3} - \frac{4}{\pi}\right) > 0.\end{aligned}$$

It follows that there is a unique $\sigma \in (0, 1/2)$ such that $F(\sigma) = 0$. Suppose now that $\mu > \sigma$, and that ϕ is positive with $t^{\mu}\phi(t)$ decreasing. Let $\hat{\phi}$ denote the cosine transform of ϕ, so that

$$\hat{\phi}(x) = \int_0^{\infty} \phi(t) \cos xt \, dt, x > 0.$$

Then $F(\mu) > 0$ and

$$\hat{\phi}(1) = \int_0^{3\pi/2} \phi(t) \cos t \, dt + \int_{3\pi/2}^{\infty} \phi(t) \cos t \, dt;$$

the last term is clearly positive since ϕ decreases, and, because $t^{\mu}\phi(t)$ decreases, we have

$$\int_0^{3\pi/2}\phi(t)\cos t\,dt = \left(\int_0^{\pi/2} + \int_{\pi/2}^{3\pi/2}\right) t^{\mu}\phi(t)(t^{-\mu}\cos t)\,dt$$

$$> (\pi/2)^{\mu}\phi(\pi/2)\int_0^{3\pi/2} t^{-\mu}\cos t\,dt = (\pi/2)^{\mu}\phi(\pi/2)F(\mu) > 0.$$

Thus, $\hat{\phi}(1) > 0$. If $\phi_x(t) = \phi(t/x)$, then ϕ_x satisfies the same conditions as ϕ, and, since $\hat{\phi}(x) = x^{-1}\hat{\phi}_x(1)$, we obtain $\hat{\phi}(x) > 0$ as required. On the other hand, if $\mu < \sigma$, let $\phi(t) = t^{-\mu}$ for $0 < t < 3\pi/2, \phi(t) = 0$ for $t \geqslant 3\pi/2$. Then $t^{\mu}\phi(t)$ decreases, but $\hat{\phi}$ fails to be positive since $\hat{\phi}(1) = F(\mu) < 0$.

Also solved by the proposer who shows that σ is approximately 0.30844.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Geometric Theory of Dynamical Systems: An Introduction. By Jacob Palis, Jr., and Welington de Melo. Springer-Verlag, New York, 1982. xii + 198 pp.

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The solution of the initial value problem in differential equations:

$$\dot{x} = X(x);$$
$$x = x_0 \quad \text{when} \quad t = 0$$

is given by a curve of maps

$$x = f^t(x_0).$$

The standard existence and uniqueness theorems yield the formulas

$$f^t \circ f^s = f^{t+s}$$
$$f^0 = \text{identity}$$

which say that the maps $\{f^t\}$ form a one-parameter group. *Topological dynamics* is the study of such one-parameter groups of homeomorphisms; *differentiable dynamics* restricts attention to one-parameter groups of diffeomorphisms on a smooth manifold M. Thus the former theory is more general, the latter is closer to its origins (i.e., differential equations). In the former theory the generator X plays no role, in the latter it is a vector field on M. A somewhat simpler theory results if the continuous parameter t is replaced by an integer n; the one parameter group now consists of all iterates of some fixed diffeomorphism f. In this review we shall consider this case ("discrete dynamical systems") though the book under review actually considers both discrete and continuous systems.

The ultimate (though generally unrealistic) aim of any mathematical theory is to classify the objects under study up to an appropriate equivalence relation. In this subject an appropriate equivalence relation (though certainly not the only one) is given by C^s-conjugacy: two systems f and g are called *C^s-conjugate* if and only if there is a C^s-diffeomorphism h intertwining them:

$$g = h \circ f \circ h^{-1}. \tag{1}$$

When $s = 0$ we call f and g *topologically conjugate* (a C^0-diffeomorphism is by definition a homeomorphism). The reason this equivalence is suggested is that it respects iteration:

$$g^n = h \circ f^n \circ h^{-1}$$

for all $n \in \mathbb{Z}$ so that g and f will (if conjugate) have the same asymptotic (i.e., long time) behavior.

Let us consider some simple examples. First suppose $f: \mathbb{R} \to \mathbb{R}, f(x) = ax$. Then f has a fixed point at zero and for $x \in \mathbb{R}$ we have either $\lim f^n(x) = 0$ (if $a < 1$) or $\lim f^{-n}(x) = 0$ (if $a > 1$) as $n \to \infty$. Which systems $g: \mathbb{R} \to \mathbb{R}$ are C^s conjugate to f? Such a g must have a unique fixed point (which we may as well take to be 0); if $s \geqslant 1$ equation (1) implies $g'(0) = f'(0)$ and it is not hard to show that for $s \geqslant 2$ these conditions are sufficient as well as necessary. On the other hand for $s = 0$ it is easy to show that g will be C^0 conjugate to f provided that g has a unique fixed point (say at 0) and $0 < g'(0) < 1$ (if $0 < a < 1$) or $1 < g'(0)$ (if $1 < a$). (E.g., for $f(x) = 2x, g(x) = 8x$, take $h(x) = x^3$.) Roughly speaking, for $s = 0$ all systems g close to f (in an appropriate sense) will be C^0 conjugate to f, while for $s \geqslant 2$, the systems g near f possess a 1-dimensional invariant (viz., $g'(0)$) which determines their C^s conjugacy class.

Now consider a map $f: [0,1] \to [0,1]$ with exactly two fixed points $f(0) = 0$, $f(1) = 1$ and $f(x) < x$ for $0 < x < 1$. We take $0 < f'(0) < 1$ and $1 < f'(1)$. By the previous paragraph (if f is C^2) we may find diffeomorphisms $\alpha: [0,1) \to [0,\infty)$ and $\beta: (0,1] \to [0,\infty)$ with

$$(\alpha \circ f \circ \alpha^{-1})(x) = ax$$

$$(\beta \circ f \circ \beta^{-1})(y) = by$$

with $a = f'(0), b = f'(1)$, and if we normalize by taking $\alpha'(0) = \beta'(1) = 1$, they are unique. We thus associate an invariant $\gamma = \beta \circ \alpha^{-1}$ to f; it is a diffeomorphism $\gamma: (0,\infty) \to (0,\infty)$ subject to the restriction

$$\gamma(ax) = b\gamma(x)$$

but to no other restriction; we see the space of C^1 conjugacy classes of systems g near f is infinite dimensional. On the other hand, if $g: [0,1] \to [0,1]$ is another system satisfying $g(0) = 0$, $g(1) = 1$ and $g(y) < y$ for $0 < y < 1$ we can construct a C^0 conjugacy by taking any homeomorphism

$$h: \left[f\left(\tfrac{1}{2}\right), \tfrac{1}{2}\right] \to \left[g\left(\tfrac{1}{2}\right), \tfrac{1}{2}\right]$$

and extending it to an homeomorphism of $[0,1]$ by

$$h(f^n(x)) = g^n(h(x)) \quad \left(f\left(\tfrac{1}{2}\right) \leqslant x \leqslant \tfrac{1}{2}\right);$$

$$h(0) = 0; \quad h(1) = 1.$$

Now let us call $f: M \to M$ *C^r-C^s-rigid* if and only if there is a neighborhood N of f in the C^r topology (in the space of all dynamical systems) such that each $g \in N$ is C^s-conjugate to f; call f *structurally stable* if and only if it is C^1-C^0-rigid. Both our examples are structurally stable and neither is C^r-C^1-rigid. (The latter is in some sense "less rigid" than the former in that its conjugacy class has "infinite codimension" rather than "codimension one".) It is unknown whether there exist C^r-C^s-rigid systems with M compact and $r \geqslant 2, s \geqslant 1$.

The work of Smale and his school in the sixties and seventies greatly advanced our understanding of differentiable dynamics from this point of view. In particular, it emerged that structurally stable systems are (if suitably defined) generalizations of the map of the interval considered above and a wealth of examples and theories beyond structural stability were discovered. It also emerged that structurally stable systems are not as ubiquitous as first thought and much current research is devoted to the study of systems which are definitely *not* structurally stable.

The book under review provides a good introduction to this theory. It treats the (C^0) linearization problem mentioned above but in n-dimensions. It also discusses the theory of structurally stable (continuous time) systems on 2-manifolds as developed by Peixoto. An indication of the more advanced theory is given, and there is a bibliography with 122 entries. The book contains ample detail, plenty of examples, and 62 figures illustrating the text. Both authors have themselves made important contributions to the subject, and it is to be hoped that their effort will attract new graduate students to a beautiful subject.

How to Tell the Liars from the Statisticians. By Robert Hooke. Marcel Dekker, New York and Basel, 1983. xv + 173 pp.

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Life is subject to a variety of random events: a weather change, an illness, an unexpected financial failure, and ultimately a time for dying. If these events can be measured, statistical methods will allow us to extract a semblance of order from their apparent arbitrariness. Hooke's book points out the range and importance of statistics, and lists some of its most frequent misuses. The aim of showing "how statistical reasoning affects nearly all aspects of our lives" is achieved with the help of amusing illustrations by Joseph M. Liles, and without once lapsing into a formula. Hooke discourses on some 76 topics involving statistics, from drug laws, through smoking and cancer, the cost of living, quality control, political polls and psychological tests, to random digits and computer simulation. He explains basic statistical principles simply, and disposes of many common statistical fallacies. For the citizen with a nonnumerate background, this is a delightful way of learning how to distinguish statistical liars from genuine statisticians.

Let me outline an example of Hooke's simple expository style: his discussion of a paradox in discrimination statistics. In a particular university, the admission statistics appear to indicate that the acceptance rate for women (37%) in the Graduate School is lower than that for men (43%). Is there discrimination? The two Graduate Departments A and B involved both report that their acceptance rates are the same for both sexes. The table below indicates the way in which such a paradox can arise (see pp. 26–27). The explanation

Table 1. Admissions to Graduate School

	WOMEN		MEN	
	Applied	Admitted (%)	Applied	Admitted (%)
Department A	50	25 (50%)	100	50 (50%)
Department B	100	30 (30%)	50	15 (30%)
University Total	150	55 (37%)	150	65 (43%)

lies in the fact that men and women do not apply to different departments at the same rate. Mathematical sophisticates may find such examples trivial, but the general public is often baffled by misinterpretations of similar data, and the controversies to which they lead. The elucidation of such paradoxes renders the public a service, none the less valuable for its apparent simplicity.

Since statitistics in some form enters into every citizen's life, the enterprise of enlightening the public is decidedly worthwhile. I recall trying to make the same point by going in detail through *The New York Times* of Saturday, May 22, 1982. I estimated that 32.6% of its total available news space of 31 pages contained information involving statistical material of some kind. The breakdown of this material is given in the following table.

Table 2. Information involving statistical material, The New York Times, 22 May 1982

Category	Percentage of 31 pages
Financial Reports	20.2%
Economic Problems	4.4%
Social Issues	3.5%
Political Arithmetic	1.7%
Sporting Records	1.7%
Weather Information	1.1%
TOTAL	32.6%

With nearly one third of the newspaper based on some form of statistical information, I felt the subject could simply not be ignored. The logical consequence seemed to be that some statistical

training at a basic level should form part of every citizen's education.

The level of such statistical training is not consistent throughout the U.S.A. There is little uniformity in the amount of statistics taught in secondary schools; in some states, statistical topics form part of the curriculum, in others not. There are, however, some excellent experimental projects in statistical education for children, such as CEMREL-CSMP in St. Louis. Other countries have made a more sustained attack on the problem. In Britain, for example, statistics is an established subject which can be studied for the General Certificate of Education, the graduating diploma from secondary schools. To help teachers of geography, biology, the sciences, social science, economics and other subjects to use statistical ideas and methods, a magazine *Teaching Statistics* has been published by the Teaching Statistics Trust, Sheffield, since 1979. Several other countries are also attempting to provide more time for statistics in their school curricula; for details, see [**1**].

At the undergraduate level, many statistics courses of different types are offered at colleges throughout the U.S.A. These are of variable quality, many of them taught by faculty in other disciplines, ranging from mathematics to psychology or social science. This is often a positive factor, but there are cases where the teachers have an imperfect or outdated knowledge of the subject. The fact that statistics is a rapidly developing field means that there is a serious need for statistical refresher courses for both statisticians and nonstatisticians. Regrettably, there are only too few of these.

There are 64 universities in the U.S.A. listed as providing doctoral programs in statistics; yet, with a few notable exceptions, statistics remains one of the poorly supported disciplines on university campuses. The relative youth of the subject, and the competing claims of other disciplines, including mathematics and computer science, do not quite justify this neglect. The mean size of the statistics faculty in these 64 universities with an average of 20,000 students is 12, hardly an adequate number to train graduates and undergraduates, respond to the numerous requests for statistical services, and provide the essential refresher courses for teachers of statistics in schools and colleges. This insufficient support of the discipline is eventually reflected in the frequency with which ordinary citizens are gulled by the unscrupulous misuse of statistical doublespeak. "Figures don't lie, but liars can figure" quotes Hooke in his opening sentence.

The International Statistical Institute, a society consisting of academic, government and industrial statisticians has drawn attention to the problem of statistical education by sponsoring the First International Conference on Teaching Statistics (ICOTS) in Sheffield, England, in 1982 [**3**]. Nearly 500 conference participants from all over the world discussed various facets of the topic: statistical instruction in schools, the preparation of statistics curricula, the teaching of statistics to nonstatisticians, teaching materials, the training of statistics teachers, the use of calculators, and the teaching of specific topics such as survey sampling or economic statistics. A second ICOTS is to be held in 1986; that will be the time to note whether serious progress has been made since 1982.

What Hooke has done very deftly in his amusing book is to focus our attention on the problem of statistical education for the ordinary citizen. For this, we owe him thanks. Within the context of public numeracy, statistics is important; without some understanding of it, no citizen can peruse a newspaper intelligently, or fault the misrepresentations of statistical liars. It is my sincere hope that this book and the earlier [**2**], [**4**], [**5**] (among others) are heralds of a sustained campaign for statistical literacy, not only in the U.S.A. but throughout the world.

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The Future of College Mathematics: Proceedings of a Conference/Workshop on the First Two Years of College Mathematics. Edited by Anthony Ralston and Gail S. Young. Springer-Verlag, New York, 1983. ix + 278 pp.

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Considerable and well-justified anxiety exists about the state of mathematics education today, especially at the precollege level. Some aspects of this sorry state are shared by education in general, but in any case it would not be appropriate to attempt here a full discussion of its causes. Nevertheless, it would be wrong to miss any opportunity to point out that a prime cause is the unattractiveness of the career of secondary school teacher to bright graduates who could achieve double the starting salary, higher prestige and more job security in high-technology industry.

Here, however, it is more relevant to discuss a symptom of the malaise—the unpreparedness of entering freshmen students for the traditional first year of the basic college mathematics course. Certainly, this generalization is subject to many exceptions, but it is rapidly becoming an extremely frequent phenomenon. The unpreparedness takes many forms—poor computational skills, an inability to manipulate algebraic expressions, an inability to think geometrically, poor intuition, no conception of the real line as an archimedean ordered field with a compatible structure as a complete metric space. Thus, increasingly, our universities and colleges are having to provide remedial, precalculus courses for students with such deficiencies—Dick Anderson, in his contribution to the volume under review, quoted the figure of 40% of the freshman class of 1981 at Louisiana State University enrolled in remedial arithmetic classes!

It should also be said that the problem is exacerbated by the poor verbal skills of many students. This reviewer recently had the experience of students complaining that he had set a very difficult question in an examination; it turned out that the difficulty arose, exclusively, from the use of the expression "Hence, or otherwise,..."—there was no other complaint. Communication between teacher and student is uncertain and imperfect in the presence of such real language difficulties.

Running parallel with this problem of student unpreparedness is another development which does not appear to be causally related to it but which also militates against the accessibility and acceptability of the traditional calculus sequence. The availability of the computer must cause us to rethink our core curriculum in mathematics. It renders out of date much of the current content, and it should surely be changing the emphases we give to certain traditional topics. On the positive side, it brings into prominence certain aspects of mathematics which have not hitherto been regarded as central, and these should surely find their place in the curriculum.

Moreover, computers are popular and already familiar to most students before they enter college. Further, a competence in handling computers is viewed by very many students, and their parents, as the best, though by no means the certain, guarantee of future gainful employment. Undergraduate programs in computer science are springing up and attracting substantial enrollments. The computer science major is growing rapidly; indeed, the Volterra-Lotke predator-prey model (naturally, suitably discretized) is plainly appropriate for the study of the relationship between the computer science major and the mathematics major.

On the other hand, the mathematics department continues, in general, to be responsible for teaching mathematics to all students contemplating majors in disciplines having a mathematics requirement, so that it is increasingly necessary to consider the needs and appetites of those students oriented towards computer science and towards the many science and engineering disciplines which accord to the computer a fundamental role.

We will argue below that the two developments discussed above which led to an awareness of the need to reform the core curriculum are, in fact, related; but let us, for now, simply consider the effect of their conjunction on the design of a suitable mathematics curriculum for the first two years of the college program. The effect is the creation of a situation in which traditional calculus

courses are unpopular, unappreciated and frequently inaccessible to students, and in which courses oriented toward the computer are in great demand. It is thus right and proper—indeed, essential—that we consider this issue and consider what changes would be appropriate. This volume represents the thinking of 30 mathematicians brought together in a conference/workshop in the summer of 1982, under the auspices of the Sloan Foundation, to consider various aspects of the problem, the needs of various students, and the feasibility of reform. Others had undertaken related studies before [**1**], [**2**], [**3**], [**4**]; this group, like the CUPM panel [**2**], gave itself the charge actually to propose detailed curricula.

In broad terms, it may be argued—and is cogently argued by many contributors to this volume—that it is no longer appropriate to devote so much time in those first two years to the calculus, this is, to the mathematics of continuity, since the computer has brought into prominence the complementary field of discrete mathematics—notably, combinatorics and graph theory, linear algebra, probability theory and the theory of algorithms. Moreover, certain topics, like the nature and application of inductive definitions and proofs, now need to be presented in a machine context, where they become much more readily intelligible.

Some would go further in dismantling the traditional curriculum and would find ammunition for doing so in Herb Wilf's persuasive article on symbolic manipulations and algorithms. Plainly we are on the threshold of an era where *any* mechanical mathematical process will, in fact, be carried out by a machine. Less radical reforms are embodied in the curricula given as appendices to the volume; one proposal envisages a year's calculus and a year's discrete mathematics in sequence (the reader should also consult the contribution by Donald Bushaw), while another foresees a more integrated two-year sequence (the contribution by Fred Roberts has much relevance here).

It would be fair to say that the unpreparedness of entering freshmen for a course in the calculus is not seen by all as an unmitigated disaster. It provides the opportunity, some would say the necessity, to reform the program in a direction which in any case makes sense. Arguments have appeared—in this volume and elsewhere—that there may even be an advantage in being able to begin certain topics from scratch at the university level, since students will not be burdened with misconceptions and experiences of failure in their precalculus courses which, currently, affect adversely their success rate and their enthusiasm for the calculus. Indeed, it is here that one perceives the link between the two developments discussed at the start of this review—the predominance of the computer provides the positive aspect of the stimulus to computer science and discrete mathematics, while the unpreparedness of students for courses following up their high school experiences provides the negative aspect.

All the contributors to this volume have thought long and hard about the issues involved and their pieces are well worth reading (perhaps a little invidiously, this reviewer would like to single out the article by Lida Barrett, which says so many sensible, apposite things about the practical considerations and difficulties involved in carrying out a reform.) Some of the contributions deal with the needs of students in particular disciplines, while others consider the impact of reform on various aspects of mathematics education—the training of teachers, high school mathematics, remedial mathematics. Many of the contributors attest to the lack of suitable texts; perhaps they would agree that [**5**] is one such.

The discussion, overall, is careful and exemplary. Nevertheless, it can still not be considered comprehensive. It must be remembered that attendance at the workshop was by invitation, and whether by chance or intention, the viewpoint of those who continue to regard the calculus as the basis of the mathematical sciences was not represented. If one consults [**3**], one finds Anthony Ralston, a key member of the present group, arguing strongly for reform, "... the undergraduate curriculum, particularly its first two years, needs an overhaul for virtually everyone But the dominance of calculus and, more generally, its relations ... is ending, if it is not already over." But one also finds Peter Lax declaring "... I hope to refute the claims that the new thrusts in mathematics, theoretical and applied, are in the discrete area. I claim otherwise." Particularly disturbing to this reviewer was the absence of item [**1**] from the bibliography of any contributor to this volume. (Item [**1**] was, however, prominently cited in the Tucker report [**2**], which is itself

mentioned frequently in this volume.) There was no overlap between the membership of this conference and that of the NRC Committee on Applied Mathematics Training; but it is surprising that a group of mathematicians considering the issue of curricular reform could ignore the deliberations, over a two-year period, of a committee including in its membership Felix Browder, Steve Smale, Dan Wagner and Shmuel Winograd, to name but four.

Perhaps symptomatic of the same homogeneity of opinion is the arrogation of the description "modern mathematics" to describe computer-related mathematical studies. This exclusivity is especially prominent in the contributions of Bill Lucas and Fred Roberts; one salutes them both for what they do advocate—this reviewer particularly appreciated their recognition of the vital "fun" element in good, attractive, serious mathematics—but one denies that, for example, differential geometry, real and complex analysis, and topology are not modern just because they existed in the precomputer age. (It is, incidentally, very annoying—and quite unfair—to find it implied that there is no fun in these subjects today!)

This reviewer was also made uncomfortable by the argument for "top-down" reform, as represented in Ralston's contribution. To hope that the high schools will somehow adapt to a fundamental reform carried out autonomously at the college level is utterly utopian. Certainly we cannot, and should not, simply await a "bottom-up" reform movement, but it is surely perfectly plain that the problem of articulation with precollege mathematics teaching, carefully addressed in this volume by Stephen Maurer and Julian Weissglass, is crucial. Let us not fool ourselves—the difficulties of today's students with the calculus will be matched by the difficulties of tomorrow's students with discrete mathematics unless we can ensure appropriate precollege mathematics preparation.

It is only fair to add that not every contributor sees inadequate preparation as a factor in the perceived unsatisfactory nature of our calculus courses. Donald Kreider, discussing the demand for computer science courses, writes, "Mere programming courses will not suffice. These students need and want courses that match the intellectual substance of calculus, and that apply to computer science mathematical methods using combinatorics, sequences and series, recursion, mathematical induction, and elementary statistics and probability." Need and *want*?! Would that it were so! Could it be that Kreider was simultaneously conducting his own miniconference on "The Future of Dartmouth College Mathematics"?

Finally, something should be said on the relation of computer science, programming and mathematics. First, it will increasingly be the case that students will arrive in college already well versed in computer programming. It is dubious, in any case, whether programming (like cycling) is best learnt through a course; but the argument will soon be rendered academic by the student's high school and general precollege experience. Second, advanced placement programs in computer science are likely to prove increasingly popular, so that it is reasonable to envisage that many science and engineering students, not majoring in computer science, will already have acquired the prerequisite computer science competence. Third, there is no general agreement—even among the contributors to this workshop—as to whether computer science is a mathematical science; but there is a general feeling—here articulated by Stephen Garland—that "... the computer science community as a whole places low emphasis on mathematics." It is thus no surprise that the computer science major is perceived by most students as an easy option compared to the mathematics major, and this feature of the problem clearly imports a political element into the debate. If we offer basic courses in discrete mathematics in the core curriculum and eliminate considerable parts of the traditional calculus course, we may well be doing the right thing for not wholly noble, though valid, reasons. We should not blind ourselves to the obvious fact that calculus is a difficult subject, difficult to teach effectively, especially to students of widely different background, talent and motivation. The reforms advocated in this volume could be viewed as a part of our attempt to reoccupy the center of undergraduate studies in the exact sciences. This reviewer believes that mathematics departments should, both in their interests and those of their students, endeavor to keep or regain a measure of influence over the design and delivery of undergraduate courses in computer science. We will thus attract more students—and our

consciences will be clear. For industry is coming to value, among its recruits from the universities, young people well versed in computer technology and educated to a capacity for mathematical thinking. And thus our continued existence as a key component of academe, now in doubt, will be assured—at least so long as academe itself continues to exist in recognizable form.

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LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405.

Editor:

In a recent note, H. Haruki and S. Haruki [this MONTHLY, 90 (1983) 464–466] evaluated the definite integral

$$f(a) = \int_0^1 \frac{\log(x^2 - 2x\cos a + 1)}{x}\,dx \qquad (0 \leqslant a \leqslant 2\pi). \tag{1}$$

The proof of their result

$$f(a) = -(a - \pi)^2/2 + \pi^2/6$$

is unnecessarily long. From (1), by differentiation under the integral sign, we obtain

$$f'(a) = 2\sin a \int_0^1 \frac{dx}{x^2 - 2x\cos a + 1} = \pi - a,$$

so that

$$f(a) = \pi a - \frac{a^2}{2} + C.$$

The constant $C = -\pi^2/3$ is easily determined from the functional equation

$$f(a/2) + f(\pi - a/2) = f(a)/2$$

proved by Haruki and Haruki.

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Editor:

Robert C. Thompson's article "Author vs. Referee: A Case History for Middle Level Mathematicians" (this MONTHLY, December 1983, pp. 661–668) gives me an excuse to raise an issue that has long troubled me.

We all know that the referee's name is not usually submitted to the author of a mathematical paper. But Professor Thompson's experience, as well as my own, indicates that authors' names *are* sent to referees (cf. p. 664, item 4, where Thompson's referee writes, "I suggest not publishing the enclosed paper of R. C. Thompson..."). Of course, this practice is not restricted to the MONTHLY but permeates most journals, apparently; Thompson's paper was sent to the *Rocky Mountain Journal of Mathematics*.

Why the lack of symmetry (referee knows author but not vice versa)? More seriously, is not the objectivity of refereeing impaired if the referee knows who the author is? Does this practice not give the paper written by a known author from a large university a different chance of being accepted than the same paper written by a newcomer from a small college? At the very least, should not anonymity be a *right* of the prospective author if he wishes?

Clearly the *editor* must know the names of both referee and author, but shouldn't it stop there? This discipline of ours, which prides itself on objectivity, has run aground on an unnecessary reef. Let the prospective paper stand on its own merits *without* the buttress or hindrance of the author's name and institution! To the retort, "Other disciplines are guilty, too!" I answer simply (1) two wrongs do not one right make in any modular arithmetic, and (2) all other journals do NOT do this (the *Journal of Finance*—official publication of the American Finance Association—is one counterexample).

Has my thinking gone astray? Am I missing something?

Donald P. Minassian
Department of Mathematical Sciences
Butler University
Indianapolis, IN 46208

P.S. Would Thompson's paper have been accepted if his name had been lacking? We'll never know.

ANSWERS TO REBUSES ON PAGES 436–437

1. Square root

2. Partial derivative

3. Cantor with the middle third removed

4. Odd permutation

5. Cubic equation

6. Morse inequality

7. This is definitely Fermat's *last* theorem.



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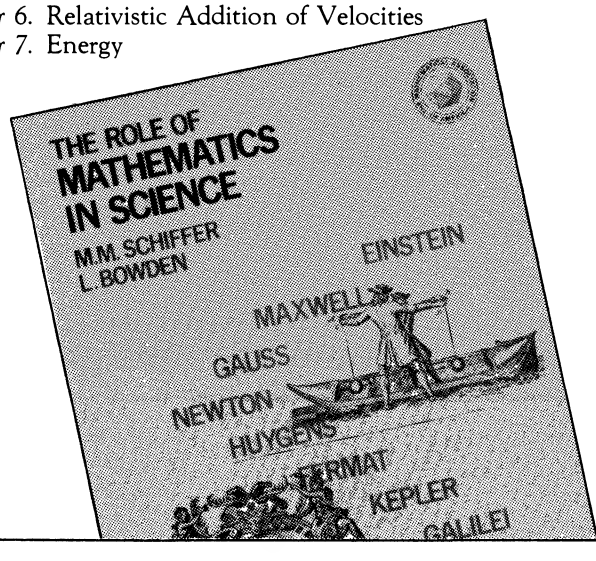
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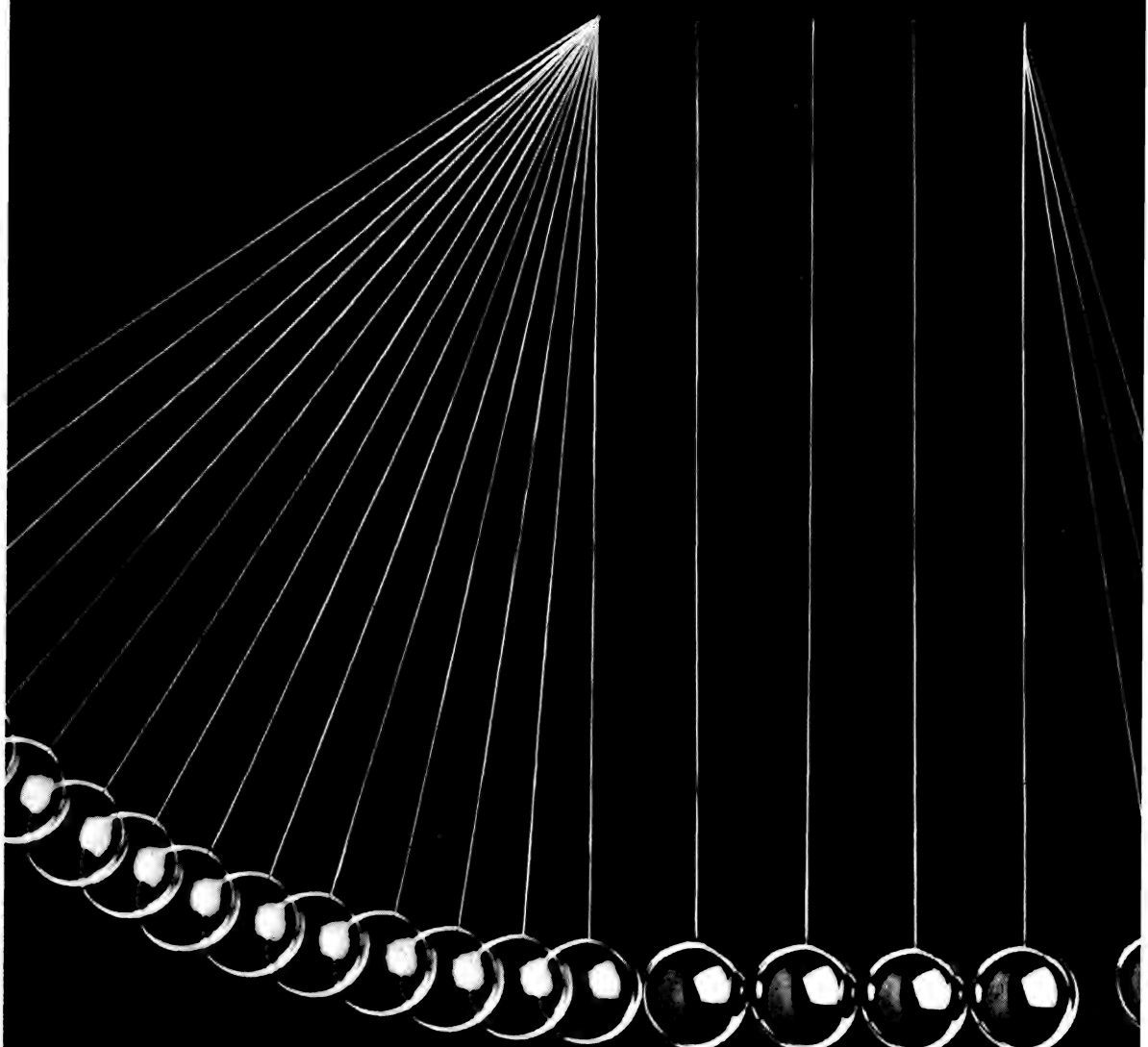
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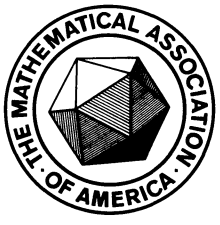
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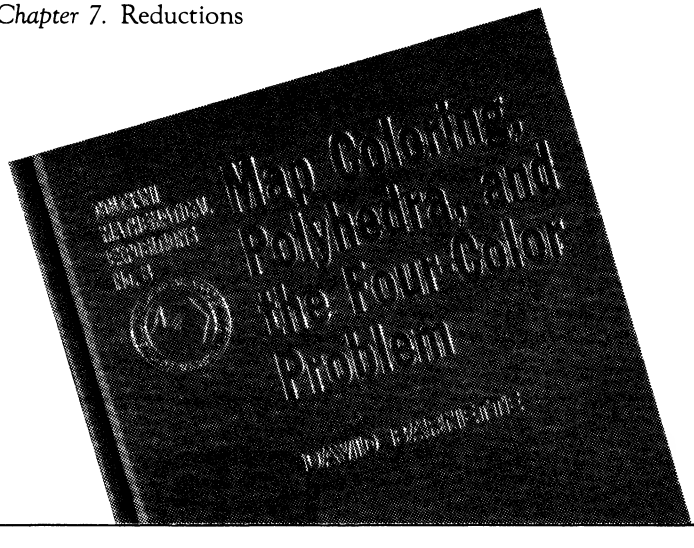
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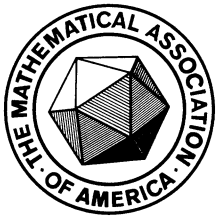
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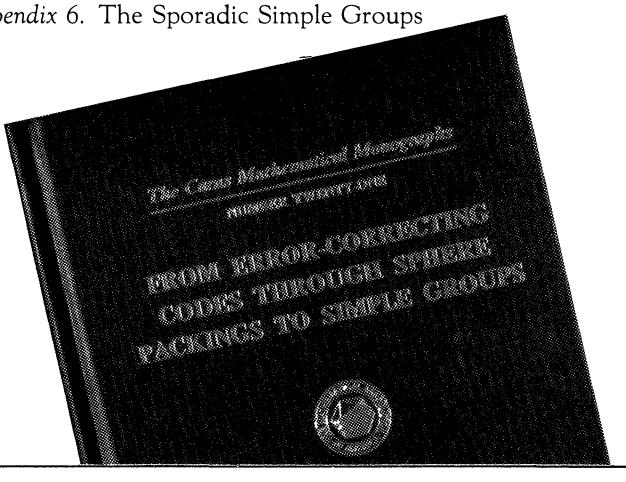
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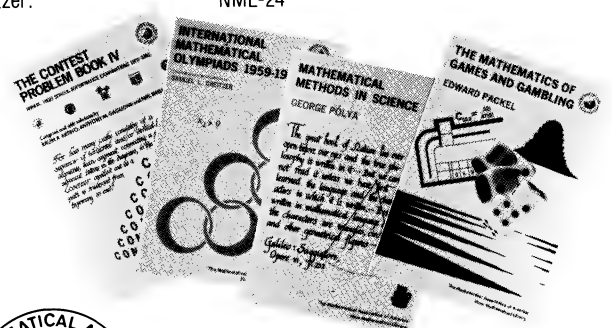
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THE TOILET PAPER PROBLEM

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1. Introduction. The toilet paper dispensers in a certain building are designed to hold two rolls of tissues, and a person can use either roll.

There are two kinds of people who use the rest rooms in the building: *big-choosers* and *little-choosers*. A big-chooser always takes a piece of toilet paper from the roll that is currently larger; a little-chooser always does the opposite. However, when the two rolls are the same size, or when only one roll is nonempty, everybody chooses the nearest nonempty roll. When both rolls are empty, everybody has a problem.

Let us assume that people enter the toilet stalls independently at random, with probability p that they are big-choosers and probability $q = 1 - p$ that they are little-choosers. If the janitor supplies a particular stall with two fresh rolls of toilet paper, both of length n, let $M_n(p)$ be the average number of portions left on one roll when the other roll first empties. (We assume that everyone uses the same amount of paper, and that the lengths are expressed in terms of this unit.) For example, it is easy to establish that

$$M_1(p) = 1, \quad M_2(p) = 2 - p, \quad M_3(p) = 3 - 2p - p^2 + p^3; \quad M_n(0) = n; \quad M_n(1) = 1.$$

The purpose of this paper is to study the asymptotic value of $M_n(p)$ for fixed p as $n \to \infty$. We will see that the generating function $\sum_n M_n(p) z^n$ has a surprisingly simple form, from which the asymptotic behavior can readily be deduced. Along the way we will encounter several other interesting facts.

2. Recurrence Relations. Let us begin by generalizing the problem slightly, using the notation $M_{mn}(p)$ to stand for the mean number of portions left when one roll empties, if we start with m on one roll and n on the other. Thus

$$\begin{aligned} M_n(p) &= M_{nn}(p); \\ M_{m0}(p) &= m; \\ M_{nn}(p) &= M_{n(n-1)}(p), \qquad \text{if } n > 0; \\ M_{mn}(p) &= pM_{(m-1)n}(p) + qM_{m(n-1)}(p), \qquad \text{if } m > n > 0. \end{aligned}$$

The value of $M_n(p)$ can be computed for all n from these recurrence relations, since no pairs (m', n') with $m' < n'$ will arise.

It is convenient to visualize the recurrence by drawing certain arcs between adjacent lattice points in the plane, where the arc from (n, n) to $(m - 1, n)$ has weight p and from (m, n) to $(m, n - 1)$ has weight q, for all $0 < n < m$; the arc from (m, n) to $(n, n - 1)$ has weight 1 for all $n > 0$; and there are no other arcs. Then $M_{mn}(p)$ is the sum, over all $k \geqslant 1$, of k times the sum of the weights of all paths from (m, n) to $(k, 0)$, where the weight of a path is the product of the individual arc weights.

A path that starts at the diagonal point (n, n) must go first to $(n, n - 1)$; then it either returns to the diagonal at $(n - 1, n - 1)$ or goes to $(n, n - 2)$, etc. Let c_k be the number of paths from (n, n) to $(n - k, n - k)$ whose intermediate points do not touch the diagonal, and let d_{nk} be the number of paths from $(n, n - 1)$ to $(k, 1)$ whose points do not ever touch the diagonal. A path that starts at (n, n) either returns to the diagonal for the first time at some point $(n - k, n - k)$, or never returns to the diagonal at all; it follows that

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$$M_n(p) = c_1 pM_{n-1}(p) + c_2 p^2 qM_{n-2}(p) + \cdots + c_{n-1}p^{n-1}q^{n-2}M_1(p) + L_n(p)$$
$$= \sum_{0<k<n} c_k p^k q^{k-1} M_{n-k}(p) + L_n(p);$$
$$L_n(p) = \sum_{2\leqslant k\leqslant n} k d_{nk} p^{n-k} q^{n-1}, \quad \text{for } n \geqslant 2; \qquad L_1(p) = 1.$$

(Each path from (n, n) to $(n-k, n-k)$ has weight $p^k q^{k-1}$ if no intermediate diagonal points are involved, since the step to $(n, n-1)$ has weight 1 and then there are k steps of weight p and $k-1$ of weight q, in some order. Similarly, each diagonal-avoiding path from $(n, n-1)$ to $(k, 1)$ has weight $p^{n-k}q^{n-2}$.)

The coefficients c_k are the well-known Catalan numbers, and the coefficients d_{nk} are the well-known numbers that arise in the classical ballot problem; see, for example, [2, III.1], [3, exercise 2.2.1–4]. We can discover the required values by observing that d_{nk} is the number of decreasing paths from $(n, n-1)$ to $(k, 1)$ minus the number of decreasing paths from $(n, n-1)$ to $(1, k)$, where a "decreasing path" is any path that decreases either the left component or the right component by unity at each step. This follows because there is a 1-1 correspondence between all decreasing paths from $(n, n-1)$ to $(k, 1)$ that do touch the diagonal and all decreasing paths from $(n, n-1)$ to $(1, k)$; the idea [1] is to reflect the path about the diagonal, starting after the place where it first touches a diagonal point. Since the number of decreasing paths from (a, b) to (c, d) is $\binom{a+b-c-d}{a-c} = \binom{a+b-c-d}{b-d}$ for all $a \geqslant c$ and $b \geqslant d$, we have

$$d_{nk} = \binom{2n-k-2}{n-2} - \binom{2n-k-2}{n-1} = \binom{2n-k-2}{n-2}\frac{k-1}{n-1}.$$

Furthermore $c_{n-1} = d_{n2}$, hence

$$c_n = \binom{2n-2}{n-1}\frac{1}{n}.$$

3. Special power series. The generating function for Catalan numbers

$$C(z) = c_1 z + c_2 z^2 + \cdots = \sum_{n\geqslant 1} \binom{2n-2}{n-1}\frac{1}{n} z^n = \frac{1-\sqrt{1-4z}}{2}$$

can be derived in many ways. For our purposes it seems best to make use of the general identity

$$(*) \qquad \sum_{k\geqslant 0} \binom{2k+w}{k} z^k = \frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^w.$$

This well-known identity holds for all complex numbers w; it can be proved easily by contour integration: The coefficient of z^k in the Maclaurin expansion of the right-hand side is

$$\frac{1}{2\pi i}\oint \frac{1}{\sqrt{1-4z}}\left(\frac{1-\sqrt{1-4z}}{2z}\right)^w \frac{dz}{z^{k+1}} = \frac{1}{2\pi i}\oint \frac{dt}{(1-t)^{w+k+1}t^{k+1}}$$

if we make the substitutions $t = \frac{1}{2}(1-\sqrt{1-4z})$, $z = t - t^2$, $dz = (1-2t)\,dt$. The latter integral is the residue of the integrand, i.e., the coefficient of t^k in $(1-t)^{-w-k-1}$, namely $\binom{-w-k-1}{k}(-1)^k = \binom{2k+w}{k}$. (A more elementary proof can be found in [3, exercise 1.2.6–26].)

The derivative of $C(z)/z$ with respect to z is $C(z)^2/(z^2\sqrt{1-4z})$; hence we can replace w by $w+1$ in $(*)$ and integrate, obtaining the companion formula

$$\sum_{k\geqslant 0} \frac{w}{k+w}\binom{2k+w-1}{k} z^k = \left(\frac{1-\sqrt{1-4z}}{2z}\right)^w.$$

Again, this result is valid for all complex w, if we evaluate the coefficient by continuity when $k+w=0$. The case $w=1$ of this formula reduces to the generating function for Catalan numbers stated earlier.

These power series converge for $|z| < 1/4$, because the righthand side is singular only when z is infinite or $\sqrt{1-4z}$ is singular. It is interesting to consider what happens when $z = pq$, $p \geqslant 0$, $q \geqslant 0$, and $p + q = 1$: We have $1 - 4z = (p-q)^2$, hence

$$\sqrt{1-4z} = |p-q| = \max(p,q) - \min(p,q),$$

and we obtain the interesting formula

$$C(pq) = \sum_{n\geqslant 1} \binom{2n-2}{n-1} \frac{1}{n} p^n q^n = \min(p,q).$$

We have $pq < 1/4$ unless $p = q = 1/2$; the formula holds also in the latter case, by Abel's limit theorem.

4. Generating functions. Let us now set

$$M(z) = \sum_{n\geqslant 1} M_n(p) z^n; \qquad L(z) = \sum_{n\geqslant 1} L_n(p) z^n.$$

The recurrence relation for $M_n(p)$ in section 2 is equivalent to

$$M(z) - L(z) = q^{-1} C(pqz) M(z),$$

and we also have

$$\begin{aligned} L(z) &= z + \sum_{2\leqslant k\leqslant n} \frac{q^{n-1}}{n-1} k(k-1) p^{n-k} \binom{2n-k-2}{n-2} z^n \\ &= \sum_{j,k\geqslant 0} \frac{q^{j+k-1}}{j+k-1} k(k-1) p^j \binom{2j+k-2}{j} z^{j+k} \\ &= \sum_{k\geqslant 0} q^{k-1} k z^k \sum_{j\geqslant 0} \frac{k-1}{j+k-1} \binom{2j+k-2}{j} (pqz)^j. \end{aligned}$$

By the identity in section 3, the latter sum is

$$\begin{aligned} &= \sum_{k\geqslant 0} q^{k-1} k z^k \left(\frac{1-\sqrt{1-4pqz}}{2pqz} \right)^{k-1} \\ &= z \sum_{k\geqslant 0} k p^{1-k} C(pqz)^{k-1} = \frac{p^2 z}{(p - C(pqz))^2}. \end{aligned}$$

We can now eliminate $L(z)$ and solve for $M(z)$, obtaining a "closed form" for the desired generating function:

$$M(z) = z \left(\frac{p}{p - C(pqz)} \right)^2 \left(\frac{q}{q - C(pqz)} \right).$$

Such a simple form for $M(z)$ is unexpected; but in fact, we can do even more! We have

$$(p - C(pqz))(q - C(pqz)) = pq - C(pqz) + C(pqz)^2 = pq(1-z),$$

because $C(z) - C(z)^2 = z$. Hence the denominator of $M(z)$ can be vastly simplified:

$$M(z) = \frac{z}{(1-z)^2} \left(\frac{q - C(pqz)}{q} \right).$$

This is the product $(z + 2z^2 + 3z^3 + \cdots) \cdot (1 - c_1 pz - c_2 p^2 q z^2 - c_3 p^3 q^2 z^3 - \cdots)$, so the coefficient of z^n can be written

$$M_n(p) = n - (n-1)c_1 p - (n-2)c_2 p^2 q - \cdots - 1 \cdot c_{n-1} p^{n-1} p^{n-2}.$$

When a formula turns out to be so simple, it must have a simple explanation. But the author hasn't been able to think of any direct proof. For some reason, $M_n(p)$ is not only the expected size of the remaining roll when one roll empties, it is also the expected value of the "first return to the diagonal," in the following sense: Suppose the two toilet paper rolls start in the full state (n, n), and they are used by big-choosers and little-choosers until the empty state $(0, 0)$ is reached; and suppose that the rolls first become equal in size again at state $(n-k, n-k)$. Then the average value of k is $M_n(p)$. (This follows from our formula for $M_n(p)$, because $c_k p^k q^{k-1}$ is the probability of first return to $(n-k, n-k)$ for each $k < n$, and $1 - c_1 p - \cdots - c_{n-1} p^{n-1} q^{n-2}$ is the probability that the diagonal is not encountered until state $(0, 0)$ is reached.)

Is there an easy way to prove that the same expected value occurs in both problems? The distributions are different, but the mean values are the same.

5. The limiting behavior. Now that $M(z)$ has been put into a fairly simple form, we are ready to deduce the asymptotic value of $M_n(p)$ for fixed p as $n \to \infty$.

Let's assume first that $p \neq q$. Then $4pq < 1$, and the function $C(pqz) = \frac{1}{2}(1 - \sqrt{1 - 4pqz})$ is analytic for $|z| < 1/(4pqz)$; so it is analytic in a neighborhood of $z = 1$. In fact, a simple computation proves that its Taylor series at the point $z = 1$ involves the Catalan numbers once again:

$$C(pqz) = \min(p, q) + (\max(p, q) - \min(p, q))C\left(\frac{pq(z-1)}{(p-q)^2}\right).$$

(This formula generalizes our previous observation that $C(pq) = \min(p, q)$.)

If $q < p$, our formula for $M(z)$ reduces to

$$\begin{aligned} M(z) &= \frac{z}{(1-z)^2}\frac{q-p}{q}C\left(\frac{pq(z-1)}{(p-q)^2}\right) \\ &= \frac{z}{1-z}\frac{p}{p-q} - z\left(c_2\frac{p^2q}{(p-q)^3} + c_3\frac{p^3q^2}{(p-q)^5}(z-1) + \cdots\right) \\ &= \frac{z}{1-z}\frac{p}{p-q} + f(z), \end{aligned}$$

where $f(z)$ is analytic in the region $|z| < 1/(4pq)$. This determines the value of $M_n(p)$ quite accurately:

THEOREM 1. *Let r be any value greater than $4pq$. Then*

$$M_n(p) = \begin{cases} p/(p-q) + O(r^n), & \text{if } q < p; \\ ((q-p)/q)n + p/(q-p) + O(r^n), & \text{if } q > p. \end{cases}$$

(The constants implied by O in these formulas depend on p and r, but not on n.)

Proof. If $q < p$, the value of $M_n(p)$ is the coefficient of z^n in $M(z)$, which is $p/(p-q)$ plus the coefficient of z^n in $f(z)$. But $f(z)$ converges absolutely when $z = 1/r$, hence its nth coefficient is $O(r^n)$.

If $q > p$, the stated result follows from the formula for $q < p$, using the identity

$$qM_n(p) + pn = pM_n(q) + qn$$

which is an immediate consequence of the formula for $M_n(p)$ in section 4. QED.

For example, if $p = 2/3$ and $q = 1/3$, so that big-choosers outnumber little-choosers by 2 to 1, the average size of the remaining roll will be very close to 2, when n is large; but when $p = 1/3$ and $q = 2/3$ the average will be approximately $\frac{1}{2}n + 1$.

This agrees with our intuition: If little-choosers predominate, the size of the larger roll will tend

to be proportional to n, when the smaller roll is used up. But if big-choosers are in the majority, the larger roll will tend to be reduced to a bounded size, independent of the initial size n.

6. The transition point. But what about the boundary case, $p = q$? Does it lead to lengths of order n, or order 1, or something in between?

This is actually the simplest case to analyze, because $p = q = 1/2$ is equivalent to saying that everybody is a random-chooser; the problem reduces to a fairly simple "random walk." In fact, we are essentially dealing here with "Banach's match box problem" as discussed by Feller [2, IX.3(f)]. According to our general formula, the generating function in this case is simply

$$M(z) = \frac{z}{(1-z)^{3/2}},$$

so there is a solution in closed form:

$$M_n\left(\frac{1}{2}\right) = \binom{-3/2}{n-1}(-1)^{n-1} = \frac{2n}{4^n}\binom{2n}{n}.$$

By Stirling's approximation we have the following result:

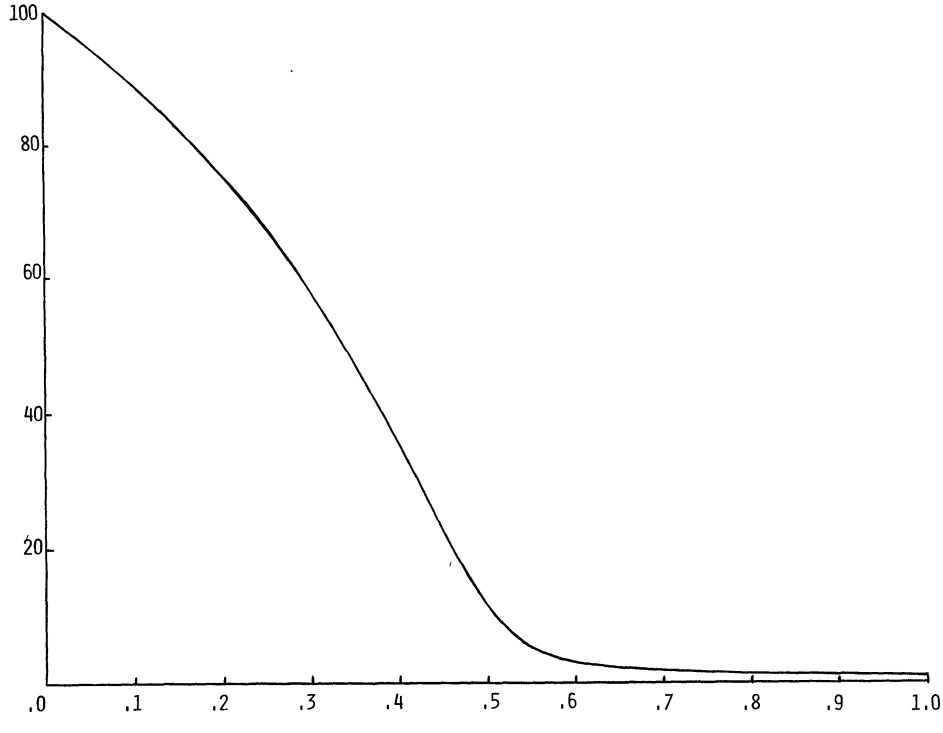
THEOREM 2.

$$M_n(p) = 2\sqrt{\frac{n}{\pi}} - \frac{1}{4}\sqrt{\frac{1}{\pi n}} + O(n^{-3/2}),$$

when $p = q$.

The function $M_n(p)$ is a polynomial in p of degree $2n-3$, for $n \geqslant 2$, and it decreases monotonically from n down to 1 as p increases from 0 to 1. The remarkable thing about this decrease is that it changes in character rather suddenly when p passes $1/2$.

We can't use the formulas of Theorem 1 when p is too close to $1/2$, even if n is extremely



large. For example, if $n = 10^{10}$ and $p = \frac{1}{2} \pm 10^{-20}$, both approximations in Theorem 1 give the ridiculous estimate $M_n(p) \approx \frac{1}{4} \times 10^{20}$. Indeed, we know that $M_n(1/2)$ is of order $\sqrt{n}$, so the approximations can be valid only when $|p - \frac{1}{2}|$ is of order $1/\sqrt{n}$ at least.

The slope of $M_n(p)$ at $p = 1/2$ can be calculated by differentiating $M(z)$ with respect to p and extracting the coefficient of z^n. The derivative is

$$-\frac{z}{(1-z)^2}\frac{d}{dp}\left(\frac{C(p(1-p)z}{1-p}\right) = -\frac{z}{(1-z)^2} \times \left(\frac{(1-2p)zC'(p(1-p)z)}{1-p} + \frac{C(p(1-p)z)}{(1-p)^2}\right)$$

and at $p = 1/2$ this equals $-2z(1-z)^{-2} + 2z(1-z)^{-3/2}$. Hence

$$M_n'(1/2) = -2n + 2M_n(1/2);$$

this is consistent with $M_n(p)$ dropping from n to a small value as p goes from 0 to $1/2$. The graph of $M_{100}(p)$ is shown on page 469.

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THE LOGIC OF PROVABILITY

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The subject of this article is the way in which an ancient branch of logic, first investigated by Aristotle and known as *modal logic*, has recently been found to shed light on a branch of logic of much later date, the mathematical study of mathematics itself, a study begun by David Hilbert and brought to fruition by Kurt Gödel.

The fundamental concepts studied in modal logic are those of *necessity* and *possibility*: a statement is called "necessary" if it *must* be true, and "possible" if it *might* be true. Thus, since there might be a war in the year 2000, the statement "there will be a war in 2000" is possible, but it is not necessary, as there might not be a war then. On the other hand, the statement "there will

George Boolos: After an undergraduate degree in mathematics at Princeton, where my supervisor was Raymond Smullyan, and a graduate degree in philosophy at Oxford, I became the first person ever to receive a Ph.D. in philosophy from MIT, writing a thesis on hierarchy theory under Hilary Putnam. I taught for three years at Columbia and in 1969 returned to MIT, where I am now a professor of philosophy. In addition to a book on the topic of this article, *The Unprovability of Consistency*, and a textbook, *Computability and Logic* (co-authored with Richard Jeffrey), I have written a number of articles in logic and philosophy.

be a war in 2000 or there won't" is necessary: it must be that there either will or won't be a war then. "Necessary" and "possible" are interdefinable: a statement is necessary if it is not possible that it not be true; similarly, a statement is possible if it is not necessary that it not be true. In brief, necessary = not-possible-not and possible = not-necessary-not. And of course, whatever is necessary is true, and whatever is true is possible.

Probably because of the metaphysical character of the notions of necessity and possibility, their obscurity and remoteness from experience, modal logic has always been a subject more or less on the periphery of logic. Aristotle himself, who developed the theory of the syllogism ("All As are Bs; some Cs are As; therefore, some Cs are Bs", etc.) in an almost perfect form, also worked on a theory of modal syllogisms, in which the premises and conclusions contain expressions meaning "necessary" and "possible." Scholarly commentators, however, have regarded his modal theory as confused, fragmentary, and otherwise defective. The logical properties of necessity and possibility were much studied by medieval logicians such as Abelard, and these notions figured importantly in the thought of Leibniz. In our own time, the American philosopher-logicians C.I. Lewis and Saul Kripke have been the most important contributors to modal logic. Despite their work, the subject has not been considered to be of central interest to contemporary logic.

In contrast, no work is closer to the heart of logic than Kurt Gödel's theorems on the incompleteness of arithmetic, contained in his celebrated 1931 paper "On formally undecidable propositions of *Principia Mathematica* and related systems I." Partly because of the popular success of such books as Nagel and Newman's *Gödel's Proof* and Hofstadter's *Gödel, Escher, Bach*, the leading ideas, methods, and results of Gödel's paper have become quite well known, but it will be helpful to review them briefly.

Gödel's incompleteness theorems concern those rigorous presentations of mathematical theories known as *axiomatic*, or *formal*, systems, among them the widely studied systems PA (*Peano Arithmetic*) and ZF_1 (*Zermelo-Fraenkel Set Theory*). PA is intended to describe the sequence of natural numbers $0, 1, 2, \ldots$. Its axioms are elementary principles of addition and multiplication, e.g., $x + 0 = x$, as well as statements expressing the principle of mathematical induction. ZF is a much stronger system than PA, from whose axioms almost all known mathematics can be derived; it is currently regarded as *the* standard system of set theory.

Gödel's first theorem states that in any "reasonable" axiomatic system in which a small amount of the ordinary arithmetic of the natural numbers can be developed, certain statements can be formulated in the language of the system that can neither be proved nor disproved in the system. To disprove a statement S is merely to prove not-S, the negation of S; a statement is called (*formally*) undecidable in a system if it is neither provable nor disprovable in the system (and is thus a statement as to whose truth or falsity the system has no opinion); and a system is called *incomplete* if undecidable statements can be formulated in its language. Thus the first incompleteness theorem says that any reasonable axiomatic system of arithmetic is incomplete. ("Reasonableness" will be discussed later.)

When it first appeared, Gödel's second incompleteness theorem was considered more surprising than the first theorem and is less well understood even today. It states, roughly, that no consistent axiomatic system can prove its own consistency: if an axiomatic system is consistent, then the statement of the system's language expressing the system's consistency is not provable in the system. (Henceforth the words "in the system" will be omitted but understood after "provable.")

Although the subject matter of any of the axiomatic systems referred to in the incompleteness theorems is, ostensibly, the numbers $0, 1, 2, \ldots$ (and possibly other mathematical objects), by means of "coding," or "Gödel numbering," that subject matter can be taken to include the expressions of the language: via a systematic assignment of Gödel numbers to all such expressions, statements apparently about numbers can be taken to refer to the expressions coded by those numbers. The notion "provable statement" can then be expressed in the system: a formula can be constructed that applies to a number if and only if it is the Gödel number of a provable

statement. With an eye to the connections with modal logic, we shall write this formula:

$$\Box(x);$$

"$\Box$" is the sign standardly used in modal logic to mean "necessarily." To understand the next stage of the reasoning in the proofs of the incompleteness theorems we need a quick review of some elementary logic.

In standard Boolean (or "propositional") logic are found signs $-$, &, and $\vee$, for the notions *not*, *and*, *or*, as well as "atomic," i.e., non-compound, statements $P, Q, R, \ldots$, which can assume the Boolean values *true* and *false*. (Boolean values are often called "truth-values".) The sign $\rightarrow$ is commonly used for *if... then* , defined to mean "not... or ." $(P \rightarrow Q)$ is thus true if P is false or Q is true; it is false if P is true and Q is false. Another common sign is $\leftrightarrow$, meaning "if and only if" (two-way "if... then ").

A compound statement built up from $P, Q, \ldots$ by means of $-$, &, $\vee$, $\rightarrow$, and $\leftrightarrow$ is called a *tautology* if it comes out true no matter what Boolean values are assigned to its atomic components $P, Q, \ldots$. For example,

$$((P \rightarrow Q) \rightarrow ((P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow R))$$

is a tautology. Statements $A, B, \ldots, C$ are said to *imply* a statement D if

$$((A \,\&\, B \,\&\, \ldots \,\&\, C) \rightarrow D)$$

is a tautology, or what comes to the same thing, if

$$(A \rightarrow (B \rightarrow \cdots \rightarrow (C \rightarrow D) \cdots))$$

is a tautology. E.g., $(P \rightarrow Q)$ and $(P \rightarrow (Q \rightarrow R))$ imply $(P \rightarrow R)$. Finally, statements are called *equivalent* if they imply each other.

The most spectacular aspect of Gödel's proof was the construction of a statement in the language of the system that asserts its own unprovability. More precisely, Gödel showed how to construct a statement G such that the compound statement

$$(G \leftrightarrow -\Box(G))$$

is provable. Here $\Box(x)$ is the formula expressing provability mentioned above, $\Box(G)$ is the statement that says that (the Gödel number of) G is (the Gödel number of) a provable statement, and $-\Box(G)$ is the negation of this statement. Thus, according to the system, G is true if and only if it is not provable. (In what follows, we shall often eliminate some parentheses and write, e.g., "$\Box G$" instead of "$\Box(G)$.")

The statement G is an analogue of the "liar" sentence:

"attached to its own quotation is false"
attached to its own quotation is false.

This English sentence, like the more familiar paradoxical sentence "*This* very sentence is false," asserts its own falsity (because it *is* the result of attaching the (nonsense) phrase "attached to its own quotation is false" to its own quotation). If in place of the adjective "false," we substitute (say) "short," we obtain a sentence asserting its own shortness. In like manner, we can construct from any English adjective a sentence asserting that it itself possesses the property denoted by the adjective. It was by applying to the formula $-\Box(x)$ devices that are available in the system to do the work done above by "attached" and "quotation" that Gödel showed how to construct a statement G such that $(G \leftrightarrow -\Box G)$ is provable. It should be emphasized that the procedure Gödel used, called *diagonalization*, is quite general and works for any *arbitrary* formula $\Delta(x)$ at all, and not just $-\Box(x)$. Thus given an arbitrary formula $\Delta(x)$ of the system, we can always construct a statement S such that

$$(S \leftrightarrow \Delta(S))$$

is provable.

Axiomatic systems like the ones studied in Gödel's paper satisfy the following three conditions:

1. Every tautology is provable.
2. If $(R \to S)$ is provable, then if R is provable, S is provable.
3. If S is provable, then $\Box S$ is provable.

The second condition asserts that the rule of inference known as *modus ponens*–from $(R \to S)$ and R, infer S–holds in the system. It follows from conditions 1 and 2 that any statement implied by provable statements is also provable: if $R_1, \ldots, R_n$ are all provable and together imply S, then

$$(R_1 \to \cdots \to (R_n \to S) \cdots)$$

is a tautology and hence provable; successive applications of modus ponens then show that S is provable. The third condition gives expression to the idea that *a proof proves itself to be a proof*. Thus if S has a proof in the system, that proof can be proved in the system to be a proof of S, and the statement $\Box S$, which says that S is provable, is itself provable. It usually requires a certain amount of detailed technical argumentation to establish that a given system satisfies condition 3.

Using these three conditions, we can easily see that if the system is consistent, Gödel's G is not provable. For suppose G provable; then by 3, $\Box G$ is also provable; but $-\Box G$ is implied by G and the provable statement $(G \leftrightarrow -\Box G)$; so $-\Box G$ is provable in addition to $\Box G$; thus both halves of a contradiction are provable and the system is inconsistent. Therefore if the system *is* consistent, G is *not* provable.

Assume now that anything provable (again, in the system) is true (as is presumably the case for the systems most commonly studied). Then, because $(G \leftrightarrow -\Box G)$ is provable, it is true; and since G is not provable, $-\Box G$, which expresses the unprovability of G, is also true. G is therefore true as well and is thus an example of *a true statement that is not provable*. Furthermore, since G is true, $-G$ is false and therefore not provable. Thus neither G nor $-G$ is provable, G is an undecidable system, and the system is incomplete, as the first incompleteness theorem asserts.

These arguments work for *any* G such that $(G \leftrightarrow -\Box G)$ is provable: if the system is consistent, no such G is provable; if anything the system proves is true (a much stronger assumption than consistency), any such G is true. And Gödel's diagonalization technique guarantees that in any system of arithmetic such statements G can always be constructed.

The status of a statement asserting its own unprovability is thus easily settled. But what if a statement asserts its own *provability*? The technique of diagonalization, which enabled us to construct a statement G such that $(G \leftrightarrow -\Box G)$ is provable, also enables us to construct a statement H such that $(H \leftrightarrow \Box H)$ is provable. Is such an H true and provable, as it asserts, or false and unprovable, despite what it asserts? Even if we assume that anything provable is true, no easy answer suggests itself. We seem to be in a situation reminiscent of Raymond Smullyan's knight (truth-teller) and knave (liar) puzzles: knights and knaves alike will claim to be knights. Here we have a statement H which claims to be provable. Is it a truth-teller or a liar?

Leon Henkin raised precisely this question in 1952. It was settled in 1955 by the discovery by M. H. Löb of what must be the strangest theorem in all logic. Löb showed that for any statement S,

$$\text{if } (\Box S \to S) \text{ is provable, then so is S.}$$

To appreciate the utter strangeness of Löb's theorem, notice that the hypothesis of the theorem, viz., that $(\Box S \to S)$ is provable, states a condition on S which might seem to be vacuously satisfied by every single S. For if a system proves nothing but truths, as we may suppose is the case with PA and ZF, then statement $(\Box S \to S)$, which says that S is true if provable, will certainly be true; and since this true statement $(\Box S \to S)$ would appear to be completely obvious, how, one might wonder, could it *fail* to be provable? Of course if S is provable, then $(\Box S \to S)$ will certainly be provable (by condition 2), since the tautology $(S \to (\Box S \to S))$ is provable; but one might have supposed the apparently obvious $(\Box S \to S)$ to be provable in a vast number of cases in which S is not provable. Löb's theorem tells us that this supposition is as wrong as it can

be, that $(\Box S \rightarrow S)$ is provable only in the rare circumstance in which S itself is actually provable.

Löb's theorem immediately answers Henkin's question in the affirmative: If $(H \leftrightarrow \Box H)$ is provable, then so is $(\Box H \rightarrow H)$, which is implied by it, and thus by Löb's theorem, so is H. H is a knight.

Löb demonstrated the theorem by observing that two further conditions are satisfied by axiomatic systems of the type we have been considering:

4. Every statement $(\Box(R \rightarrow S) \rightarrow (\Box R \rightarrow \Box S))$ is provable.

5. Every statement $(\Box S \rightarrow \Box\Box S)$ is provable.

Condition 4, in effect, says that condition 2 can be *proved* to hold of each pair of statements R, S. Condition 5, similarly, says that condition 3 can be proved to hold of each statement S. Löb then took as $\Delta(x)$ the formula $(\Box(x) \rightarrow S)$, built up from the formula $\Box(x)$ and the statement S, and applied the diagonalization technique to $(\Box(x) \rightarrow S)$, to construct a statement I such that

$$(I \leftrightarrow (\Box I \rightarrow S))$$

is provable.

The rest of the proof is short but intricate:

(1)	$(I \leftrightarrow (\Box I \rightarrow S))$	By diagonalization.
(2)	$(I \rightarrow (\Box I \rightarrow S))$	Implied by (1).
(3)	$\Box(I \rightarrow (\Box I \rightarrow S))$	From (2), by 3.
(4)	$(\Box(I \rightarrow (\Box I \rightarrow S)) \rightarrow (\Box I \rightarrow \Box(\Box I \rightarrow S)))$	An instance of 4.
(5)	$(\Box I \rightarrow \Box(\Box I \rightarrow S))$	Implied by (3) and (4).
(6)	$(\Box(\Box I \rightarrow S) \rightarrow (\Box\Box I \rightarrow \Box S))$	An instance of 4.
(7)	$(\Box I \rightarrow (\Box\Box I \rightarrow \Box S))$	Implied by (5) and (6).
(8)	$(\Box I \rightarrow \Box\Box I)$	An instance of 5.
(9)	$(\Box I \rightarrow \Box S)$	Implied by (7) and (8).
(10)	$(\Box S \rightarrow S)$	Hypothesis of the theorem.
(11)	$(\Box I \rightarrow S)$	Implied by (9) and (10).
(12)	I	Implied by (1) and (11).
(13)	$\Box I$	From (12), by 3.
(14)	S	Implied by (11) and (13).

It is convenient to introduce into logic the signs $\top$ and $\bot$, meaning *true* and *false* (and pronounced "tee" and "eet"). $\top$ automatically counts as a tautology, as does $-\bot$. Since $(P \rightarrow \bot)$ is false if P is true and true if P is false, $(P \rightarrow \bot)$ is equivalent to $-P$. Further, since $(\bot \rightarrow -P)$, $(\bot \rightarrow -P)$, and $(P \rightarrow (-P \rightarrow \bot))$ are all tautologies, if $\bot$ is provable, then so are P and $-P$, and if both P and $-P$ are provable, $\bot$ is too. Thus $\bot$ is provable if and only if both halves of any contradiction are, and we may therefore *define* a system to be consistent if and only if $\bot$ is not one of its provable statements. (In place of $\bot$ we could have used any disprovable statement, such as $0 = 1$ or $2 + 2 = 5$.)

The sign $\bot$ having been introduced, the statement mentioned in the second incompleteness theorem becomes, simply, $-\Box\bot$. The second theorem can thus be put: if $\bot$ is not provable, then neither is $-\Box\bot$.

Strikingly, as G. Kreisel observed, the second incompleteness theorem is an immediate consequence of Löb's theorem. For if $\bot$ is not provable, then by Löb's theorem (taking $S = \bot$), neither is $(\Box\bot \rightarrow \bot)$, and thus neither is $-\Box\bot$, which is equivalent to $(\Box\bot \rightarrow \bot)$! (Conversely, Kripke has shown how to deduce Löb's theorem from the second incompleteness theorem.)

The facts about provability, consistency, and undecidability in axiomatic systems which we have been examining can be attractively systematized by means of modal logic, to which we now turn.

Inspired by the attempt of Whitehead and Russell to provide, in *Principia Mathematica*, axiomatic foundations for the whole of mathematics, C. I. Lewis initiated the modern develop-

ment of modal logic by publishing a series of papers between 1910 and 1920 in which were set out a number of deductive calculi whose purpose was to characterize axiomatically the logical properties of necessity and allied notions ("calculi," as opposed to "systems," because the atoms $p, q, r, \ldots$ of these calculi are uninterpreted letters, rather than meaningful statements). In the most common contemporary streamlinings of Lewis's calculi, the sign "$\Box$" is the one new basic symbol to be added to the apparatus of Boolean logic. Axioms and rules of inference are then laid down, whereupon it becomes a mathematical question to determine which formulae are derivable from the axioms by repeated applications of the rules of inference. Which axioms and rules are selected will depend, in part, on the concept that "$\Box$" is intended to represent. If it is intended to represent *necessity*, then the axioms of these modern modal calculi will almost always include all tautologies and all formulae of the forms $(\Box(A \to B) \to (\Box A \to \Box B))$ and $(\Box A \to A)$, the justification for the last being: whatever is necessary is true. Often, all formulae of the form $(\Box A \to \Box\Box A)$ are also assumed as axioms. The rules of inference are almost always the two rules *modus ponens* and *necessitation*: from A, derive $\Box A$. The rule of necessitation enables one to derive the formula $\Box A$ if one has (already) *derived* the formula A; it should be stressed that it by no means licenses the unconditional derivation of all formulae of the form $(A \to \Box A)$ ("whatever is true is necessary").

The reader will certainly have noticed resemblances between the modal calculi referred to in the preceding paragraph and conditions 1–5, which are met by the axiomatic systems we have been considering. In fact, a calculus designed to axiomatize the properties of provability can be set out in similar fashion:

Atoms: $p, q, r, \ldots$.
Formation Rules: All atoms are formulae.
$\bot$ and $\top$ are formulae.
If A is a formula, so are $-A$ and $\Box A$.
If A and B are formulae, so are $(A \,\&\, B)$, $(A \vee B)$, $(A \to B)$, and $(A \leftrightarrow B)$.
Axioms: a. All tautologies.
b. All formulae of the form $(\Box(A \to B) \to (\Box A \to \Box B))$.
c. All formulae of the form $(\Box A \to \Box\Box A)$.
Inference Rules: d. Modus ponens: from $(A \to B)$ and A, B follows.
e. Necessitation: from A, $\Box A$ follows.
f. The Löb rule: from $(\Box A \to A)$, A follows.
Derivability: A formula B is said to be derivable from formulae $A_1, \ldots, A_n$ if there is a finite sequence of formulae, each of which
either is an axiom,
or is one of the formulae $A_1, \ldots, A_n$,
or follows by modus ponens from formulae that occur earlier in the sequence,
or follows by necessitation from a formula that occurs earlier in the sequence,
or follows by the Löb rule from a formula that occurs earlier in the sequence,
and in which the last formula is B.

We shall call this calculus "GL", after Gödel and Löb. We abbreviate "B is derivable from $A_1, \ldots, A_n$" as "$A_1, \ldots, A_n \vdash B$." Thus, for example, $(p \to q) \vdash (\Box p \to \Box q)$, as the sequence:

$$(p \to q), \Box(p \to q), (\Box(p \to q) \to (\Box p \to \Box q)), (\Box p \to \Box q)$$

shows. The first formula of this sequence is the formula $(p \to q)$, the second follows from the first by necessitation, the third is a formula of the form $(\Box(A \to B) \to (\Box A \to \Box B))$, and the fourth,

$(\Box p \to \Box q)$, follows from the second and third by modus ponens. A more interesting and important example is

$$(p \leftrightarrow -\Box p) \vdash (p \leftrightarrow -\Box \bot).$$

A derivation of $(p \leftrightarrow -\Box \bot)$ from $(p \leftrightarrow -\Box p)$ can be given in which the Löb rule is *not* used; here is one:

(1)	$(p \leftrightarrow -\Box p)$	
(2)	$(p \to -\Box p)$	Implied by (1).
(3)	$\Box(p \to -\Box p)$	From (2), by e.
(4)	$(\Box(p \to -\Box p) \to (\Box p \to \Box - \Box p))$	An instance of b.
(5)	$(\Box p \to \Box - \Box p)$	Implied by (3) and (4).
(6)	$(\Box p \to \Box\Box p)$	An instance of c.
(7)	$(-\Box p \to (\Box p \to \bot))$	A tautology.
(8)	$\Box(-\Box p \to (\Box p \to \bot))$	From (7), by e.
(9)	$(\Box(-\Box p \to (\Box p \to \bot)) \to (\Box - \Box p \to \Box(\Box p \to \bot)))$	An instance of b.
(10)	$(\Box - \Box p \to \Box(\Box p \to \bot))$	Implied by (8) and (9).
(11)	$(\Box p \to \Box(\Box p \to \bot))$	Implied by (5) and (10).
(12)	$(\Box(\Box p \to \bot) \to (\Box\Box p \to \Box \bot))$	An instance of b.
(13)	$(\Box p \to (\Box\Box p \to \Box \bot))$	Implied by (11) and (12).
(14)	$(\Box p \to \Box \bot)$	Implied by (6) and (13).
(15)	$(\bot \to p)$	A tautology.
(16)	$\Box(\bot \to p)$	From (15), by e.
(17)	$(\Box(\bot \to p) \to (\Box \bot \to \Box p))$	An instance of b.
(18)	$(\Box \bot \to \Box p)$	Implied by (16) and (17).
(19)	$(\Box p \leftrightarrow \Box \bot)$	Implied by (14) and (18).
(20)	$(p \leftrightarrow -\Box \bot)$	Implied by (1) and (19).

The similarities between GL and the Lewisian calculi described above and the differences between them, notably the absence of the popular axiom scheme $(\Box A \to A)$ and the presence of the unfamiliar Löb rule, are explained by the motivation behind GL, which is to give an axiomatic characterization of *provability*, a notion with some properties similar to, but others very different from, the properties which necessity is commonly supposed to possess. (The idea that provability "ought" to behave like necessity is doubtless the main cause of the oddness of the look to Löb's theorem.) The primary "structural" difference between the two notions is that although the fact that whatever is necessary is true is so obvious that it must be regarded as necessary, the fact that whatever is provable is true just cannot, in general, be proved.

Indeed the construction of a statement G such that $(G \leftrightarrow -\Box G)$ is provable might have alerted us to the possibility that statements of the form $(\Box S \to S)$ are not always provable. For since $(((p \to q) \& (q \leftrightarrow -p)) \to q)$ is a tautology (!), $(\Box G \to G)$ and $(G \leftrightarrow -\Box G)$ imply G. And since $(G \leftrightarrow -\Box G)$ is provable, if $(\Box G \to G)$ is provable, G is too, contra consistency. Thus if the system is consistent, $(\Box G \to G)$ is not provable.

The precise connection between the calculus GL and an axiomatic system like PA or ZF (say PA for the sake of definiteness) is this: SUPPOSE that statements $p^*, q^*, \ldots$ of PA are correlated in some way to the atoms $p, q, \ldots$ of modal logic. For any formula A of modal logic, define A^* to be the statement of PA that results when the correlated statements $p^*, q^*, \ldots$ are (respectively) substituted in A for the atoms $p, q, \ldots$ and the sign $\Box$ of modal logic is interpreted as the formula $\Box(x)$ discussed above expressing provability in PA. (Thus if $A = (B \to C)$, $A^* = (B^* \to C^*)$; if $A = (-B)$, $A^* = -(B^*)$; and if $A = (\Box B)$, where $B^* = S$, then $A^* = \Box S$, i.e., $\Box(S)$, the statement that S is provable. So for example, if $A = (\Box q \to p)$, $p^* = G$, and $q^* = H$, then

$A^* = (\Box H \to G)$, i.e., $(\Box(H) \to G)$.) THEN, as conditions 1–5 and Löb's theorem tell us, if $A_1^*, \ldots, A_n^*$ are all provable in the system PA and $A_1, \ldots, A_n \vdash B$, then B^* is also provable in PA. (The same result holds for the system ZF.)

The demonstration of this fact is quite elementary: if $A_1, \ldots, A_n \vdash B$, then there is a finite sequence of formulae each of which is an axiom or one of the As or follows by one of the inference rules from an earlier formula or two earlier formulae, and whose last formula is B. If $A_1^*, \ldots, A_n^*$ are all provable, then, as we can see by proceeding inductively along the sequence, for *every* formula D occurring in the sequence, D^* is provable, and thus B^* is provable. The axioms of GL are taken care of by conditions 1, 4, and 5, and formulas that follow from earlier formulas by an inference rule are handled by condition 2 (if the rule is modus ponens), condition 3 (necessitation), or Löb's theorem (the Löb rule). Thus, e.g., if D follows from the earlier $(C \to D)$ and C by modus ponens and C^* and $(C^* \to D^*)$ are provable, then since $(C \to D)^* = (C^* \to D^*)$, D^* is provable, by condition 2.

Using GL, we can give another proof of the second incompleteness theorem that is closer in spirit to Gödel's original proof. Construct a statement G such that $(G \leftrightarrow -\Box G)$ is provable. We have seen that G is not provable if the system is consistent. Let $A = (p \leftrightarrow -\Box p)$ and $B = (p \leftrightarrow -\Box \bot)$. Let $p^* = G$. Then $A^* = (G \leftrightarrow -\Box G)$ and $B^* = (G \leftrightarrow -\Box \bot)$. Since A^* is provable and $A \vdash B$ (in virtue of the derivation given above), $B^* = (G \leftrightarrow -\Box \bot)$ is provable. Thus if the system is consistent, $-\Box \bot$ is not provable, for otherwise G would be provable, as $(G \leftrightarrow -\Box \bot)$ *is* provable. The core of this proof of the second theorem is the observation that $(p \leftrightarrow -\Box p) \vdash (p \leftrightarrow -\Box \bot)$, for whose demonstration the Löb rule is not required. Thus in this proof of the second theorem, no appeal is made to Löb's theorem, proved some twenty years after Gödel's paper appeared.

Let us look one last time at the *first* incompleteness theorem. If $\Box \bot$ (no "$-$") is not provable, then neither is $-G$, for otherwise $\Box \bot$ would be implied by the provable statements $(G \leftrightarrow -\Box \bot)$ and $-G$; moreover if $\Box \bot$ is not provable, neither is $\bot$ (for if $\bot$ is provable, anything is), and thus the system is consistent and G is unprovable as well. Thus if $\Box \bot$ is unprovable, G is an undecidable statement. And if the system is "reasonable," then $\Box \bot$ is certainly unprovable; otherwise *either* the system is inconsistent *or* the system is consistent but makes the daffy claim that it is inconsistent.

A modal formula B is said to be derivable in GL if it is derivable from the *empty* set of As, i.e., if there is a finite sequence each formula of which either is an axiom or follows from (an) earlier formula(e) by an inference rule and whose last formula is B. As before, if B is derivable in GL, B^* is provable in PA. A noteworthy theorem, due to Robert Solovay and several other logicians, states that there is a correlation $*$ such that for all modal formulae B, if B is not derivable in GL, then B^* is not provable in PA. Thus *a modal formula B is derivable in* GL *if, and only if, B* is provable in* PA *for all correlations* $*$. By finding out facts about the modal calculus GL we are enabled to find out facts about provability in axiomatic systems like PA or ZF. As we shall see, GL has some further notable properties.

The first of these is that it is possible (indeed quite easy) to program a computer to determine whether or not an arbitrary formula of modal logic is derivable in GL. Logicians describe this state of affairs by saying that GL is decidable. (A different usage from that of "undecidable statement.") In contrast, neither PA nor ZF is decidable.

Secondly, GL can be given an attractive analysis via a mathematically "sanitized" version of Leibniz's fantasy of the actual world as one among a number of "possible worlds." Solovay's proof of his theorem makes essential use of a possible-worlds analysis of GL.

Thirdly, GL has an alternative formulation in which modus ponens is the sole rule of inference. Call the formula $\Box D$ the *box* of D. Then the axioms of the reformulation of GL are all tautologies, all formulae of the forms

$$(\Box(A \to B) \to (\Box A \to \Box B)), (\Box A \to \Box\Box A), (\Box(\Box A \to A) \to \Box A),$$

together with the boxes of all these formulae. Exactly the same formulae are derivable in this calculus as are derivable in the original version of GL. If we extend this reformulation of GL, in which neither necessitation nor the Löb rule is a rule of inference, by taking as new axioms all formulae $(\Box A \rightarrow A)$ (but *not* their boxes), then, as Solovay also showed, a formula A is derivable in the extended calculus if and only if A^* is *true* under every correlation of statements $p^*, q^*, \ldots$ of PA with atoms $p, q, \ldots$ of modal logic. These two calculi highlight the difference we have noted between truth and provability: $(\Box p \rightarrow p)^*$ is always true, but provable if and only if p^* is provable.

The most remarkable further fact about GL, however, is the beautiful fixed point theorem, due to D. H. J. de Jongh and G. Sambin. The fixed point theorem in effect shows how GL renders it possible to dispense with a certain type of self-referential characterization, such as "a statement that says that it (itself) is unprovable," in favor of descriptions involving no such self-reference. We have already seen an instance of the kind of phenomenon with which the theorem deals, but to state the theorem precisely, we will need one definition now and one more later.

We define "$A \dashv\vdash B$" to mean that "$A \vdash B$" and "$B \vdash A$."

We have seen that $(p \leftrightarrow -\Box p) \vdash (p \leftrightarrow -\Box \bot)$, from which we can infer that for any statement S, if $(S \leftrightarrow -\Box S)$ is provable, so is $(S \leftrightarrow -\Box \bot)$. With a little more work we can show that

$$(p \leftrightarrow -\Box p) \dashv\vdash (p \leftrightarrow -\Box \bot)$$

and similarly infer that $(S \leftrightarrow -\Box S)$ is provable if and only if $(S \leftrightarrow -\Box \bot)$ is provable. In like manner, we can easily show that

$$(p \leftrightarrow \Box p) \dashv\vdash (p \leftrightarrow \top)$$

and infer that $(S \leftrightarrow \Box S)$ is provable if and only if $(S \leftrightarrow \top)$ is. Some other similar facts about GL are that

$$(p \leftrightarrow \Box - p) \dashv\vdash (p \leftrightarrow \Box \bot),$$

from which we can deduce that a statement says that it is disprovable if and only if it says that the system is inconsistent;

$$(p \leftrightarrow -\Box - p) \dashv\vdash (p \leftrightarrow \bot),$$

whence it follows that a statement says that it is consistent (undisprovable) if and only if it says that $0 = 1$;

$$(p \leftrightarrow (\Box p \rightarrow q)) \dashv\vdash (p \leftrightarrow (\Box q \rightarrow q)),$$

from which it follows that the statement I constructed by Löb for the given S says that S is true if S is provable; and

$$(p \leftrightarrow (\Box p \rightarrow \Box - p)) \dashv\vdash (p \leftrightarrow (\Box\Box \bot \rightarrow \Box \bot)),$$

from which it follows that a statement says that it is disprovable-if-provable if and only if it says that if the system proves its own inconsistency, then it is inconsistent. Some further examples:

$$(p \leftrightarrow (\Box - p \,\&\, q)) \dashv\vdash ((p \leftrightarrow (\Box - q \,\&\, q)),$$

$$(p \leftrightarrow (\Box(p \rightarrow q))) \dashv\vdash (p \leftrightarrow \Box q),$$

and

$$(p \leftrightarrow (q \,\&\, ((\Box p \rightarrow q) \rightarrow \Box - p))) \dashv\vdash (p \leftrightarrow (\Box - q \,\&\, q)).$$

Observe the pattern. In each of the formulae $(p \leftrightarrow A(p))$ to the left of "$\dashv\vdash$" other atoms than p may occur in $A(p)$, but *whenever p occurs in* $A(p)$, *it occurs inside a* $\Box$. In the corresponding formula $(p \leftrightarrow B)$ to the right of "$\dashv\vdash$" p does not occur in B at all and, moreover, B contains no atoms not found in $A(p)$. ($\top$ and $\bot$ are logical signs and not atoms.)

Thus the formulae $(p \leftrightarrow A(p))$ are "self-referential," since p "is defined in terms of" itself, while the corresponding formulae $(p \leftrightarrow B)$ are not.

We define a formula $A(p)$ of modal logic to be *modal in* the atom p if whenever p occurs in $A(p)$, it occurs inside a $\Box$. Thus the formulae $-\Box p, \Box p, (\Box p \rightarrow q), \Box(p \rightarrow q)$, are all modal in p, but the formulae $p, -p, (\Box p \rightarrow p), (p \leftrightarrow -\Box p)$ are not. To formulae that are modal in p, *formulae* of PA (or some other system) can be correlated: $-\Box(x)$ is correlated to $-\Box p, (\Box(x) \rightarrow q^*)$ to $(\Box p \rightarrow q)$, etc. (The atom p corresponds to the *variable* x in $\Box(x)$.)

The fixed point theorem of de Jongh and Sambin asserts that if a formula $A(p)$ is modal in the atom p, then there is a formula B—called a *fixed point* of $A(p)$—such that B does not contain p, B contains only such atoms $q, r, \ldots$ as are found in $A(p)$, and $(p \leftrightarrow A(p)) \dashv\vdash (p \leftrightarrow B)$. For example, if $A(p) = (\Box p \rightarrow q)$, we may take $B = (\Box q \rightarrow q)$. B does not contain p, and the only atom in B, q, occurs in $A(p)$. If $A(p) = \Box - p$, then we may take $B = \Box \bot$. Again, B does not contain p, and since it contains no atoms at all, it contains no atoms not found in $A(p)$. Similarly for the other examples. The content of the fixed point theorem can be put: if $A(p)$ is modal in p, then an "equation" $(p \leftrightarrow A(p))$ can always be solved for p in GL by a modal formula B built up from the variables in $A(p)$ other than p. Moreover, the solution is unique in the sense that if B' is another solution, then $(B \leftrightarrow B')$ is derivable in GL. (The term "fixed point" of course derives from the usage according to which x is a *fixed point* of a function f if $f(x) = x$, i.e., if f leaves x "fixed.")

Diagonalization assures us that for any formula $A(p)$ modal in p and any correlation $q^*, r^*, \ldots$ of statements with atoms $q, r, \ldots$, there will always exist a statement S such that if we let $p^* = S$, then $(p \leftrightarrow A(p))^*$ is provable. The fixed point theorem supplies us with a B not containing p such that $(p \leftrightarrow A(p)) \dashv\vdash (p \leftrightarrow B)$, and thus with a B such that $(p \leftrightarrow B)^*, = (S \leftrightarrow B^*)$, is provable, independently of the correlation S.

An algorithm for computing a fixed point of any formula modal in p can be given and is usually provided by a proof of the fixed point theorem. An interesting strengthening of the theorem states that every $A(p)$ modal in p has a fixed point of low complexity: the maximum number of nested $\Box$s in the fixed point need be no greater than the number of different formulae $\Box D$ contained in $A(p)$.

There are many other connections between modal logic and the theory of formal systems which we have not discussed here, and many problems remain to be solved, of which perhaps the most interesting is that of the incorporation of the quantifiers "all" and "some" into the logic of provability. But in view of the work of Gödel and Löb, it now seems fair to say, as one distinguished logician has put it, that we know the first thing about provability.

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SOME NEW TYPES OF CLOSURE PROPERTIES IN THE PLANE

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The essential results of this paper are:

(1) Every plane set, containing at least three noncollinear points, which contains the center of the circumcircle of the triangle determined by each noncollinear triple of its points, is dense in the plane.

(2b) Every plane set, containing at least three noncollinear points, which is symmetric about the line determined by each pair of points of the set, is dense in the plane.

(3) Let P', Q', R', be three given noncollinear points in the plane. Let A be a plane set, containing more than one point, such that for each two points P, Q in A, the point R for which the triangles PQR and $P'Q'R'$ have the same angles at corresponding vertices and the same counterclockwise orientation of vertices, is also in A.

(3a) Then the convex hull of A is the plane.

(3b-3e) Various results are given concerning cases in which A is or is not dense in the plane.

1. Let us define a subset A of the plane to be "*center-closed*" if it contains at least three non-collinear points and if for any triplet (P, Q, R) of such points of A it contains the circumcenter of the triangle (P, Q, R), i.e., the center of the circle defined by P, Q and R.

If any cartesian coordinate system is applied to the plane, and if $F \subset \mathbb{R}$ is any field, then the set of all points whose coordinates both come from F is a center-closed set. In particular, if F is not the reals, e.g., the rationals, then the resulting set is not the whole plane.

The reader may further prove that, if a center-closed set contains a straight line or a circle segment, it must be identical with the plane.

We now show that

every center-closed subset of the plane is a dense subset of the plane.

DEFINITION. Let $z(P, Q, R)$ denote the center of the circle passing through the points $P = (p_1, p_2)$, $Q = (q_1, q_2)$ and $R = (r_1, r_2)$.

By analytic geometry,

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$$z(P,Q,R) = \frac{\left(\begin{vmatrix} 1, p_1^2 + p_2^2, p_2 \\ 1, q_1^2 + q_2^2, q_2 \\ 1, r_1^2 + r_2^2, r_2 \end{vmatrix}, \begin{vmatrix} 1, p_1, p_1^2 + p_2^2 \\ 1, q_1, q_1^2 + q_2^2 \\ 1, r_1, r_1^2 + r_2^2 \end{vmatrix}\right)}{2\begin{vmatrix} 1, p_1, p_2 \\ 1, q_1, q_2 \\ 1, r_1, r_2 \end{vmatrix}}.$$

Let A be a plane set, containing at least three noncollinear points, such that whenever P, Q and R are noncollinear points of A, then $z(P,Q,R)$ is also a point of A. Let B be the closure of A. Note that the formula for $z(P,Q,R)$ is a continuous function of P, Q, and R (the denominator is not zero unless P, Q, and R are collinear), so for all noncollinear points P, Q, and R of B, $z(P,Q,R)$ must also be an element of B. Thus B is center-closed.

LEMMA. *B is convex.*

Proof of Lemma. Otherwise, there must be a line l containing a point P not in B such that l contains points of B on either side of P. Since B is closed, we may take Q and R to be the points of B on l which are closest to P on either side of P. Then if S is the midpoint of segment QR, we know that S is not a point of B. If T is any point of B that is not on l, then $z(Q,R,T)$ is on the perpendicular bisector l' of QR. Since B is closed, we may take U to be the point of B on l' that is closest to S on the same side of S as $z(Q,R,T)$. There must be points of B on l' on the other side of S from U (for example, $z(Q,U,R)$). Let V be the point of B on l' closest to S on this side. The perpendicular bisector l'' of segment UV contains at least one pair of points of B symmetric about l', for example, $z(Q,U,V)$ and $z(R,U,V)$. Since B is closed we may take W and X to be the pair of points of B on l'', symmetric about l', that is closest to l'. Now if segment WX is shorter than or equal to segment UV, then $z(W,U,X)$ is a point of B on l' between U and V, contradicting the construction of U and V. And if WX is longer than UV, the pair of points $z(U,V,X)$ and $z(U,V,W)$ is a pair of points of B and l'', symmetric about l', closer to l' than W and X, contradicting the construction of W and X. This contradiction proves the lemma.

Since B is convex it must contain the interior of at least one triangle. So if P is any point of the plane, a circle with center P passing through the interior of this triangle will contain at least three points of B, and so P must be a point of B. Since P was arbitrary, B must be the whole plane.

Thus we have proved that A is dense in the plane.

2. Let us define a subset A of the plane to be "*mirror-closed*" if it contains at least three non-collinear points and if for each triplet (P,Q,R) of non-collinear points of A it contains the point S that lies symmetrically to R about the line through P and Q. This property is just the symmetry of A with respect to all lines running through any two points of A.

The properties of mirror-closed sets are similar to those of the center-closed sets we discussed in the last chapter. As in Section 1, one can easily prove that there is a mirror-closed set different from the plane and that a mirror-closed set is identical with the plane if it contains a straight line or a circle segment. We leave these proofs as a task for the readers. As in Section 1, we now show in two steps that A is a dense subset of the plane.

(a) *The convex hull of a mirror-closed set is identical with the plane.*

Let A denote a mirror-closed set. It is clear that it suffices to show that for a suitable cartesian x-y-coordinate system and for suitable positive real numbers a and b A contains all points of the rectangular lattice $L_{a,b} = \{(ma, nb);\ m, n \in \mathbb{Z}\}$.

Let P,Q,R be three non-collinear points of A. We construct the mirror-image S of R with

respect to the line through P and Q and write $S = (PQ) \cdot R$ for short. Then $S \in A$. Since P, Q, R are not collinear, $\overline{RS}$ cannot contain both P and Q. Let us assume $P \notin \overline{RS}$. Let $T = (RS) \cdot P$. Then $T \in A$ and the points P, R, S, T are the vertices of a rhombus (Fig. 1).

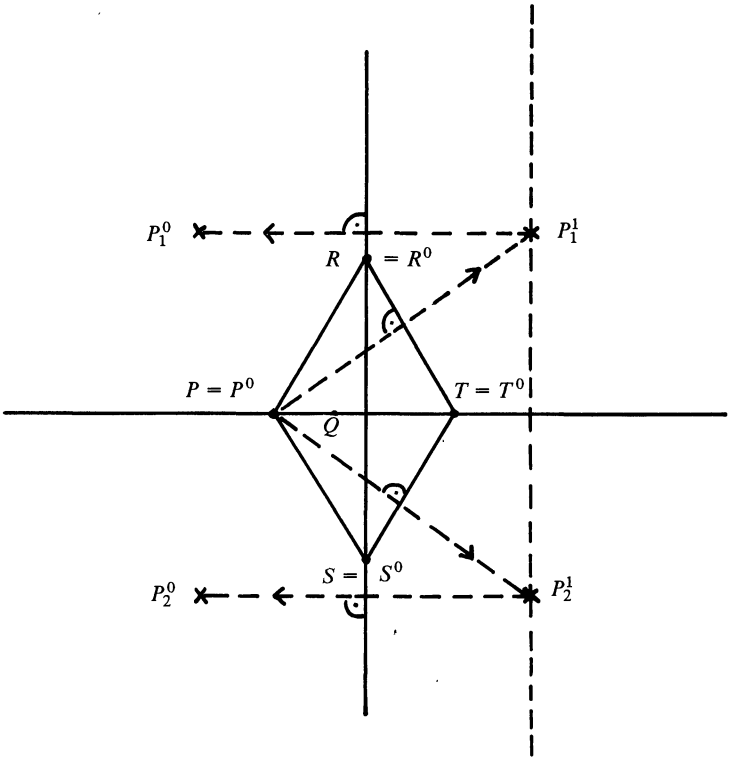


FIG. 1

Since

$$360° = \sphericalangle(TRP) + \sphericalangle(STR) + \sphericalangle(PST) + \sphericalangle(RPS) = 2 \cdot \sphericalangle(TRP) + 2 \cdot \sphericalangle(STR),$$

one of the angles $\sphericalangle(TRP)$ and $\sphericalangle(STR)$ is not greater than 90°. By symmetry we can assume $\sphericalangle(TRP) \leqslant 90°$. Then

$$90° > \sphericalangle(TPR) = \sphericalangle(RTP) \geqslant 45°,$$

from which we easily conclude that neither $P_1^1 = (RT) \cdot P$ nor $P_2^1 = (ST) \cdot P$ is collinear with R and S. By symmetry $\overline{P_1^1 P_2^1}$ is parallel to $\overline{RS}$.

Let $P_1^0 = (RS) \cdot P_1^1$, $P_2^0 = (RS) \cdot P_2^1$, $P_0^0 = (P_1^0 P_1^1) \cdot P_2^0$, and $P_0^1 = (P_1^0 P_1^1) \cdot P_2^1$. By repeated use of mirror-closure, using symmetry about lines parallel to the sides of the rectangle $P_0^0 P_0^1 P_1^1 P_1^0$, we can construct the points $P_j^i \in A$, $i, j \in \mathbb{Z}$, that define a rectangular lattice $L_{a,b}$, where $a = d(P_0^0, P_0^1)$ and $b = d(P_0^0, P_1^0)$.

(b) *Every mirror-closed set is a dense subset of the plane.*

Let A be a mirror-closed set and B the closure of A. By the continuity of $(PQ) \cdot R$ as a function of P, Q, and R, we conclude that B is mirror-closed. By (a) it suffices to show that B is convex.

If B is not convex, there must be a line l containing a point X not in B such that l contains points of B on either side of X. Since B is closed, we may take P and Q to be the points of B on

l which are closest to X on either side of X.

With these points P and Q and an arbitrary third point R of B not collinear with P and Q, we construct the lattice $L_{a,b}$ as in part (a). By the mirror-closure of B we conclude that B contains $L_{a,b}$.

Taking the notations of part (a) (Fig. 1), we see that $D_1^k := (PP_1^k) \cdot Q$ and $D_2^k := (PP_2^k) \cdot Q$ are points of B. If k is great enough, D_1^k and D_2^k lie on the same side of the midperpendicular of P and Q as Q. Then $D := (D_1^k D_2^k) \cdot Q$ is a point of B between P and Q, contradicting the construction of P and Q.

3. Let r be a positive real number and α an angle, $0 < \alpha < 180°$. We define a subset A of the plane to be "(r, α)-closed", if A contains at least two points and if for any two points P, Q of A it contains the point R which is defined by the following three conditions: $d(P, R) = r \cdot d(P, Q)$, $\sphericalangle(QPR) = \alpha$, P, Q, R are the corners of a triangle written counterclockwise. Thus in a (r, α)-closed set each pair (P, Q) of points of A together with their associated third point R define a triangle similar to and oriented the same way as the special triangle $(P', Q', R') \subset \mathbb{R}^2$, where $P' = (0, 0)$, $Q' = (1, 0)$ and $R' = (r\cos(\alpha), r\sin(\alpha))$. Thus each point $R' = (x, y)$ with $y > 0$ defines a certain (r, α)-closure.

With the notations as above, we observe that A in general does not contain the point S symmetric to R with respect to the midperpendicular of $\overline{PQ}$, since an exchange of P and Q in the definition yields only the point symmetric to R with respect to the midpoint of $\overline{PQ}$.

Let us denote a subset A of the plane "*triangle-closed*", if it is (r, α)-closed for some r and α. It is somewhat surprising that triangle-closed sets are not dense subsets of the plane in general. (However, their convex hull is always the plane; see part (a).) The simplest example is the square lattice, but we shall become familiar with some more examples in (c) and (d).

Our main task concerning triangle-closures is to find all pairs (r, α) (respectively, all R') such that every (r, α)-closed set is dense in the plane. This problem is partly answered in parts (b) and (c). For example, $r < 1$ is such a condition.

(a) *The convex hull of a triangle-closed set is the plane.*

Proof. Let A be triangle-closed, let S and T be any two points of A, and let d denote their distance from each other. Let U be an arbitrary point of A, a distance at least $d/2$ from S. For the sake of notation, we refer to S as S_1. We define a sequence $S_1, S_2, S_3, \ldots$, as follows: given S_n and U, we apply the hypothesis to find a point S_{n+1} so that the angles of the triangle $S_n U S_{n+1}$ will be the same as the corresponding angles of the reference triangle $P'Q'R'$, and these triangles will have the same counterclockwise orientation of their vertices as listed above. Since the triangles $S_n U S_{n+1}$ all have the same angle at U and all are subsets of the convex hull of A (since their vertices are points of A), then a finite number of these triangles serve to surround U with some neighborhood of radius r which is a subset of the convex hull of A. Since for all such U the distance from S to U is at least $d/2$, it is easy to show that for all such U the set of values of r has a positive lower bound. A similar argument applies to all points $\tilde{U} \in A$ a distance at least $d/2$ from T. Obviously every point of A is such a point U or such a point $\tilde{U}$, and so there is some positive r_0 such that for all points $W \in A$ the convex hull of A contains the disc $B_{r_0}(W)$. Since A is a planar set, any point V in its convex hull lies in some triangular region whose vertices are points of A, and so $B_{r_0}(V)$ lies in the convex hull of A. This shows that no point of the plane can be a boundary point of the convex hull of A, which in turn shows that the convex hull of A must be the entire plane.

(b) *With the notations defined at the beginning, let r and α be such that $d(P', R') < 1$ or $d(Q', R') < 1$ (or both, see Fig. 2). Then every (r, α)-closed set is a dense subset of the plane.*

Let A be a (r, α)-closed set, where r and α are as described. Since we know from (a) that the convex hull of A is the plane, for the density of A it suffices to show that the midpoint of two

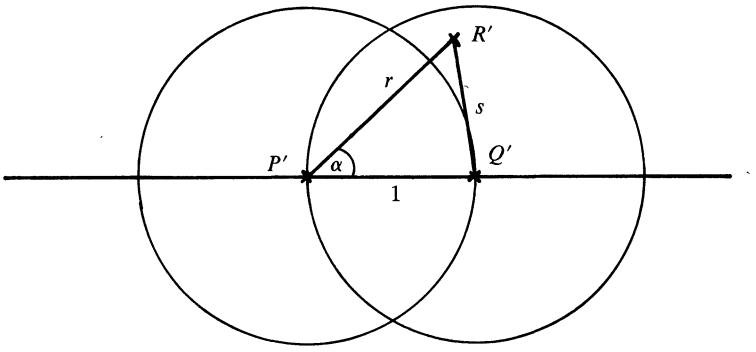


FIG. 2

points of $A \cup H(A)$ is a point of $A \cup H(A)$; $H(A)$ denotes the set of accumulation points of A.

For this it suffices to show that the midpoint of two points of A is a point of $H(A)$.

So let P and Q be two points of A. Let M denote their midpoint and a their distance. A contains the unique point R_1 associated to (P, Q) by the (r, α)-closure; $\sphericalangle(R_1PQ) = \alpha$, $d(P, R_1) = r \cdot a$. Let $d(Q, R_1) = s \cdot a$ for some $s \in \mathbb{R}$. Then by assumption we have $r < 1$ or $s < 1$. Moreover, with $\beta = \sphericalangle(PQR_1)$ we have $0 < \alpha, \beta < 180°$.

We first assume $r < 1$. We define $R_2, \ldots, R_n \in A$ just like in part (a). Thus we find a sequence $\{R_i\}_{i \in \mathbb{N}} \subset A$ such that $d(P, R_i) = r^i \cdot a$ for all $i \in \mathbb{N}$.

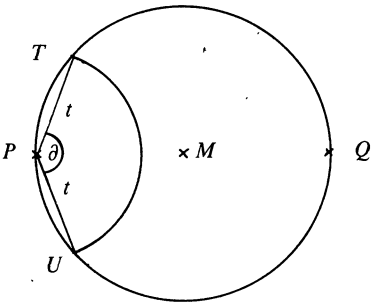


FIG. 3

Let ∂ be such that $\alpha < \partial < 180°$. We define the points T and U by $d(T, M) = d(U, M) = a/2$ and $\sphericalangle(TPM) = \sphericalangle(MPU) = \partial/2$ (Fig. 3). Let t denote the distance of P from T. There is a natural number k such that $r^k a < t$. Then infinitely many R_i with $i > k$ must lie in the sector $\{S; \sphericalangle(SPU) < \partial, d(S, P) < t$, the points S, P, U define a triangle written counter-clockwise$\}$, because $\sphericalangle(R_{i+1}PR_i) = \alpha < \partial$. Let R_j be one of these points. Then $d(R_j, M) = q \cdot a/2$ for some $q < 1$.

Performing all steps a second time point-symmetric to M yields a point $\tilde{R}_j \in A$. The midpoint of R_j and $\tilde{R}_j$ is M.

As to (P, Q) we now apply the procedure above to $(R_j, \tilde{R}_j)$. Continuing this way inductively we find two sequences of points of A converging to M, showing that M is an accumulation point of A.

The proof for the case $s < 1$ is symmetric to the preceding proof.

(c) *Let A be a $(1, \alpha)$-closed set. Then in general A is a dense subset of the plane. The only exceptions appear for $\alpha = 60°$, $\alpha = 90°$, and $\alpha = 120°$.*

Let P and Q be two arbitrary distinct points of A. As in the proof of (3b), we construct the

sequence $R_1, R_2, R_3, \ldots$ of points of A from P and Q by applying the definition of (r, α)-closure. In this case, however, since $r = 1$, this sequence does not converge to P but instead each R_i lies on the circle centered at P through Q. For each such R_i, define its associated angle g_i to be the angle measured counterclockwise from the ray PQ to the ray PR_i, so that $0° \leqslant g_i < 360°$.

(1) If for any R_i, $0° < g_i < 60°$ or $300° < g_i < 360°$, then $d(R_i, Q) < d(P, Q)$. Since a corresponding construction can be applied to any pair of points of A in place of P and Q, A is triangle-closed in a way which implies that A is dense by result (3b).

(2) If for any R_i, $60° < g_i < 72°$, then $300° < g_{5i} < 360°$, which implies that A is dense by case (1).

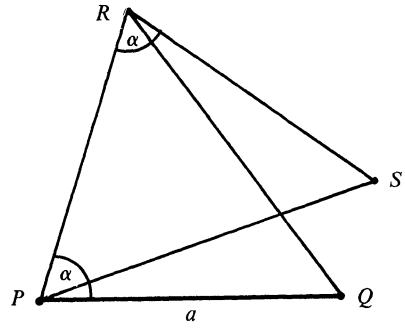


FIG. 4

(3) If for any R_i, $g_i = 72°$, then we can apply the definition of $(1, 72°)$-closure to R_1 and P to find a point S in A, as in Fig. 4. It can be shown that $d(S, Q) < d(P, Q)$, and so A is triangle-closed in a way which, by result (3b), implies that A is dense.

(4) If for any R_i, $72° < g_i < 90°$, then either $0° < g_{5i} < 60°$ or $300° < g_{4i} < 360°$ (or both), which implies that A is dense by case (1).

(5) If for any R_i, $90° < g_i < 120°$, then either $0° < g_{4i} < 60°$ or $300° < g_{3i} < 360°$ (or both), and A is dense by case (1).

(6) If for any R_i, $120° < g_i < 140°$, then $0° < g_{3i} < 60°$, so by case (1) A is dense.

(7) If for any R_i, $g_i = 140°$, then $g_{5i} = 340°$, and A is dense by case (1).

(8) If for any R_i, $140° < g_i < 150°$, then $60° < g_{3i} < 90°$, and by cases (2), (3), and (4), A is dense.

(9) If for any R_i, $g_i = 150°$, then $g_{5i} = 30°$, and A is dense by case (1).

(10) If for any R_i, $150° < g_i < 180°$, then $300° < g_{2i} < 360°$, and A is dense by case (1).

So, in particular, if α is any angle between 0° and 180°, other than 60°, 90°, or 120°, then R_1 will satisfy one of the cases (1)–(10) above; hence A is dense.

What about the cases $\alpha = 60°, 90°, 120°$? It is easy to see that if we take two arbitrary points of the plane $\mathbb{R}^2$, say $P = (0,0)$ and $Q = (1,0)$, the $(1, \alpha)$-closure of $\{P, Q\}$ is an isometric lattice for $\alpha = 60°$, a square lattice for $\alpha = 90°$, and a hexagonal lattice for $\alpha = 120°$ (Figs. 5, 6 and 7). (The $(1, \alpha)$-closure of A denotes the smallest $(1, \alpha)$-closed set containing A.)

(d) *By an argument symmetric to that in* (c) *one sees that* $R' \in C_1(Q')$ *defines a triangle closure such that* A *is dense for each* (r, α)*-closed set* A *if and only if the angle* $\sphericalangle(R'Q'P')$ *is not equal to* 60°, 90° *or* 120°.

We now have examined all R' in the union $D = \overline{B_1(P')} \cup \overline{B_1(Q')}$ of closed discs. *It is not clear what happens for arbitrary* R'. Are all exceptions (points R' whose corresponding triangle closure does not imply density) isolated points?

However, we shall see that there are infinitely many exceptions.

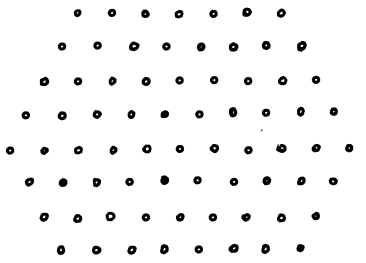


FIG. 5

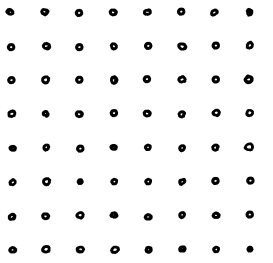


FIG. 6

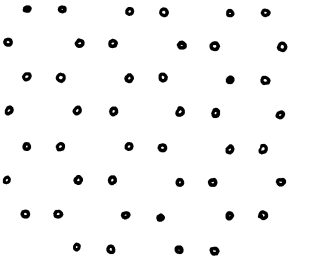


FIG. 7

(e) *Let L^1 denote the square lattice that is the $(1, 90°)$-closed hull of $P' = (0,0)$, $Q' = (1,0) \subset R^2$ and L^2 the isometric lattice that is the $(1, 60°)$-closure of $\{P', Q'\}$. Then each point R' of L^1 or L^2 which does not lie on the x-axis defines a (r, α)-closure property which does not induce density* (i.e., not every (r, α)-closed set is a dense subset of the plane).

This is obvious since L^1 is a non-dense (r, α)-closed set if $R' \in L^1$ and L^2 is an example if $R' \in L^2$.

As an illustration we prove that for any natural k the set

$$L_k = \{P_i + n(k^2, k) + m(0, k^3 + k);\ n, m \in \mathbb{Z},\ i = 1, 2, 3, 4\},$$

where $P_1 = (0,0)$, $P_2 = (1,0)$, $P_3 = (1, -k)$, $P_4 = (0, k)$, is $(k, 90°)$-closed. The author believes that indeed L_k is the $(k, 90°)$-closure of the points $(0,0)$ and $(1,0)$, but could prove this only for the case $k = 1$ (square lattice) and the case $k = 2$. The set L_2 is sketched in Fig. 8.

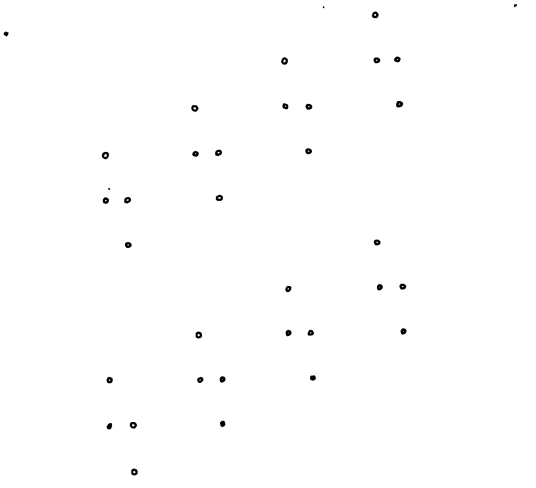


FIG. 8

Proof. We first prove that L_k is $(k, 90°)$-closed. If

$$A = P_i + n(k^2, k) + m(0, k^3 + k), \qquad B = P_j + n'(k^2, k) + m'(0, k^3 + k)$$

are two arbitrary points of L_k, the point C associated to A and B by the $(k, 90°)$-closure is $C = B + k(-y, x)$, where $(x, y) = A - B$. We have to show $C \in L_k$.

Let P_{ix} be the x-coordinate of P_i, etc. We easily compute that with

$$D_{ij} = \big(P_{jx} + k(P_{jy} - P_{iy}), P_{jy} + k(P_{ix} - P_{jx})\big)$$

we have

$$C = D_{ij} + \big(2n' - n + (m' - m)(k^2 + 1)\big)(k^2, k) + (n - n' + m)(0, k^3 + k).$$

Thus, $C \in L_k$ if and only if $D_{ij} \in L_k$. Busy computing yields

$$D_{11} = P_1, \quad D_{12} = P_3, \quad D_{13} = P_3 - (k^2, k), \quad D_{14} = P_1 + (k^2, k),$$

$$D_{21} = P_4, \quad D_{22} = P_2, \quad D_{23} = P_2 - (k^2, k), \quad D_{24} = P_4 + (k^2, k),$$

$$D_{31} = P_1 + (k^2, k), \quad D_{32} = P_3 + (k^2, k), \quad D_{33} = P_3, \quad D_{34} = P_1 + 2(k^2, k),$$

$$D_{41} = P_4 - (k^2, k), \quad D_{42} = P_2 - (k^2, k), \quad D_{43} = P_2 - 2(k^2, k), \quad D_{44} = P_4,$$

showing that $D_{i,j} \in L_k$ for all i and j.

It remains to show that L_2 is the $(2, 90°)$-closure H_2 of the points $P_1 = (0,0)$ and $P_2 = (1,0)$. By the first part of the proof it suffices to show that H_2 contains L_2. Of course, $P_3, P_4 \in H_2$. Since L_2 is the union of the images of $S = \{P_1, \ldots, P_4\}$ under all translations defined by the vectors

$$\{n(4,2) + m(0,10);\ n, m \in \mathbb{Z}\},$$

it suffices to show that $S - (4,2) \subset H_2$ and $S + (0,10) \subset H_2$. This entertaining construction is left to the reader.

The reader is invited also to devise and investigate similar geometric closure properties of plane (and higher-dimensional) sets.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the forty-fourth William Lowell Putnam Mathematical Competition, held on December 3, 1983, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its winning team were: Bradley W. Brock, Charles J. Cuny, and Alan G. Murray; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of **Washington University**, St. Louis, Missouri. The members of its team were: Paul H. Burchard, Edward A. Shpiz, and Richard A. Stong; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of the **University of Waterloo**, Waterloo, Ontario, Canada. The members of its team were: David W. Ash, W. Ross Brown, and Bev I. Cope; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of its team were: Gregg N. Patruno, Daniel J. Scales, and Kevin M. Walker; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **University of Chicago**, Chicago, Illinois. The members of its team were Keith A. Ramsay, Michael P. Spertus, and David S. Yuen; each was awarded a prize of fifty dollars.

The five highest-ranking individual contestants, in alphabetical order, were **David W. Ash**, University of Waterloo; **Eric D. Carlson**, Michigan State University; **Noam D. Elkies**, Columbia University; **Michael J. Larsen**, Harvard University; and **Gregg N. Patruno**, Princeton University. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest-ranking individuals, in alphabetical order, were *Thomas O. Andrews*, Yale University; *Joel Friedman*, Harvard University; *Alan G. Murray*, California Institute of Technology; *Richard A. Stong*, Washington University, St. Louis; and *David S. Yuen*, University of Chicago. Each of these students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order received honorable mention: *University of Alberta*, with team members Arthur B. Baragar, Robert P. Morewood, and David S. Salopek; *Harvard University*, with team members Zachary M. Franco, Joel Friedman, and Michael J. Larsen; *Memorial University of Newfoundland*, with team members Quoc T. Pham, Michael J. Sandys-Wunsch, and Arthur P. Smith; *Queen's University*, with team members Neale Ginsburg, Teddy Hsu, and Michael J. Swain; and *Yale University*, with team members Thomas O. Andrews, Alan S. Edelman, and Nathaniel E. Glasser.

Honorable mention was achieved by the following thirty-four individuals, named in alphabetical order: *Bruce W. K. Brandt*, Harvard University; *Bradley W. Brock*, California Institute of Technology; *W. Ross Brown*, University of Waterloo; *Paul H. Burchard*, Washington University, St. Louis; *Pang-Chieh Chen*, California Institute of Technology; *John J. Chew*, University of Toronto; *Charles J. Cuny*, California Institute of Technology; *David B. Delaney*, Case Western Reserve University; *Stephen A. DiPippo*, Brown University; *Yong Yao Du*, University of Waterloo; *Benji N. Fisher*, Harvard University; *Daniel J. Goldstein*, University of Chicago; *Frederic M. Gourdeau*, Université Laval; *Everett W. Howe*, California Institute of Technology; *Paul S. Hsieh*, Massachusetts Institute of Technology; *Teddy Hsu*, Queen's University; *Jung C. Im*, California Institute of Technology; *Russell G. Impagliazzo*, Wesleyan University; *Eric H. Kawamoto*, California Institute of Technology; *Richard W. Kenyon*, Rice University; *Gary R. Lawlor*, Brigham Young University; *Stephen T. Mark*, Yale University; *David I. McIntosh*, University of Waterloo; *Robert P. Morewood*, University of Alberta; *Howard M. Pollack*, Harvard University; *Keith A. Ramsay*, University of Chicago; *James R. Roche*, University of Notre Dame; *James R. Russell*, Massachusetts Institute of Technology; *Daniel J. Scales*, Princeton University; *Arthur P. Smith*, Memorial University of Newfoundland; *Christopher R. Stover*, Swarthmore College; *John M. Sullivan*, Harvard University; *James C. Yeh*, Princeton University; and *Thomas M. Zavist*, Rice University.

The other individuals who achieved ranks among the top 101, in alphabetical order of their schools, were: University of Alberta, *Arthur B. Baragar*; University of British Columbia, *Lawrence D. Hammick*, *Thomas R. Stevenson*; California Institute of Technology, *Christian G. Bower*, *Jonathan S. Shapiro*; University of California, Davis, *Michael P. Quinn*; University of California, San Diego, *Peter M. De Marzo*; University of California, Santa Barbara, *Emerson S. Fang*, *John R. Kelly*; Case Western Reserve University, *Magnus R. Karlsson*, *Kevin E. Kelso*; University of Chicago, *Geoffrey R. Harris*; Colorado State University, *Jorg A. Brown*; The

Cooper Union, *Tsz Mei Ko*; Harvard University, *Frederick R. Adler*, *Glen D. Ellison*, *Alfred D. Shapere*, *David Wolland*, *David T. Wu*; Harvey Mudd College, *Arthur A. Middleton*; University of Illinois, Urbana-Champaign, *Eric K. Lossin*; Université Laval, *David Bernier*; University of Maryland, Baltimore County, *Gary S. Katzenberger*; Massachusetts Institute of Technology, *Jonathan W. Aronson*, *Andrew E. Gelman*, *Chun-Nip Lee*, *Warren D. Smith*; Memorial University of Newfoundland, *Michael J. Sandys-Wunsch*; Michigan State University, *Erin J. Schram*; University of Michigan, Ann Arbor, *Fred I. Diamond*; University of New Brunswick, *Christian Friesen*; University of North Carolina, Chapel Hill, *Leick D. Robinson*; Northwestern University, *Wayne W. Wheeler*; Oberlin College, *Mark R. Hanisch*, *Iwan Pranata*; University of Pennsylvania, *Mark E. Banilower*, *William A. Graham*; Princeton University, *Troy W. Barbee III*, *Rama R. Kocherlakota*, *Burt J. Totaro*, *Stephen A. Vavasis*, *Kevin M. Walker*; Queen's College of the City University of New York, *Boris Aronov*; Rice University, *Garrett T. Biehle*; Rose-Hulman Institute of Technology, *Daniel W. Johnson*; Stanford University, *Washington Taylor*; University of Texas, Austin, *Andrew Chin*; University of Utah, *Eric M. Weeks*; Washington University, St. Louis, *William H. Paulsen*, *Edward A. Shpiz*; University of Waterloo, *Todd A. Cardno*, *Bev I. Cope*, *Charles S. A. Timar*; University of Wisconsin, Madison, *Chris S. Jantzen*, *John H. Rickert*; University of Wisconsin, Oshkosh, *Douglas G. Kilday*; and Yale University, *Alan S. Edelman*.

There were 2055 individual contestants from 345 colleges and universities in Canada and the United States in the competition of December 3, 1983. Teams were entered by 256 institutions.

The Questions Committee for the forty-fourth competition consisted of Douglas A. Hensley (Chairman), Melvin Hochster, and Bruce Reznick; they composed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS

Problem A-1

How many positive integers n are there such that n is an exact divisor of at least one of the numbers

$$10^{40}, 20^{30}?$$

Problem A-2

The hands of an accurate clock have lengths 3 and 4. Find the distance between the tips of the hands when that distance is increasing most rapidly.

Problem A-3

Let p be in the set $\{3,5,7,11,\dots\}$ of odd primes and let

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2}.$$

Prove that if a and b are distinct integers in $\{0,1,2,\dots,p-1\}$ then $F(a)$ and $F(b)$ are not congruent modulo p, that is, $F(a) - F(b)$ is not exactly divisible by p.

Problem A-4

Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero. $\left[\text{As usual, } \binom{m}{r} = \frac{m!}{r!(m-r)!}.\right]$

Problem A-5

Prove or disprove that there exists a positive real number u such that $[u^n] - n$ is an even integer for all positive integers n.

Here $[x]$ denotes the greatest integer less than or equal to x.

Problem A-6

Let $\exp(t)$ denote e^t and

$$F(x) = \frac{x^4}{\exp(x^3)} \int_0^x \int_0^{x-u} \exp(u^3 + v^3)\, dv\, du.$$

Find $\lim_{x\to\infty} F(x)$ or prove that it does not exist.

Problem B-1

Let v be a vertex (corner) of a cube C with edges of length 4. Let S be the largest sphere that can be inscribed in C. Let R be the region consisting of all points p between S and C such that p is closer to v than to any other vertex of the cube. Find the volume of R.

Problem B-2

For positive integers n, let $C(n)$ be the number of representations of n as a sum of nonincreasing powers of 2, where no power can be used more than three times. For example, $C(8) = 5$ since the representations for 8 are:

$$8, \quad 4+4, \quad 4+2+2, \quad 4+2+1+1, \quad \text{and} \quad 2+2+2+1+1.$$

Prove or disprove that there is a polynomial $P(x)$ such that $C(n) = [P(n)]$ for all positive integers n; here $[u]$ denotes the greatest integer less than or equal to u.

Problem B-3

Assume that the differential equation

$$y''' + p(x)y'' + q(x)y' + r(x)y = 0$$

has solutions $y_1(x)$, $y_2(x)$, and $y_3(x)$ on the whole real line such that

$$y_1^2(x) + y_2^2(x) + y_3^2(x) = 1$$

for all real x. Let

$$f(x) = \left(y_1'(x)\right)^2 + \left(y_2'(x)\right)^2 + \left(y_3'(x)\right)^2.$$

Find constants A and B such that $f(x)$ is a solution to the differential equation

$$y' + Ap(x)y = Br(x).$$

Problem B-4

Let $f(n) = n + [\sqrt{n}]$ where $[x]$ is the largest integer less than or equal to x. Prove that, for every positive integer

m, the sequence

$$m, f(m), f(f(m)), f(f(f(m))), \ldots$$

contains at least one square of an integer.

Problem B-5

Let $\|u\|$ denote the distance from the real number u to the nearest integer. (For example, $\|2.8\| = .2 = \|3.2\|$.) For positive integers n, let

$$a_n = \frac{1}{n}\int_1^n \left\| \frac{n}{x} \right\| dx.$$

Determine $\lim_{n\to\infty} a_n$. You may assume the identity

$$\tfrac{2}{1}\cdot\tfrac{2}{3}\cdot\tfrac{4}{3}\cdot\tfrac{4}{5}\cdot\tfrac{6}{5}\cdot\tfrac{6}{7}\cdot\tfrac{8}{7}\cdot\tfrac{8}{9}\cdot\cdots = \frac{\pi}{2}.$$

Problem B-6

Let k be a positive integer, let $m = 2^k + 1$, and let $r \neq 1$ be a complex root of $z^m - 1 = 0$. Prove that there exist polynomials $P(z)$ and $Q(z)$ with integer coefficients such that

$$(P(r))^2 + (Q(r))^2 = -1.$$

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant i \geqslant 0$ is the number of students among the top 195 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (155, 26, 3, 0, 0, 0, 0, 0, 4, 5, 1, 1)

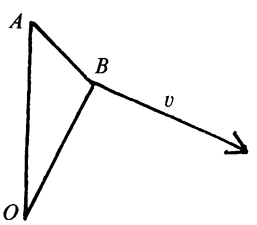
For d and m in $Z^+ = \{1, 2, 3, \ldots\}$, let $d|m$ denote that d is an integral divisor of m. For m in Z^+, let $\tau(m)$ be the number of d in Z^+ such that $d|m$. The number of n in Z^+ such that $n|a$ or $n|b$ is

$$\tau(a) + \tau(b) - \tau(\gcd(a, b)).$$

Also $\tau(p^s q^t) = (s+1)(t+1)$ for p, q, s, t in Z^+ with p and q distinct primes. Thus the desired count is

$$\begin{aligned}\tau(2^{40}\cdot 5^{40}) + \tau(2^{60}\cdot 5^{30}) - \tau(2^{40}\cdot 5^{30}) &= 41^2 + 61\cdot 31 - 41\cdot 31 \\ &= 1681 + 620 = 2301.\end{aligned}$$

A-2. (40, 21, 55, 8, 3, 8, 1, 1, 17, 16, 22, 3)



Let OA be the long hand and OB be the short hand. We can think of OA as fixed and OB as rotating at constant speed. Let v be the vector giving the velocity of point B under this assumption. The rate of change of the distance between A and B is the component of v in the

direction of AB. Since v is orthogonal to OB and the magnitude of v is constant, this component is maximal when $\measuredangle OBA$ is a right angle, i.e., when the distance AB is $\sqrt{4^2 - 3^2} = \sqrt{7}$.

Alternatively, let x be the distance AB and $\theta = \measuredangle AOB$. By the Law of Cosines,

$$x^2 = 3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos\theta = 25 - 24\cos\theta.$$

Since $d\theta/dt$ is constant, we may assume units chosen so that θ is also time t. Now

$$2x\frac{dx}{d\theta} = 24\sin\theta, \quad \frac{dx}{d\theta} = \frac{12\sin\theta}{\sqrt{25 - 24\cos\theta}}.$$

Since $dx/d\theta$ is an odd function of θ, $|dx/ds|$ is a maximum when $dx/d\theta$ is a maximum or a minimum. Since dx/ds is a periodic differentiable function of θ, $d^2x/ds^2 = 0$ at the extremes for dx/ds. For such θ,

$$12\cos\theta = x\frac{d^2x}{d\theta^2} + \left(\frac{dx}{d\theta}\right)^2 = \left(\frac{dx}{d\theta}\right)^2 = \frac{144\sin^2\theta}{x^2}.$$

Then

$$x^2 = \frac{12\sin^2\theta}{\cos\theta} = \frac{12 - 12\cos^2\theta}{\cos\theta} = 25 - 24\cos\theta,$$

and it follows that

$$12\cos^2\theta - 25\cos\theta + 12 = 0.$$

The only allowable solution for $\cos\theta$ is $\cos\theta = 3/4$ and hence $x = \sqrt{25 - 24\cos\theta} = \sqrt{25 - 18} = \sqrt{7}$.

A-3. (72, 15, 7, 1, 0, 0, 0, 0, 3, 6, 14, 77)

$$F(n) = 1 + 2n + 3n^2 + \cdots + (p-1)n^{p-2},$$
$$nF(n) = n + 2n^2 + \cdots + (p-2)n^{p-2} + (p-1)n^{p-1}.$$

Hence $(1-n)F(n) = (1 + n + n^2 + \cdots + n^{p-2}) - (p-1)n^{p-1}$ and similarly

$$(1-n)^2F(n) = 1 - n^{p-1} - (1-n)(p-1)n^{p-1} = 1 - p \cdot n^{p-1} + (p-1)n^p.$$

Modulo p, $n^p \equiv n$ by the Little Fermat Theorem and so $(1-n)^2F(n) \equiv 1 - n$. If neither a nor b is congruent to 1 (mod p), $1 - a \not\equiv 1 - b$ and there are distinct reciprocals $(1-a)^{-1}$ and $(1-b)^{-1}$(mod p); then

$$f(a) \equiv (1-a)^{-1}, f(b) \equiv (1-b)^{-1}, f(a) \not\equiv f(b) (\text{mod } p).$$

If one of a and b, say a, is congruent to 1, then $b \not\equiv 0$ (mod p) and so $f(b) \equiv (1-b)^{-1} \not\equiv 0$ (mod p) while

$$f(a) = 1 + 2 + \cdots + (p-1) = p(p-1)/2 \equiv 0 (\text{mod } p).$$

A-4. (15, 2, 7, 3, 0, 1, 0, 3, 4, 5, 29, 126)

Let $\binom{m}{r} = 0$ for $r > m$ and for $r < 0$. For $i = 0, 1, 2$ let

$$T_i(m) = \binom{m}{i} - \binom{m}{i+3} + \binom{m}{i+6} - \binom{m}{i+9} + \cdots.$$

We note that $S(m) = T_2(m) + 1$. Since $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$,

$$T_2(m) = T_2(m-1) + T_1(m-1), T_1(m) = T_1(m-1) + T_0(m-1),$$
$$T_0(m) = T_0(m-1) - T_2(m-1).$$

Let the backwards difference operator ∇ be given by $\nabla f(n) = f(n) - f(n-1)$. Then

$$\nabla T_2(m) = T_1(m-1),\ \nabla T_1(m) = T_0(m-1),\ \nabla T_0(m) = -T_2(m-1).$$

These imply that

$$\nabla^3 T_2(m) = \nabla^2 T_1(m-1) = \nabla T_0(m-2) = -T_2(m-3) \text{ for } m \geqslant 3.$$

Expanding $\nabla^3 T_2(m)$, this gives us

$$\text{(R)} \qquad T_2(m) = 3[T_2(m-1) - T_2(m-2)] \text{ for } m \geqslant 3.$$

When $m = 6k - 1$ with $k \geqslant 1$, we have $m \geqslant 5$. It then follows from (R) that $T_2(m) \equiv 0 \pmod 3$ and hence $S(m) \equiv 1 \pmod 3$. Thus $S(m) \neq 0$.

A-5. $(1,1,1,0,0,0,0,0,0,0,43,149)$

Inductively we define a sequence of integers $3 = a_1, a_2, a_3, \ldots$ and associated intervals $I_n = [(a_n)^{1/n}, (1 + a_n)^{1/n})$ such that $a_n \geqslant 3^n$, $a_n \equiv n \pmod 2$, the sequence $\{(a_n)^{1/n}\}$ is nondecreasing, and $I_n \supseteq I_{n+1}$. When this has been done, $\{(a_n)^{1/n}\}$, being nondecreasing and bounded, will have a limit u which is in I_n for all n. Then $(a_n)^{1/n} \leqslant u < (1 + a_n)^{1/n}$ will imply that $a_n \leqslant u^n < 1 + a_n$ and so $[u^n] = a_n \equiv n \pmod 2$ for all n.

Let $a_1 = 3$. Then $I_1 = [3,4)$. Let us assume that we have $a_1, a_2, \ldots, a_k$ and $I_1, I_2, \ldots, I_k$ with the desired properties. Let

$$J_k = \left[(a_k)^{(k+1)/k}, (1 + a_k)^{(k+1)/k}\right).$$

Then x is in I_k if and only if x^{k+1} is in J_k. The length of J_k is

$$(1 + a_k)^{(k+1)/k} - (a_k)^{(k+1)/k} \geqslant (1 + a_k - a_k)(a_k)^{1/k} = a_k^{1/k} \geqslant (3^k)^{1/k} = 3.$$

Since the length of J_k is at least 3, J_k contains an interval $L_k = [a_{k+1}, 1 + a_{k+1})$ for some integer a_{k+1} which is congruent to $k + 1 \pmod 2$. Let

$$I_{k+1} = \left[(a_{k+1})^{1/(k+1)}, (1 + a_{k+1})^{1/(k+1)}\right).$$

Since $x \in I_k$ if and only if $x^{k+1} \in J_k$, $x \in I_{k+1}$ if and only if $x^{k+1} \in L_k$, and $J_k \supseteq L_k$, one sees that $I_k \supseteq I_{k+1}$. Also

$$a_{k+1} \geqslant (a_k)^{(k+1)/k} = \left[(a_k)^{1/k}\right]^{k+1} \geqslant 3^{k+1}.$$

This completes the inductive step and shows that the desired u exists.

A-6. $(0,1,1,0,0,0,0,2,5,3,38,145)$

Under the change of variables $s = u - v$ and $t = u + v$, with the Jacobian $\partial(u,v)/\partial(s,t) = 1/2$, $F(x)$ becomes $I(x)/E(x)$ where

$$I(x) = \int_0^x \int_{-t}^{t} \exp\left\{\left(\frac{t+s}{2}\right)^3 + \left(\frac{t-s}{2}\right)^3\right\} ds\, dt$$

$$= \int_0^x \int_{-t}^{t} \exp\left(\frac{1}{4}t^3 + \frac{3}{4}ts^2\right) ds\, dt$$

and $E(x) = 2x^{-4}\exp(x^3)$. Since $I(x)$ and $E(x)$ go to $+\infty$ as x goes to $+\infty$, one can use L'Hôpital's Rule and we have $\lim_{x\to\infty} F(x) = \lim_{x\to\infty}(I'/E')$ where

$$I' = \int_{-x}^{x} \exp\left(\frac{1}{4}x^3 + \frac{3}{4}xs^2\right) ds = \exp(x^3/4)\int_{-x}^{x} \exp(3xs^2/4)\, ds$$

and $E' = (6x^{-2} - 8x^{-5})\exp(x^3)$. In the integral for I', make the change of variable $s = w/\sqrt{x}$,

$ds = dw/\sqrt{x}$, to obtain

$$I' = \frac{\exp(x^3/4)}{\sqrt{x}} \int_{-x\sqrt{x}}^{x\sqrt{x}} \exp(3w^2/4)\, dw.$$

Now

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{I'}{E'} = \lim_{x\to\infty} \frac{\int_{-x\sqrt{x}}^{x\sqrt{x}} \exp(3w^2/4)\, dw}{(6x^{-3/2} - 8x^{-9/2})\exp(3x^3/4)}.$$

We can, and do, use L'Hôpital's rule again to obtain

$$\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \frac{2(3/2)x^{1/2}\exp(3x^2/4)}{[(27/2)x^{1/2} + \cdots]\exp(3x^2/4)} = \frac{2}{9}.$$

B-1. (5, 15, 41, 0, 0, 0, 0, 64, 53, 6, 10, 1)

The diameter of S must be 4 and S must be centered at the center of C. The set of points inside C nearer to v than to another vertex w is the part of that half-space, bounded by the perpendicular bisector of the segment vw, containing v which lies within C. The intersection of these sets is a cube C' bounded by the three facial planes of C through v and the three planes which are perpendicular bisectors of the edges of C at v. These last 3 planes are planes of symmetry for C and S. Hence R is one of 8 disjoint congruent regions whose union is the set of points between S and C, excepting those on the 3 planes of symmetry. Therefore

$$8\,\mathrm{vol}(R) = \mathrm{vol}(C) - \mathrm{vol}(S) = 4^3 - \frac{4\pi}{3} \cdot 2^3,$$

$$\mathrm{vol}(R) = 8 - \frac{4\pi}{3}.$$

B-2. (33, 5, 5, 2, 0, 1, 0, 0, 32, 13, 43, 61)

A representation for $2n$ is of the form

$$2n = e_0 + 2e_1 + 4e_2 + \cdots + 2^k e_k,$$

the e_i in $\{0, 1, 2, 3\}$, and with e_0 in $\{0, 2\}$. Then $e_1 + 2e_2 + \cdots + 2^{k-1}e_k$ is a representation for n if $e_0 = 0$ and is a representation for $n - 1$ if $e_0 = 2$. Since all representations for n and $n - 1$ can be obtained this way,

$$C(2n) = C(n) + C(n - 1).$$

Similarly, one finds that

$$C(2n + 1) = C(n) + C(n - 1) = C(2n).$$

Since $C(1) = 1$ and $C(2) = 2$, an easy induction now shows that $C(n) = [1 + n/2]$.

B-3. (76, 5, 4, 2, 0, 0, 0, 3, 0, 3, 5, 97)

To satisfy the equation, each y_i must have at least 3 derivatives. Here Σ will be a sum with i running over 1, 2, 3. We have $\Sigma y_i^2 = 1$ and $\Sigma(y_i')^2 = f$. Differentiating, one has $\Sigma 2y_i y_i' = 0$ and $\Sigma 2y_i' y_i'' = f'$. Differentiating $\Sigma y_i y_i' = 0$ leads to $\Sigma y_i y_i'' + \Sigma(y_i')^2 = 0$ so $\Sigma y_i y_i'' = -f$. Differentiating this gives us $\Sigma y_i' y_i'' + \Sigma y_i y_i''' = -f'$. This and $\Sigma y_i' y_i'' = f'/2$ leads to $\Sigma y_i y_i''' = -3f'/2$. Multiplying each term of

$$y_i''' + py_i'' + qy_i' + ry_i = 0$$

by y_i and summing gives us

$$-3f'/2 - pf + q \cdot 0 + r = 0.$$

Thus $f' + (2/3)pf = (2/3)r$ and so $A = 2/3 = B$.

B-4. (41, 21, 19, 3, 0, 0, 3, 0, 6, 4, 17, 81)

We can let $m = k^2 + j$, where k and j are integers with $0 \leqslant j \leqslant 2k$, since the next square after k^2 is $k^2 + 2k + 1$; let this j be the *excess* for m. We note that $[\sqrt{m}] = k$ and $f(m) = k^2 + j + k$. If the excess j is 0, m is already a square. Let A consist of the m's with excess j satisfying $0 \leqslant j \leqslant k$ and B consist of the m's with $k < j \leqslant 2k$. If m is in B,

$$f(m) = k^2 + j + k = (k+1)^2 + (j - k - 1),$$

with the excess $j - k - 1$ for $f(m)$ satisfying $0 \leqslant j - k - 1 \leqslant k + 1$, and hence $f(m)$ is either a square or is in A. Thus it suffices to deal with the case in which m is in A. Then $[\sqrt{m+k}] = k$ and

$$f^2(m) = f(f(m)) = f(m+k) = m + 2k = (k+1)^2 + (j - 1).$$

Hence $f^2(m)$ is either a square or an integer in A with excess smaller than that of m. Continuing, one sees that $f^r(m)$ is a square for some r with $0 \leqslant r \leqslant 2j$.

B-5. (77, 16, 14, 0, 0, 0, 0, 0, 9, 4, 16, 59)

By definition of a_n and $\|u\|$,

$$\begin{aligned} a_n &= \sum_{k=1}^{n-1} \frac{1}{n}\left[\int_{2n/(2k+1)}^{n/k}\left(\frac{n}{x} - k\right)dx + \int_{n/(k+1)}^{2n/(2k+1)}\left(k + 1 - \frac{n}{x}\right)dx\right] \\ &= \sum_{k=1}^{n-1}\left[\ln\frac{2k+1}{2k} - \frac{1}{2k+1} + \frac{1}{2k+1} - \ln\frac{2k+2}{2k+1}\right] \\ &= \ln\prod_{k=1}^{n-1}\frac{(2k+1)^2}{2k(2k+2)} = \ln\left[\frac{3}{2}\cdot\frac{3}{4}\cdot\frac{5}{4}\cdot\frac{5}{6}\cdots\frac{(2n-1)}{(2n-2)}\cdot\frac{(2n-1)}{2n}\right]. \end{aligned}$$

Since

$$\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots = \frac{\pi}{2}$$

and $\ln x$ is continuous for $x > 0$, $\lim_{n\to\infty} a_n = \ln(4/\pi)$.

B-6. (3, 0, 0, 1, 0, 0, 0, 1, 1, 20, 16, 153)

Since $r \neq 1$ and $r^m - 1 = (r-1)(r^{m-1} + r^{m-2} + \cdots + 1) = 0$, one has $r^{m-1} + r^{m-2} + \cdots + 1 = 0$ and so

$$\begin{aligned} -1 &= r(1 + r + r^2 + \cdots + r^{m-2}), \\ -1 &= r(1+r)(1+r^2)(1+r^4)\cdots(1 + r^{(m-1)/2}), \\ -1 &= (r + r^2)(1+r^2)(1+r^4)\cdots(1 + r^{(m-1)/2}). \end{aligned}$$

Since $r + r^2 = r^{m+1} + r^2$ with $m + 1 = 2(2^{k-1} + 1)$, each of the factors in the last expression for -1 is a sum of two squares. Their product can be expressed as a sum of two squares by repeated application of the identity

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

This converts -1 into $P^2 + Q^2$ with each of P and Q a polynomial in r with integer coefficients.



He is not known for equality. (See p. 514.)

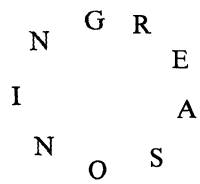


—Janet Tremain

4. ALGEBRA = ARBEGLA

—Dale Bachman

5. space space space

6.


—Carl Morris

7. ROUPG

Answers on p. 521.

133. MISCELLANEA

...a remarkable change is at present passing over the mathematical work done in the universities and colleges of this country. Courses that a short time ago were offered in only a few of our leading universities are now not uncommon in institutions of college rank. They are often given by men who have taken advanced degrees in mathematics at Göttingen, Berlin, Paris, or other leading universities abroad, and they are awakening a great interest in the modern field. A recent investigation (1903) showed that 67 students in ten American institutions were taking courses in the theory of functions, 11 in the theory of elliptic functions, 94 in projective geometry, 26 in the theory of invariants, 45 in the theory of groups, and 46 in the modern advanced theory of equations, courses which only a few years ago were rarely given in this country.

—David Eugene Smith, *History of Modern Mathematics*, Wiley, New York, 1906, pp. 75–76.

ANSWER TO PHOTO ON PAGE 496

H. A. Schwarz, of lemma and inequality fame.

PROGRESS REPORTS

EDITED BY THOMAS BANCHOFF AND RICHARD MILLMAN

It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

APPROXIMATION BY POLYNOMIALS: INTERPOLATION AND OPTIMAL NODES

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The idea of "approximating" an arbitrary function using a linear combination of nice functions has been around for a very long time and has been extensively studied. This paper is related to the problem of approximating a given function by an ordinary polynomial.

More specifically, let f be a continuous function on $[a, b]$, i.e., an element of $C[a, b]$, and set $\|f\| = \max_{a \leqslant x \leqslant b} |f(x)|$. If Π_n is the subspace of all polynomials of degree at most n, then the best (uniform) approximation Bf in Π_n to f on $[a, b]$ is defined by

$$\|f - Bf\| = \inf\{\|f - p\| : p \in \Pi_n\} = \operatorname{dist}(f, \Pi_n).$$

Several efficient algorithms are used routinely for machine computation of the best approximation to a given f. Before the computer era, however, it was generally very difficult to find optimal polynomial approximations.

Perhaps because it offered an alternative to the best approximation, early in this century a number of well-known mathematicians focused their attention on the theoretical properties of another approximation method, polynomial interpolation. What is polynomial interpolation? Simply put, it's an approximation method that results in a polynomial that matches (or "interpolates") a given function f at specified data points (called "nodes"). In particular, let $a = t_0$, $b = t_n$ and

$$T = \{t = \langle t_1, \ldots, t_{n-1} \rangle \in R^{n-1} : \quad a < t_1 < \cdots < t_{n-1} < b\}.$$

Corresponding to each t in T, there is a basis of polynomials $\{L_k(x)\}_{k=0}^n$, where

$$L_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} (x - t_j)/(t_i - t_j), \qquad i = 0, 1, \ldots, n.$$

Since $L_i(t_i) = 1$ and $L_i(t_j) = 0$ if $i \neq j$, the polynomial that interpolates f at the nodes $\{t_j\}_{j=0}^n$ is $(P_t f)(x) = \sum_{i=0}^n f(t_i) L_i(x)$.

The polynomial P_t is very easy to construct, and in many cases $P_t f$ is uniformly close to f when n is large and the nodes are "nicely" distributed over $[a, b]$. Unfortunately, the interpolating polynomial can also oscillate wildly and bear little resemblance to f.

For example, in 1901 Runge showed that if $r(x) = 1/(1 + x^2), -5 \leqslant x \leqslant 5$, and if $t_j = -5 + 10j/n$, $j = 0, 1, \ldots, n$, then $\|r - P_t r\| \to \infty$ as $n \to +\infty$. The "bunching" of nodes at the end points of the interval may substantially alter the phenomenon exhibited by the Runge example. If, for instance, the nodes are chosen to be the zeros of the Chebyshev polynomial of degree $n + 1$, adjusted to the interval $[a, b]$ so that the first and last zero coincide with the end points of the interval, then $\|r - P_t r\| \to 0$ as $n \to +\infty$.

In view of the difference a change of nodes makes in the Runge example, it is natural to ask if there is a set of nodes $\{t_j^{(n)}\}_{j=0}^n$, independent of f, for which $\|f - P_t f\| \to 0$ as $n \to +\infty$. Faber answered this question in 1914 when he demonstrated that for any set of nodes $\{t_j^{(n)}\}_{j=0}^n$, there exists a function f in $C[a, b]$ for which $\sum_{i=0}^n f(t_i^{(n)}) L_i^{(n)}$ fails to converge to f as $n \to +\infty$.

Faber's work notwithstanding, it is worth restating that $P_t f$ does converge for many f in $C[a, b]$. At the same time, Runge's example points out the importance of node placement in polynomial interpolation. Is there a "best choice" of nodes that is independent of the function, f? To formulate this question more precisely, a little more background is useful.

Given the nodes $\{t_j\}_{j=0}^n$, we observe that the mapping P_t: $C[a, b] \to \Pi_n$ by $f \to P_t f$ is a projection with

$$\|P_t\| = \sup\{\|P_t f\| : f \in C[a, b] \text{ and } \|f\| = 1\} = \max_{x \in [a, b]} \sum_{i=0}^n |L_i(x)|.$$

We also observe that since $P_t Bf = Bf$, we have

$$\|f - P_t f\| = \|f - Bf - P_t f + P_t Bf\| = \|(I - P_t)(f - Bf)\|.$$

Thus $\|f - P_t f\| \leqslant (1 + \|P_t\|) \operatorname{dist}(f, \Pi_n)$. The utility of this inequality can be demonstrated by again appealing to the nodes consisting of the adjusted Chebyshev zeros. In this case, $\|P_t\| = O(\operatorname{Log}(n + 1))$, and if f is k times differentiable, than $\operatorname{dist}(f, \Pi_n) = O(n^{-k})$. Thus for these nodes, $\|f - P_t f\| = O(n^{-k} \operatorname{Log}(n + 1))$. Hence the adjusted Chebyshev zeros are an excellent choice for nodes.

Referring again to the inequality, we see that an appropriate definition for the "best choice of nodes" is a choice of nodes, if any, that minimizes $\|P_t\|$.

Is there a set of nodes t^* in T for which $\|P_{t^*}\| = \inf\{\|P_t\| : t \in T\}$? If so, is this set of "optimal" nodes unique, and is it characterized by some special property? Two classical conjectures of approximation theory deal with this problem of optimal interpolation nodes. Let

$$\Lambda_t(x) = \sum_{i=0}^n |L_i(x)|, x \in [a, b],$$

and set

$$\lambda_i(t) = \max_{t_{i-1} \leqslant x \leqslant t_i} \Lambda_t(x), \qquad i = 1, 2, \ldots, n.$$

The function Λ_t is said to equioscillate when $\lambda_1(t) = \lambda_2(t) = \cdots = \lambda_n(t)$; that is, when Λ_t has equal maxima on the intervals $[t_{i-1}, t_i], 1 \leqslant i \leqslant n$. In 1931 S. Bernstein conjectured that $\|P_t\|$ is minimal when the Lebesgue function $\Lambda_t(x)$ equioscillates. Later, in 1947, P. Erdős conjectured that there is exactly one choice of t for which Λ_t equioscillates, and that $\min_i \lambda_i(t)$ achieves its maximum over T when Λ_t equioscillates.

Since their formulation, these intimately related conjectures have been the subject of many investigations by prominent approximation theorists. Even so, the conjectures remained unresolved until 1978, when both Bernstein's and Erdős's conjectures were confirmed. Considering the forty-seven years between formulation and resolution, it is perhaps surprising that Bernstein's conjecture (including uniqueness of the optimal nodes) was established independently by Kilgore

[3] and deBoor and Pinkus [2] in the same year. The latter two authors also established the Erdős conjecture.

Although the two papers use different techniques to prove Bernstein's conjecture, both appeal to an earlier Kilgore theorem:

$$\textit{If } \|P_{t^*}\| = \inf\{\|P_t\| : t \in T\}, \textit{ then } \Lambda_{t^*} \textit{ equioscillates.}$$

Bernstein's conjectures is the converse of this theorem. Both papers also use sophisticated arguments that exploit the properties of the partial derivatives of $\lambda_i(t)$ with respect to t_j, $1 \leqslant j \leqslant n-1, 1 \leqslant i \leqslant n$. For example, properties of these partial derivatives play a key role in the deBoor and Pinkus proof of their main theorem:

The map $\Gamma\colon T \to R^{n-1}$ *defined by* $\Gamma(t) = \langle \lambda_{i+1}(t) - \lambda_i(t) \rangle_{i=1}^{n-1}$ *is a homeomorphism of* T *onto* R^{n-1}.

Bernstein's conjecture (including uniqueness) is implied by this theorem.

Although they have been computed for $n \leqslant 15$, an explicit formula for the optimal nodes is not yet known. It has been shown that $\|P_{t^*}\| = O(\mathrm{Log}(n+1))$ when t^* is optimal. Recalling the asymptotic order of the norm of the operator P_t constructed from the nodes consisting of the adjusted Chebyshev zeros, one could conclude for practical purposes that the adjusted Chebyshev zeros are optimal nodes.

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2. C. deBoor and A. Pinkus, Proof of the conjectures of Bernstein and Erdős concerning optimal nodes for polynomial interpolation, J. Approx. Theory, 24 (1978) 289–303.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N IN4.*

A PROBLEM ON RATIONAL SUBSETS OF THE FREE GROUP

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Let F be a free group of finite rank. The family of **rational subsets** of F is the smallest family of subsets of F containing the finite subsets and closed under the following three operations:

$$A \cup B = \{c | c \in A \text{ or } c \in B\}$$

$$AB = \{ab | a \in A \text{ and } b \in B\}$$

$$A^* = \{a_1 a_2 \cdots a_n | a_i \in A \text{ for each } i\}.$$

(Note that when $n = 0$, we obtain the identity element in A^*.)

A subset S of F is called **nondisjunctive** if there exists a nontrivial normal subgroup N of F such that S is a union of cosets of N. If such an N exists and is of finite index in F, we say that S is a subset of **finite index**. We make the following conjecture:

CONJECTURE. *Every nondisjunctive, rational subset S of F is of finite index.*

The definition of rational subsets comes from formal language theory. Formal language theory is mainly concerned with the study of various families of subsets of the free monoid. (Let us recall that a **monoid** is a set equipped with an associative multiplication and with an identity element; and that the free monoid of rank n is isomorphic with the smallest subset of the free group of rank n, closed under multiplication, and containing the identity and n generators of the group, but none of their inverses.) The family of rational subsets of free monoids—often called the family of **regular languages**—is the first one to have been studied. The first result of the theory, Kleene's Theorem, says that a rational subset of a free monoid of finite rank is the union of classes of a congruence with finite index (cf., for instance, the treatise of Eilenberg [4]). Kleene's Theorem does not hold in general for monoids other than the free ones. If true, the conjecture stated above will give a kind of Kleene's Theorem for free groups.

On the other hand, another family of languages that is of interest, the family of context-free languages, appeared to be connected to free groups, through the Chomsky-Schützenberger Theorem for instance (cf. [3]). The conjecture above is indeed a part of a conjecture on the characterization of the deterministic context-free languages that are recognized by a free group (cf. [7]).

In [7] we noted that the conjecture holds in two particular cases:

PROPOSITION 1. *The conjecture holds if S is a subgroup of F.*

PROPOSITION 2. *The conjecture holds if S is a finite union of cosets of a nontrivial normal subgroup.*

Proposition 1 is the immediate consequence of two results. The first one is due to Anisimov and Seifert and can also be found in [3] (Theorem 2.7):

THEOREM 1 [1]: *A subgroup of a group G is a rational subset of G if and only if it is finitely generated.*

The second result is due to Greenberg and can also be found in [6] (Proposition 3.11).

THEOREM 2 [5]: *If a finitely generated subgroup of a free group F contains a nontrivial normal subgroup of F, then it has finite index in F.*

Clearly the simple juxtaposition of Theorems 1 and 2 gives Proposition 1.

In order to prove Proposition 2, let H be a nontrivial normal subgroup of the free group F, T a finite set of F, and $S = \cup_{t \in T} Ht$; we first observe that $U = \cap_{t \in T} St^{-1}$ is closed under product and that the canonical image of U in F/H is finite and thus a subgroup of F/H. The set U itself is thus a subgroup. If S is a rational set of F, so is every set St^{-1} and therefore U is a rational subset by the following result of Benois (cf. also [3] Corollary 2.10):

THEOREM 3 [2]: *The intersection of a finite family of rational subsets of the free group F is a rational subset of F.*

Since U contains H, U is of finite index in F by Theorem 1. Since by construction U is a finite union of cosets of H, H is of finite index in F.

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NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.
For instructions about submitting Notes for publication in this department see the inside front cover.

DOUBLE LIMITS AND MATCHED ASYMPTOTIC EXPANSIONS*

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Interchanging the order of two limiting processes is only permitted, as we all know, under certain restrictive assumptions dependent on the problem. One situation in which such an interchange plays an important role occurs in the theory of *singular perturbations* of ordinary differential equations (see, for instance, O'Malley [6]). We shall describe this rather general situation, in which the inversion of a double limit is permissible.

In Section 1, some basic concepts from the theory of perturbations are recalled, *asymptotic expansions* are introduced, and some of their properties are reviewed. We shall be interested in asymptotic power series expansions in a small parameter ε, with coefficients which are functions of a spatial variable x. These expansions will take different forms over two intervals of the range of the variable x. The "matching" of these two forms can be carried out if the two intervals overlap; the validity of this matching can be derived under rather general conditions from Kaplun's Extension Theorem [1], [4]. This theorem is widely used, but neither its formulation nor its proof is stated very clearly in the existing literature.

In Section 2, we present the usual formulation of the Extension Theorem, along with a precise formulation and a proof which is rigorous. This proof uses the theory of uniform convergence, which underlies many cases in which two limiting processes can be interchanged. Such an interchange is presented as a corollary of the Extension Theorem.

1. Singular Perturbations, Boundary Layers and the Matching Technique. Perturbation problems arise when a differential equation depends on a small parameter ε. As ε changes, so do the solutions $y \equiv y(x, \varepsilon)$ of the equation. The perturbation is *regular* if the dependence of y on ε is smooth, uniformly in the variable x. If the dependence is not uniform, i.e., "anomalous things" happen over a certain x-interval as $\varepsilon \to 0$, the perturbation is *singular*. This fact can usually be related to the multiscale nature of the phenomenon under investigation.

For the sake of definiteness, consider an initial- or a boundary-value problem for a scalar linear ODE of order $n \geqslant 2$, containing a small parameter $\varepsilon > 0$, in an interval $a \leqslant x \leqslant b$. Some smoothness is assumed for the coefficients of the equation; n initial or boundary conditions are prescribed. We are interested in the behavior of its solutions as $\varepsilon \to 0$.

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Telegraphic Reviews

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T:	Textbook	13-18:	Grade Level
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P:	Professional Reading	** :	Special Emphasis
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General, T(13-14: 2, 3), L. Mathematics for the Management, Life, and Social Sciences, Second Edition. Larry J. Goldstein, David C. Lay, David I. Schneider. Prentice-Hall, 1984, xxi + 791 pp, $29.95. [ISBN: 0-13-562512-2] Survey of algebra, linear algebra and linear programming, probability and a taste of statistics, and calculus, motivated by applications. This edition features expanded algebra review, traditional treatment of math of finance, discussion of duality in max/min linear programming problems. For courses with a one year format. RB

General, P☆, L☆. The TeX Book. Donald E. Knuth. Addison-Wesley, 1984, ix + 483 pp, $15. [ISBN: 0-201-13448-9] A fascinating, fun-to-read user's guide (complete with jokes and technical puns) to the "final" version of TeX, Knuth's typesetting system for mathematical books that has been underwritten by the American Mathematical Society. Intended for both novices and "TeX hackers," (alias TeXperts), this guide is not optimal for someone (such as a department secretary) who just wants to learn quickly how to use the system. It provides layer upon layer of cleverly-illustrated examples of TeX behavior, from diacritical marks to complex mathematical displays. LAS

General, S(12-14), L☆☆. The Role of Mathematics in Science. M.M. Schiffer, Leon Bowden. New Math. Lib., No. 30. MAA, 1984, xi + 207 pp, $14 (P). [ISBN: 0-88385-630-1] A delightful mathematical excursion through parts of classical science--primarily mechanics--beginning with Archimedes' axiom of a balanced lever, concluding with the Lorentz transformation and special relativity. Intended to illustrate and teach the role of mathematics as a sixth sense for scientific intuition, "to give brain-borne counsel where bone-borne guidance fails." Originally written as an SMSG pamphlet, it has been extensively revised and mathematically elevated to use more intricate (sometimes calculus-based) reasoning. LAS

General, S(13-14). Matemáticas Superiores en Ejercicios y Problemas. P.E. Dankó, A.G. Popov, T.Ya. Kozhévnikova. MIR Pub, 1983. Parte 1, 363 pp; Parte 2, 455 pp. Spanish translation of Russian equivalent of a Schaum outline. 2960 routine problems on analytic geometry, calculus, linear algebra, linear programming, differential equations, probability, complex analysis, calculus of variations. KS

Elementary, T(13: 1). Algebra for College Students. Jerome E. Kaufmann. Prindle, Weber & Schmidt, 1984, xi + 644 pp. [ISBN: 0-87150-464-2] Intermediate algebra with some college algebra topics, at the intermediate algebra level. "There is a common thread throughout the book, namely, learn a skill, then use the skill to help solve equations and inequalities, and then use equations and inequalities to solve word problems." Emphasis on examples, plenty of drill. RB

Mathematics Appreciation, S(13), L. Numbers, Their History and Meaning. Graham Flegg. Shocken Books, 1983, x + 295 pp, $14.95. [ISBN: 0-8052-3847-6] Written for the general reader. An historical and cultural development of topics such as counting, numerals, calculation, variables, recreational arithmetic and irrational numbers. Interesting reading. CEC

Precalculus, T(13: 1). College Algebra with Analytical Geometry. Charles C. Carico. Wiley, 1984, ix + 382 pp, $22.95. [ISBN: 0-471-88748-X] Standard college algebra text, although it seems a bit skimpy on some topics (e.g., inverse functions). Contains two "supplementary" chapters on analytic geometry--one on conic sections, the other on graphing trigonometric functions. Also has an appendix on partial fractions. BK

Education, S(15-17), L☆. Didactical Phenomenology of Mathematical Structures. Hans Freudenthal. Math. Educ. Lib. D Reidel Pub, 1983, x + 595 pp, $93.50. [ISBN: 90-277-1535-1] A "systematic" but "most chaotic" attempt to describe basic mathematical concepts (sets, numbers, fractions, geometry, topology, algebra) in relation to the phenomena for which they were created. Contains rich doses of history, experience, and insight, making this a valuable (albeit expensive) addition to the library of any serious teacher of mathematics. SG

Education, S(13), P, L. Learning Mathematics with Microg. A.J. Oldknow, D.V. Smith. Ser. in Math. & Its Applic. Halsted Pr, 1983, 268 pp, $49.95; $24.95 (P). [ISBN: 0-470-27488-3] This book is primarily aimed at the teacher of mathematics who is interested in exploring the use of microcomputers in the classroom. Includes discussions and programs which demonstrate the use of a micro in teaching matrices, geometry, logic, statistics, probability, calculus, and problem solving. Also includes a brief section on classroom management. Programs are given in BASIC. CEC

Foundations, P. Patras Logic Symposion. Ed: George Metakides. Stud. in Logic & Found. of Math., V. 109. Elsevier North-Holland, 1982, ix + 391 pp, $65. [ISBN: 0-444-86476-8] Proceedings of ASL summer meeting, held August 18-22, 1980 in Patras, Greece. Research papers from many areas of logic. KS

Foundations, P. Lecture Notes in Mathematics-1019: Cabal Seminar 79-81. Ed: A.S. Kechris, D.A. Martin, Y.N. Moschovakis. Springer-Verlag, 1983, 284 pp, $13 (P). [ISBN: 0-387-12688-0] Twelve papers from Caltech-UCLA Logic Seminar; half deal with extent of definable scales, remainder with topics in set theory and descriptive set theory. KS

Combinatorics, P. Lecture Notes in Mathematics-1036: Combinatorial Mathematics X. Ed: L.R.A. Casse. Springer-Verlag, 1983, xi + 419 pp, $18.30 (P). [ISBN: 0-387-12708-9] This volume contains the text of seven invited addresses and twenty-three contributed talks from the Tenth Australian Conference on Combinatorial Mathematics held at the University of Adelaide from August 23-27, 1982. CEC

Combinatorics, P. A Contribution to the Nonexistence of Perfect Codes. M.R. Best. Math Centrum, 1983, vi + 99 pp, Dfl. 13,20 (P). [ISBN: 90-6196-253-6] A slightly-revised edition of the author's Ph.D. thesis (under J.H. van Lint) in which the possibility of hitherto unknown t-perfect codes is reduced to the cases t = 1, 2, 6 or 8. GHM

Number Theory, P. Lecture Notes in Mathematics-988: Kohomologie arithmetisch definierter Gruppen und Eigensteinreihen. Joachim Schwermer. Springer-Verlag, 1983, 170 pp, $10.50 (P). [ISBN: 0-387-12292-3] Well-organized book about the connection between the cohomology of arithmetic groups and automorphic forms. In particular, it develops Eisenstein series along lines of von Selberg and Langlands. BK

Number Theory, P. Lecture Notes in Mathematics-1040: Local Analysis of Selberg's Trace Formula. Anton Good. Springer-Verlag, 1983, 128 pp, $7.40 (P). [ISBN: 0-387-12713-5] Functional equations of certain meromorphic functions connect algebraic data of discrete groups with spectral data of differential operators. Bibliography included. CEC

Group Theory, P. Lecture Notes in Mathematics-1026: Group Rings of Finite Groups Over p-adic Integers. Wilhelm Plesken. Springer-Verlag, 1983, 151 pp, $9.50 (P). [ISBN: 0-387-12728-3] A study of the basic algebraic properties (two-sided ideals, irreducible lattices, and projective indecomposable lattices) over the rings named in the title. Presumes knowledge of maximal orders and basic modular representation theory. SG

Algebra, T?(15-16: 1, 2). Applied Abstract Algebra. Ki Hang Kim, Fred W. Roush. Ser. in Math. & Its Applic. Halsted Pr, 1983, 265 pp, $24.95 (P). [ISBN: 0-470-27441-7] Text for two-semester course. Includes binary relations, semigroups, groups, vector spaces, rings, group representations, field theory plus applications to finite state machines, Polya enumeration, coding theory and more in 250 pages; consequently presentation is condensed, sometimes cryptic. Poorly written in many places; proofs include numerous "so's" and expository paragraphs provide more confusion than clarification. Three levels of exercises after each section. KS

Algebra, P. Emmy Noether in Bryn Mawr. Ed: Bhama Srinivasan, Judith D. Sally. Springer-Verlag, 1983, viii + 182 pp, $28. [ISBN: 0-387-90838-2] Proceedings of symposium in honor of Emmy Noether's one hundredth birthday, held at Bryn Mawr, March 17-19, 1982. Includes eight articles on recent work in areas of interest to Noether, brief articles on Noether and her work, and complete bibliography of her publications. KS

Calculus, T(13). Technical Mathematics with Calculus. Paul Calter. Prentice-Hall, 1984, xvii + 1017 pp, $28.95. [ISBN: 0-13-898312-7] Intended for students at technical schools and colleges. The first three-fourths of the book is devoted to high school algebra, geometry, and precalculus. The remainder covers one-variable calculus. SG

Complex Analysis, T(18), S, P**.** Univalent Functions. Peter L. Duren. Grund. der Math. Wissenschaften, B. 259. Springer-Verlag, 1983, xiv + 382 pp, $46. [ISBN: 0-387-90795-5] Reviews basic principles of complex analysis and introduces elementary and classical theory of univalent functions in first two chapters. Remaining nine chapters cover various methods used to attack the Bieberbach conjecture, including: Loewner's parametric method, Grunsky inequalities, exponentiated Grunsky inequalities, Baernstein's maximal function, variational methods. Extremely lucid and well-motivated with numerous historical references and interesting research-type problems at end of each chapter. BH

Complex Analysis, P. Lecture Notes in Mathematics-1027: Hardy Classes on Infinitely Connected Riemann Surfaces. Morisuke Hasumi. Springer-Verlag, 1983, xii + 280 pp, $13 (P). [ISBN: 0-387-12729-1] The standard theory of Hardy classes of analytic functions is extended to a collection of infinitely connected open Riemann surfaces, the Parreau-Widom surfaces. Besides extending Hardy

class theory, these research level notes aim to increase knowledge of Riemann surfaces in general. RB

Differential Equations, P. Volterra Integral and Differential Equations. T.A. Burton. Math. in Sci. & Eng., V. 167. Academic Pr, 1983, x + 313 pp, $45. [ISBN: 0-12-147380-5] Introduction to the structure and stability properties of solutions of Volterra integral and integro-differential equations. Primarily an exposition of Liapunov's direct method. Aimed chiefly at mathematicians, physicists, and engineers versed in stability of ordinary differential equations on the line using Riemannian integration. CU

Differential Equations, T(17-18: 1), P. Constructive Methods in the Analysis of Nonlinear Systems. E.A. Grebenikov, Yu. A. Ryabov. Transl: Ram S. Wadhwa. MIR Pub, 1983, 328 pp, $9.95. Asymptotic and iterative methods for solving resonant systems of differential equations, in the spirit of Lyapunov, with emphasis on constructive solutions. Asymptotic theory is based on method of coordinate transformations; algorithmic nature of iterative methods is exploited in chapter on computer solution. RB

Differential Equations, P*. Lecture Notes in Mathematics-1044: Discretization Methods for Stable Initial Value Problems. Eckart Gekeler. Springer-Verlag, 1984, viii + 201 pp, $10.50 (P). [ISBN: 0-387-12880-8] Multistep multiderivative methods for differential systems of first and second order. Linear systems with scalar time-dependence with error bounds. Recent results on error propagation in linear multistep and nonlinear differential systems of first order. New results with up-to-date references. JK

Functional Analysis, P. Dilation Theory, Toeplitz Operators, and Other Topics. Ed: C. Apostol, et. al. Operator Theory, V. 11. Birkhauser Boston, 1983, 408 pp, $39.95. [ISBN: 3-7643-1516-4] Consists of papers presented at the Seventh International Conference on Operator Theory in Timisoara and Herculane, Romania in June 1982. BH

Differential Geometry, P. Homogeneous Structures on Riemannian Manifolds. F. Tricerri, L. Vanhecke. London Math. Soc. Lect. Notes Ser., No. 83. Cambridge U Pr, 1983, vi + 125 pp, $19.95 (P). [ISBN: 0-521-27489-3] The Ambrose-Singer theorem characterizes (connected, complete, simply connected) homogeneous Riemannian manifolds according to certain (1,2)-tensor fields called homogeneous structures. Here, homogeneous structures are classified, naturally reductive homogeneous spaces are characterized, and homogeneous spaces of dimension less than or equal to 4 and Kaplan's 6-dimensional generalized Heisenberg group are discussed. RB

Differential Geometry, T(17-18: 1), S, PP. Semi-Riemannian Geometry with Applications to Relativity. Barrett O'Neill. Pure & Appl. Math. Academic Pr, 1983, xiii + 468 pp, $45. [ISBN: 0-12-526740-1] Semi-Riemannian geometry, the study of a smooth manifold furnished with a metric tensor of arbitrary signature, generalizes both Riemannian geometry and Lorenz geometry. Exposition generally coordinate free but "does not neglect coordinates." An interesting first course in geometry for both mathematics and physics students. RB

Differential Geometry, T(15-16: 1). A First Course in Differential Geometry. Izu Vaisman. Mono. & Textbooks in Pure & Appl. Math., V. 80. Dekker, 1984, v + 169 pp, $32.50. [ISBN: 0-8247-7063-3] A manifold-oriented approach to undergraduate differential geometry. Chapter titles: Differentiable manifolds in R^n, Curves in E^2 and E^3, surfaces in E^3. A flood of terminology: approximately two new terms mentioned per page throughout. 220 standard exercises. RB

Optimization, T(16-18: 1, 2), S, P. Programmation Mathématique: Théorie et Algorithmes. Michel Minoux. Dunod, 1983. Tome 1, xxxi + 294 pp, 150 FF (P) [ISBN: 2-04-015487-6]; Tome 2, xxxi + 236 pp, 130 FF (P). [ISBN: 2-04-015542-2] Presents a panorama of the theory and algorithms of mathematical programming. Organizes entire work around several key notions. Chapters on linear programming, one-dimensional optimization, nonlinear optimization with and without constraints, integer programming, large systems, dynamic programming, optimization in infinite dimensions. Chapter bibliographies. Appendices. Index. RJA

Probability, T(15-16: 1, 2). Teoria das probabilidades e estatística matemática. V.E. Gmurman. MIR Pub, 1983, 477 pp. Portuguese translation for post-calculus probability and statistics course. Exercises with answers at end of each chapter. KS

Computer Literacy, T(13-14: 1). Fundamentals of Computers and Data Processing with Basic. Wilson T. Price. Holt, Rinehart & Winston, 1983, xv + 429 pp, (P). [ISBN: 0-03-063231-5] Introductory data processing text. First half surveys hardware, programming principles, software systems, system development. Second half covers BASIC programming. Emphasizes modular approach to programming problems. Appropriate for microcomputer or timesharing system. KS

Computer Literacy, S. Your First Computer: A Guide to Business and Personal Computing. Rodnay Zaks. Sybex, 1980, xv + 258 pp, $8.95 (P). [ISBN: 0-89588-045-8] Introduction to microcomputer hardware and software. Describes basic architecture of computer systems and points out limitations and advantages of different types of microcomputers, peripherals and languages. KS

Computer Programming, S(13). Timex/Sinclair 1000 Basic Programs in Minutes. Stanley R. Trost. Sybex, 1983, xii + 145 pp, $7.95 (P). [ISBN: 0-89588-119-5] Approximately 50 programs for financial and real estate calculations, arithmetic drill, data analysis. Includes sample output for each

program and entry instructions. KS

Computer Programming, S(13). Basic Learning Programs, Second Edition: APPLE Version. University of Wisconsin, River Falls (Academic Computer Center), 1982, 270 pp, (P). Introduction to Basic commands through short programs. Objectives are listed for each program, followed by a listing and sample run. Exercises are primarily requests to change one line at a time and observe the effect. Helpful glossary and listing of what might cause various error messages. Suitable for beginners working independently. MW

Computer Programming, S. Mathematical Problem-Solving with the Microcomputer: Projects to Increase Your BASIC Programming Skill. Stephen L. Snover, Mark A. Spikell. Prentice-Hall, 1982, xiii + 190 pp, $8.95 (P). [ISBN: 0-13-561811-8] A collection of very elementary numerical problems, most of which require fewer than 20 lines of code and half an hour to solve. Problems are adorned with cartoons, partial solutions and suggested extensions. Complete solutions are in the back of the book. Good for the complete novice with low frustration tolerance. GHM

Computer Programming, S(14-16), P, L. Advanced Pascal Programming Techniques. Paul A. Sand. Osborne/McGraw-Hill, 1984, xiii + 370 pp, $19.95 (P). [ISBN: 0-88134-105-3] Step-by-step development of several complete and useful programs (in Apple Pascal) that do "real tasks," including a calculator, Reversi player, simulation, spreadsheet, and others. Emphasizes design decisions and tool building. Touches on language features, algorithms and data structures as these issues arise in building the programs. Suggests extensions, but no independent projects. GHM

Computer Programming, T? The MBASIC Handbook. Walter A. Ettlin, Gregory Solberg. Osborne/McGraw-Hill, 1983, xi + 457 pp, $17.95 (P). [ISBN: 0-88134-102-9] Microsoft's MBASIC is similar to proprietary BASICs on Apple, Radio Shack, IBM microcomputers. This handbook supplements the manuals with verbal explanation and extensive examples. Focus on software production (but omitting algorithm development techniques--how did they do it?): searching, sorting, practical sample menu driver with applications, lengthy examples with proper internal documentation. RB

Computer Programming, T(13-18: 1), S. The Icon Programming Language. Ralph E. and Madge T. Griswold. Prentice-Hall, 1983, xviii + 313 pp, $18.95 (P). [ISBN: 0-13-449777-5] Icon is a high-level and general-purpose language containing many features for processing non-numeric data, especially strings of characters. It has many concepts found in SNOBOL 4, yet it has modern control structures and is an expression-based language with a syntax similar to Algol 68 and Pascal. Text divides into 4 parts: (i) basics of the language; (ii) advanced features such as generators, goal-directed evaluation, string scanning, co-expressions; (iii) programming techniques, including string and list processing; (iv) several large sample programs. Appendices on syntax, machine dependencies, running programs, errors, built-in operations, solution to selected exercises. References, index. RJA

Computer Programming, T(13: 1), S, L. BASIC Programming Using Structured Modules. Jonathan C. Barron. Holt, Rinehart & Winston, 1983, xiv + 418 pp, $17.95 (P). [ISBN: 0-03-059241-0] An introduction to BASIC which is more complete, more sophisticated, and more readable than the average text of this type. The text conforms to the ANSI standard and presents all statements considered to be minimal BASIC. Includes an excellent collection of programming exercises. CEC

Data Structures, P. Database Machines. Ed: H.-O. Leilich, M. Missikoff. Springer-Verlag, 1983, ix + 344 pp, $23 (P). [ISBN: 0-387-12959-6] Database machines are computer systems with specialized circuitry (hardware) designed to perform data management operations more efficiently than can be done with software on general purpose computers. These 20 papers from a workshop held in Munich, September 1983, emphasize performance evaluation and modelling. GHM

Software Systems, S(17-18), P. Abstraction, Specification and Implementation Techniques with an Application to Garbage Collection. H.B.M. Jonkers. Math. Centre Tracts, No. 166. Math Centrum, 1983, vi + 317 pp, Dfl. 41,70 (P). [ISBN: 90-6196-263-3] A case study in practical mathematical concepts and methods which may be used to form useful abstractions of design problems, to describe solutions (i.e., specify algorithms) precisely but understandably, and finally to implement the solution by applying correctness-preserving transformations to the abstract algorithm. GHM

Computer Science, T(13-14: 1), S☆, P☆, L☆. Programming Principles, An Introduction. John Motil. Allyn & Bacon, 1984, 461 pp, $27.10 (P). [ISBN: 0-205-08005-7] Structured algorithm development is primary, programming separate and secondary in this unusual introductory text. Appendix introducing Pascal in some depth is ordinarily postponed for about 1/3 term. Book's flexible format is a literal application of top-down structure and modular organization; the modules are pairs of a page of text developing a single theme and a facing page of illustrative graphic images. Interesting pedagogical ideas. RB

Computer Science, T☆(15-17: 1, 2), L. Operating System Concepts. James L. Peterson, Abraham Silberschatz. Addison-Wesley, 1983, 548 pp, $29.95. [ISBN: 0-201-06097-3] Example of the second generation of operating system texts. Emphasizes accepted basic concepts and fundamentals, not specifics of one or two systems. Chapters 1-3 explain what operating systems are and do; Chapters 4-8 discuss the classical internal algorithms and structures: CPU scheduling, memory management, device management. Chapter 9 presents the unifying concept of the computer system as a collection of cooperating sequential processes. The final chapters present advanced topics and current trends: high-level languages for writing concurrent programs, protection systems, design principles,

distributed systems. The text closes with a chapter on the most influential operating systems historically and, in effect, provides references to readings for the entire text. Chapter exercises and bibliographic notes. Bibliography. Index. RJA

Control Theory, P. Algebraic Theory for Multivariable Linear Systems. Hans Blomberg, Raimo Ylinen. Math. in Sci. & Eng., V. 166. Academic Pr, 1983, xx + 360 pp, $48. [ISBN: 0-12-107150-2] An up-to-date account of polynomial system theory including a "justification" for its introduction and applications to ordinary linear time-invariant differential and difference equations. SG

Control Theory, P. Lecture Notes in Control and Information Sciences-52: Perturbations, Approximations and Sensitivity Analysis of Optimal Control Systems. A.L. Dontchev. Springer-Verlag, 1983, iv + 158 pp, $11 (P). [ISBN: 0-387-12463-2] The classical approach to sensitivity analysis requires differentiability of the system function. After considering conditions for optimal control problems to be well posed under regular and singular perturbations, this work seeks to estimate the sensitivity of such problems under conditions weaker than differentiability. RB

Applications (Artificial Intelligence), P. A Computer System for Checking Proofs. Scott D. Johnson. Computer Science: Artificial Intelligence, No. 12. UMI Research Pr, 1982, xvii + 262 pp, $44.95. [ISBN: 0-8357-1343-1] Description of computer system used to check proofs of programs written in PL/CV, a programming logic based on PL/I. KS

Applications (Artificial Intelligence), P. Planning and Understanding: A Computational Approach to Human Reasoning. Robert Wilensky. Addison-Wesley, 1983, xiv + 168 pp, $25.95. [ISBN: 0-201-09590-4] An exploration of the nature of knowledge--how it can be represented, stored, accessed, and used--required to carry out planning--the process of selecting a course of action--and understanding--the comprehension of a situation. The major focus is on a theory of plans which deals with the structure and content of knowledge about interactions between goals, events, situations, and circumstances. Of interest to linguists, cognitive psychologists, and members of the artificial intelligentsia. SG

Applications (Artificial Intelligence), P. Computational Linguistics. Ed: Nick Cercone. Intern. Ser. in Modern Appl. Math. & Comp. Sci., V. 5. Pergamon Pr, 1983, xi + 245 pp, $37.50. [ISBN: 0-08-030253-X] A special collection of papers representing a wide range of current research in computational linguistics, from theoretical foundations to programming languages to concrete applications. GHM

Applications (Economics), P. Studies in Contemporary Economics: Neoclassical Theory Structure and Theory Development. Bert Hamminga. Springer-Verlag, 1983, ix + 174 pp, $12 (P). [ISBN: 0-387-12816-6] Presents metatheory to describe structure and development of theory in economic research. Based on history of Ohlin-Samuelson program in international trade. Argues that traditional philosophies of science are not appropriate for describing economics. KS

Applications (Engineering), T(13-14: 1), S. Boolean Algebra for Computer Logic. Harold E. Ennes. Howard W Sams, 1982, 128 pp, $7.95 (P). [ISBN: 0-672-21554-3] Introduction to Boolean algebra applied to design of computer circuits. Covers truth tables, algebraic simplification, Venn diagrams, Karnaugh maps. Numerous exercises with solutions. Assumes understanding of elementary electronics. KS

Applications (Psychology), S(15-17), L. Psychology and Mathematics: An Essay on Theory. Clyde H. Coombs. U of Michigan Pr, 1983, viii + 112 pp, $12.50. [ISBN: 0-472-10034-3] Personal reflections on the role of mathematics in the construction of psychological theories. For a detailed review, see American Scientist, V. 77, No. 2, March-April 1984, p. 208. SG

Applications (Modelling/Simulation), S(16-18), P. Informatik-Fachberichte-71: First European Simulation Congress ESC 83. Ed: W. Ameling. Springer-Verlag, 1983, xii + 653 pp, $27.60 (P). [ISBN: 0-387-12723-2] Four invited papers, 83 contributed papers classified into three parts: simulation methodology (modelling, techniques, theory, etc.), simulation tools (computer systems, architectures, software), applications of simulation to various fields. Proceedings published before the conference and available to participants. RB

Reviewers

RJA: Richard J. Allen, St. Olaf; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JDB: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; AO: Arnold Ostebee, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MT: Michael Tveite, St. Olaf; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

Southern California Section

The spring meeting of the Southern California Section was held at California State University at Los Angeles on March 3, 1984. There were about 80 registrants.

Invited Addresses:

"Bits and Pieces from the Classroom," by Leonard Gillman, University of Texas at Austin.
"The Mordell Conjecture," by Murray Shacker, University of California, Los Angeles.
"Applications of the Theory of Space Curves to DNA," by James White, University of California, Los Angeles.
"The Flowering of Mathematics in Southern California," by John Green, University of California, Los Angeles.

Panel Discussion:

"Coexisting with Computer Science," by Ronald Mieck (Moderator), University of California, Los Angeles; James Friel, California State University, Fresno; William Margulies, California State University, Long Beach.

Report:

"The California State Mathematics Entrance Examination and Its Effect on the Curriculum," by Alice King, California State Polytech University, Pomona.

Illinois Section

The sixty-third annual meeting of the Illinois Section convened at Eastern Illinois University on April 27-28, 1984 with approximately seventy members in attendance.

Invited Lecture:

"Number Mysticism," by Underwood Dudley, DePauw University.

Invited Addresses:

"Algorithmically Defined Functions," by Richard D. Anderson, Louisiana State University.
"MATLAB," by Jack J. Dongarra, Argonne National Laboratory.
"Linear Algebra and the Analysis of Variance," by Patrick D. McCray, G.D. Searle & Company.
"Unexpected Points of Contact," by John A. Evell, Northern Illinois University.
"Instabilities in Numerical Models," by Charlie Dey, Jefferson Elementary School; S.K. Dey, Eastern Illinois University.
"Mathematics in Illinois: Remember the Ladies," by Jeanne LaDuke, DePaul University.
"Relativity and Space Flight," by Julian Palmore, University of Illinois at Urbana-Champaign.
"Matching, Marriage, and Satellite Communication," by C.L. Liu, University of Illinois at Urbana-Champaign.

Seaway Section

The spring meeting of the Seaway Section was held on April 6-7, 1984 at Broome Community College, Binghamton, New York. There were 98 registrants.

Invited Lectures:

"The Plot Thickens and Other Stories," by Allen Ziebur, State University Center at Binghamton.
"Linear Algebra Without Tears--And Without Vector Spaces," by Peter J. Hilton, State University Center at Binghamton.

Contributed Papers:

"A 'Good' Proof of DIM(S) + DIM($S^{\perp}$) = DIM(V)," by Jack E. Graver, Syracuse University.
"The Power Maps As An Endomorphism in Rings," by Howard E. Bell, Brock University.
"Forbidden Patterns for Triangle Centers," by Andrew P. Guinand, Trent University.
"Tree Algorithms For Coin Flipping," by Bette Warren, State University Center at Binghamton.
"A Critical Look at Binomial Confidence Intervals," by Harold A. Still, Queen's University.
"Is Modelling Just for Computers and Vice Versa?" by Boris D. Rakover, St. John Fisher College.
"An Efficient Computational Procedure for the Symbolic Multiplication of Large Multivariate

Polynomials," by Mike Milkalajunas, Montreal, Quebec.
"Video Tape Production Made Easy," by Dan Dodway, Broome Community College.

Student Paper:

"Flow Behavior of Red Blood Cells in Capillaries," by Mark S. Carraher, Rochester Institute of Technology.

Kansas Section

The spring meeting of the Kansas Section was held on March 30-31, 1984 at Bethel College, North Newton, Kansas. There were approximately 80 members registered.

Invited Addresses:

"The Impact of Computers on the Mathematics Curriculum," by Mary Grace Kantowski, University of Florida.
"Using the Computer in Teaching Mathematics," by Mary Grace Kantowski, University of Florida.
"How to Discover and Prove Theorems: A Demonstration with Partitions," by Henry L. Alder, University of California at Davis.

Short Presentations:

"Variance Reduction and the Method of Batch Means," by Robert Boncella, Washburn University.
"On the Natural Density of the Set of Niven Numbers," by Robert Kennedy and Curtis Cooper, Central Missouri State University.
"On an Asymptotic Formula for the Niven Numbers," by Curtis Cooper and Robert Kennedy, Central Missouri State University.
"Spirals," by Saul Stahl, University of Kansas.
"The Electronic Spreadsheet in the Mathematics Classroom: Some Examples of Use," by John Atkinson, Sterling College.
"Teaching Number Theory Using Computers," by Carolyn Ehr, Fort Hays State University.
"To Build a Better Box," by Kay Dundas, Hutchinson Community College.
"How You Count Does Make a Difference," by Elwyn H. Davis, Pittsburg State University.
"An Alternative to Row Reduction for Systems of Equations Mod 2," by Ronald N. Umble, Hesston College.
"Rational Approximations of Irrationals," by Glenn Fox, Wichita State University.
"Transforming Points on Maps," by Charles R. Hamaker, University of Kansas.
"Hermite Interpolation and Pade Approximates of the Exponential Function," by Gary McGrath, Pittsburg State University.

Panel Discussion:

"Current Issues in Precollege Mathematics," by Mary Kay Corbitt (Moderator), University of Kansas; Mary Grace Kantowski, University of Florida; Bee Ellington, Shawnee Mission West High School.

Kentucky Section

The sixty-seventh annual meeting of the Kentucky Section was held on March 30-31, 1984 at Centre College, Danville, Kentucky. There were 88 registrants.

Invited Addresses:

"Articulation Means Communication and Cooperation," by Billy E. Rhoades, Indiana University.
"Is There a Discrete Mathematics Course in Your Future?" by Anthony Ralston, State University of New York at Buffalo.

Short Presentations:

"Some Sum Summaries," by James Barksdale, Western Kentucky University.
"An Introduction to Mathematical Problem Solving at NKU," by Philip McCartney, Northern Kentucky University.
"Theories of Free Fall from Aristotle to Galileo," by Daniel Curtin, Northern Kentucky University.
"The Rule of Signs," by Chris Christensen, Northern Kentucky University.
"A Discussion of Developmental Mathematics in Kentucky Colleges," by Peter Moore and Roy Matthews, Northern Kentucky University.
"Cognitive Bases for Teaching Problem Solving in Mathematics," by James L. Howard, Centre College.
"Elementary Statistics and the Computer," by Emeric Hornick, Bellarmine College.
"Computing the Volume of an Ellipsoid by Examining Cross-Sectional Slices," by James McKenney, Northern Kentucky University.

Student Papers:

"Highway Curves Using Elementary Calculus," by Jeff Brownfield, Murray State.

Panel Discussion:

"Articulation," by Brock Fugate (Moderator), University of Kentucky; Tom Khearns, Northern Kentucky; Ron Pelfrey, Fayette County Schools, Lexington; Jim Moore, Danville High School; Billy Rhoades, Indiana University.

Open Forum:

"Discrete Mathematics," by Anthony Ralston, State University of New York at Buffalo.

Maryland-D.C.-Virginia Section

The eight hundred and eleventh meeting of the American Mathematical Society was scheduled jointly with the 1984 spring meeting of the Maryland-D.C.-Virginia section of the Mathematical Association of America. This joint meeting was held April 13-14, 1984 at Virginia Commonwealth University in Richmond, Virginia.

Invited Lectures:

"Interactive Computer Graphics in Differential Geometry Teaching and Research," by Thomas F. Banchoff, Brown University.
"What Kind of Applied Mathematics Goes on in a Large Telecommunications Laboratory?" by James McKenna, AT&T Bell Laboratories.

Special Sessions:

"Applications of Mathematics from Everyday Life," by Edwin G. Landauer, General Physics Corporation.
"Learning the Structure of Mathematical Concepts and Procedures through the Development of Representations," by Carolyn Maher, Rutgers University.
"Slicing d-Space," by William Wardlaw, United States Naval Academy.
"Who Invented the Computer?" by Ben Fusaro, Salisbury State College.
"Teacning Stochastic Modeling Using Computer Simulation," by James Maher, AT&T Bell Laboratories.
"Nature's Random Whispers," by Richard Eisenman, University of North Florida.
"What's ADA All About?" by Peter Welcher, United States Naval Academy.
"An Old Math Game and a Not So Old Math Game," by Frederick Gooding, Trinity College.

Short Presentations:

"E.H. Moore and the Founding of a Mathematical Community in America 1892-1902," by Karen Parshall, Sweet Briar College.
"The Dynamics of Iterates of Maps on the Unit Interval," by Paul Massell, United States Naval Academy.
"C.S., O.R., Stat., and All That: Mathematics at the F.B.I.," by Richard Sours, Wilkes College and the F.B.I..
"How to Schedule a Car Pool," by James V. Blowers.
"Proper Subfields of the Real Numbers," by Steve Butcher (Speaker), United States Naval Academy; Wallace Hamilton and John Milcetich, University of the District of Columbia.
"Choosing Your Words Carefully: A Language Based Approach to Teaching Mathematics," by G. Edgar Parker, Pan American University.
"Generalizations on Computing the Odds of Winning at RISK," by Brian Shelburne, Sweet Briar College.
"An Algorithm for Centers of Quadric Surfaces," by William Sanders, James Madison University.
"Computer Generated Polar Coordinate Graphs," by Howard Penn, United States Naval Academy.
"Extensions of Number Systems," by Robert Hanson, James Madison University.
"Mathematics and Anti-Submarine Warfare," by Lee Whitt, Daniel H. Wagner, Associates.
"The Use of a Generalization of Snell's Law in Transmission Loss Prediction," by Howard S. Burkon, JHU/Applied Physics Laboratory.
"Using a Square Root Approximation to Introduce Concepts of Numerical Analysis," by Richard E. Allan, Virginia Commonwealth University.

Student Presentation:

"The Design and Implementation of an Interpreter for the Functional Language LISP," by Anthony C. Barrett, James Madison University.

2. M. Benois, Parties rationnelles du groupe libre, C. R. Acad. Sci. Paris Sér. A, 269 (1969) A1188–A1190; MR 42#405.
3. J. Berstel, Transductions and Context Free Languages, Teubner, 1979; MR 80j: 68056.
4. S. Eilenberg, Automata, Languages, and Machines, vol. A, Academic Press, 1974; MR 58#26604a.
5. L. Greenberg, Discrete groups of motions, Canad. J. Math., 12 (1960) 415–426; MR 22#5932.
6. R. C. Lyndon and P. E. Schupp, Combinatorial Group Theory, Springer, 1977; MR 58#28182.
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NOTES

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DOUBLE LIMITS AND MATCHED ASYMPTOTIC EXPANSIONS*

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Interchanging the order of two limiting processes is only permitted, as we all know, under certain restrictive assumptions dependent on the problem. One situation in which such an interchange plays an important role occurs in the theory of *singular perturbations* of ordinary differential equations (see, for instance, O'Malley [6]). We shall describe this rather general situation, in which the inversion of a double limit is permissible.

In Section 1, some basic concepts from the theory of perturbations are recalled, *asymptotic expansions* are introduced, and some of their properties are reviewed. We shall be interested in asymptotic power series expansions in a small parameter ε, with coefficients which are functions of a spatial variable x. These expansions will take different forms over two intervals of the range of the variable x. The "matching" of these two forms can be carried out if the two intervals overlap; the validity of this matching can be derived under rather general conditions from Kaplun's Extension Theorem [1], [4]. This theorem is widely used, but neither its formulation nor its proof is stated very clearly in the existing literature.

In Section 2, we present the usual formulation of the Extension Theorem, along with a precise formulation and a proof which is rigorous. This proof uses the theory of uniform convergence, which underlies many cases in which two limiting processes can be interchanged. Such an interchange is presented as a corollary of the Extension Theorem.

1. Singular Perturbations, Boundary Layers and the Matching Technique. Perturbation problems arise when a differential equation depends on a small parameter ε. As ε changes, so do the solutions $y \equiv y(x, \varepsilon)$ of the equation. The perturbation is *regular* if the dependence of y on ε is smooth, uniformly in the variable x. If the dependence is not uniform, i.e., "anomalous things" happen over a certain x-interval as $\varepsilon \to 0$, the perturbation is *singular*. This fact can usually be related to the multiscale nature of the phenomenon under investigation.

For the sake of definiteness, consider an initial- or a boundary-value problem for a scalar linear ODE of order $n \geqslant 2$, containing a small parameter $\varepsilon > 0$, in an interval $a \leqslant x \leqslant b$. Some smoothness is assumed for the coefficients of the equation; n initial or boundary conditions are prescribed. We are interested in the behavior of its solutions as $\varepsilon \to 0$.

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By setting $\varepsilon = 0$, one obtains the so-called *reduced* or *degenerate* equation. In regular perturbation theory, the solutions of this equation are expected to approximate the solutions of the original equation with a small nonzero ε. If ε, however, multiplies the highest-order derivative in the equation, it is clear that the solutions of the reduced equation *cannot* satisfy, in general, *all* the initial or boundary conditions.

A convenient, and historically very important, way of representing solutions of perturbation problems are (formal) power series in the smallness parameter ε. These power series have functions of x as their coefficients,

$$y(x,\varepsilon) = \sum_{n=0}^{\infty} y_n(x)\varepsilon^n. \tag{1.1}$$

Clearly, $y_0(x)$ is the solution of the reduced equation; $y_1(x)$ is the solution of the problem obtained by linearizing the dependence on ε of solutions, and so on.

In general, the series in (1.1) is *not* convergent for any finite, nonzero value of ε. However, Poincaré [**8**] found a rigorous way of expressing the usefulness of the representation (1.1). He introduced the concept of an asymptotic expansion.

DEFINITION. *Let S be the sector $\{x: 0 < |x| < x_0,\ \alpha < \arg x < \beta\}$ of the complex x-plane, $f(x)$ holomorphic in S. The formal series $\Sigma_{n=0}^{\infty} a_n x^n$ is said to be the asymptotic expansion of $f(x)$, as $x \to 0$ in S, if, for every $N \geqslant 0$,*

$$f(x) = \sum_{n=0}^{N} a_n x^n + o(x^N), \quad \text{as } x \to 0, \quad x \in S. \tag{1.2}$$

We shall write, in this case,

$$f(x) \sim \sum_{n=0}^{\infty} a_n x^n, \quad \text{as } x \to 0, \quad x \in S. \tag{1.3}$$

Convergent series expansions are also asymptotic; the converse is not true in general. Some basic properties of asymptotic expansions (as defined above) are that they can be multiplied by constants, added and multiplied term by term, and integrated termwise (in S). Termwise differentiation is also permitted in every proper subsector of S (see, e.g., [**10**]). These properties make asymptotic expansions very useful in representing solutions of ODEs, since they are so easy to handle.

The asymptotic series expansion in (1.3) is *uniquely* determined by $f(x)$ (whenever it exists): in fact, it follows from the definition that

$$a_0 = \lim_{\substack{x \to 0 \\ x \in S}} f(x); \quad a_N = \lim_{\substack{x \to 0 \\ x \in S}} x^{-N}\left[f(x) - \sum_{n=0}^{N-1} a_n x^n\right], \quad N \geqslant 1. \tag{1.4}$$

Conversely, such an expansion determines $f(x)$ only up to functions "asymptotically equal to zero." Indeed, there are infinitely many functions having the same expansion. A typical example of two such functions is provided by $f(x) \equiv 0$ and $g(x) = e^{-1/x}$, for $\operatorname{Re} x > 0$, for which we obtain $a_n = 0, \quad n = 0, 1, 2, \ldots$.

However, for *every formal* power series $\Sigma_{n=0}^{\infty} a_n x^n$ and for *every* sector S of the complex x-plane, there exists a (nonunique) function $f(x)$, *holomorphic* in S, such that (1.3) holds (Borel-Ritt theorem; see, e.g., [**10**]).

In studying solutions of ODEs which depend on a small parameter ε, the asymptotic expansions used take the form

$$f(x,\varepsilon) \sim \sum_{n=0}^{\infty} a_n(x)\varepsilon^n, \quad \text{as } \varepsilon \to 0, \quad \varepsilon \in E, \quad x \in D, \tag{1.5}$$

where E is an open sector of the complex ε-plane, with vertex at the origin, and D is some region

of the complex x-plane; $f(x, \varepsilon)$ is holomorphic in $E \times D$ and the functions $a_n(x)$ are holomorphic in D. The expansion in (1.5) may *or* may *not* be uniformly valid in x, for $x \in D$.

It turns out that, for a singularly perturbed ODE, no x-uniformly valid asymptotic expansion of the solutions in powers of ε exists. This is due to the fact that, in certain subintervals of $[a, b]$, called *boundary layers*, the solution varies rapidly with x, while in the rest of $[a, b]$ it varies slowly. Separate asymptotic expansions in powers of ε can be constructed for the slowly varying part of the solution—the *outer expansion*—and for the rapidly varying part—the *inner expansion*.

Historically, the concept of boundary layer was introduced first by Prandtl [**9**], based on purely physical reasoning, by studying the *inviscid* incompressible flow past a solid object as a limit-model of moderately *viscous* flows. We recall that fluid flows are actually described by a set of nonlinear *partial* differential equations, the Navier-Stokes equations, where the viscosity coefficient multiplies the highest derivatives. Therefore, setting formally the viscosity equal to zero, we obtain equations of *lower* order, whose solutions cannot satisfy all the relevant boundary conditions. More precisely, the inviscid flow satisfies the condition of zero normal velocity at a solid boundary. It does *not* satisfy, however, the no-slip condition of zero tangential velocity, which the viscous flow satisfies. On the other hand, from a physical point of view, it is reasonable to assume "the continuity of the mathematical model," as flows of fluids with small viscosity should behave quite similarly to the inviscid ones.

The interesting fact discovered by Prandtl was that, away from the boundary, viscous and inviscid flows behaved quite alike. Near the boundary, however, the velocity in the viscous fluid dropped rapidly to zero through a narrow layer. The inviscid fluid, on the other hand, could flow freely along the boundary, and no special "boundary layer" was observed.

The complete asymptotic analysis of this situation requires the determination of certain quantities in the outer expansion. These quantities cannot be determined directly, due to the loss of initial or boundary conditions. They can be obtained by *matching* the outer expansion to the inner one (see, e.g., [**2**], [**6**, pp. 15–18]). This matching is obtained by finding the linear transformations, independent of x but depending on ε, which map the inner and the outer expansion into each other.

2. The Extension Theorem of S. Kaplun. The success of the matching procedure between the inner and the outer expansions is guaranteed by the existence of an overlap between the regions of validity of the two expansions: such an overlap obtains as a consequence of the Extension Theorem of S. Kaplun (see [**1**, p. 77], [**4**, p. 98], under very general conditions ([**3**], e.g.).

Let us clarify the theorem and its connection to double limits. We shall consider, for simplicity, the situation in which boundary layer behavior occurs near $x = 0$.

The basic idea is sometimes roughly stated in certain applications by saying that, "if an asymptotic approximation in ε of a function $f(x, \varepsilon)$, continuous on $[0,1] \times (0, \varepsilon_0]$, is uniformly valid in an interval of the variable x, then it is uniformly valid *in a wider interval*, depending on the parameter ε." The precise requirement implied by the statement above is that the asymptotic approximation above hold uniformly in *every* interval of x, say $[\tilde{x}, 1]$, contained in $(0, 1]$. This statement can be formalized as the following

PROPOSITION (Kaplun's Extension Theorem). *Suppose that*

$$\lim_{\varepsilon \to 0+} f(x, \varepsilon) = 0, \tag{2.1}$$

uniformly in $x \in [\tilde{x}, 1]$, *for every* $\tilde{x}$, $0 < \tilde{x} < 1$. *Then there exists a continuous positive function* $\xi(\varepsilon)$, *such that* $\xi(\varepsilon) \to 0$ *as* $\varepsilon \to 0+$ *and* $\lim_{\varepsilon \to 0+} f(x, \varepsilon) = 0$ *uniformly in* $x \in [\xi(\varepsilon), 1]$.

REMARK. It is convenient to think of $f(x, \varepsilon)$ as $f = y(x, \varepsilon) - y(x, 0)$, where $y(x, 0) = y_0(x)$ in (1.1).

Let us clarify the meaning of this proposition. The hypothesis (2.1) is equivalent to the ε-δ formulation that for every $\delta > 0$, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}(\delta, \tilde{x}) > 0$, such that $|f(x, \varepsilon)| < \delta$ for all ε

with $0 < \varepsilon < \tilde{\varepsilon}(\delta, \tilde{x})$ and all $x \in [\tilde{x}, 1]$. We emphasize that $\tilde{\varepsilon}$ depends on δ *as well as* on $\tilde{x}$. However, if $\tilde{x}$ is allowed to depend on $\tilde{\varepsilon}$, then it is possible "to solve" the relation $\tilde{\varepsilon} = \tilde{\varepsilon}(\delta, \tilde{x}(\tilde{\varepsilon}))$ for $\tilde{\varepsilon}$ and restate the proposition above as a

THEOREM. *Suppose that* (2.1) *holds. Then, for every* $\delta > 0$ *there exists* $\bar{\tilde{\varepsilon}} = \bar{\tilde{\varepsilon}}(\delta) > 0$, *such that* $|f(x, \varepsilon)| < \delta$ *for all* ε *with* $0 < \varepsilon < \bar{\tilde{\varepsilon}}(\delta)$ *and for all* $x \in [\xi(\varepsilon), 1]$: *here* $\xi(\varepsilon)$ *is a continuous positive function, with* $\xi(\varepsilon) \to 0$ *as* $\varepsilon \to 0+$, *which can be constructed.*

Equivalently, we can say that there is a continuous positive function $\xi(\varepsilon)$, with $\xi(\varepsilon) \to 0$ as $\varepsilon \to 0+$, such that, if $g(x, \varepsilon) \equiv f(x, \varepsilon)\chi_{[\xi(\varepsilon),1]}$, χ_A being the characteristic function of the set A, then $\lim_{\varepsilon \to 0+} g(x, \varepsilon) = 0$ uniformly in $x \in [0, 1]$.

Here is the proof (cf. [**1**, pp. 77–78]; see also [**2**, p. 24]). Choosing the decreasing sequence $\delta_k = \tilde{x}_k = 2^{-k}$, $k = 1, 2, 3, \ldots$, we obtain the relations:

$$|f(x, \varepsilon)| < 2^{-k} \quad \text{for} \quad 0 < \varepsilon < \tilde{\varepsilon}_k \quad \text{and} \quad x \in [2^{-k}, 1],$$

where the sequence $\tilde{\varepsilon}_k$ can be chosen decreasing to zero, as $k \to \infty$.

It is then possible to construct a continuous positive function $\xi(\varepsilon)$, which is defined as $\xi(\tilde{\varepsilon}_k) = \tilde{x}_{k-1} = 2^{-(k-1)}$, $k = 1, 2, 3, \ldots$, at the points $\tilde{\varepsilon}_k$, and by smooth interpolation between these points.

As for every $\delta > 0$ there exists n such that $2^{-n} \leqslant \delta < 2^{-(n-1)}$, one can find $\tilde{\varepsilon}_n$ such that $|f(x, \varepsilon)| < 2^{-n} \leqslant \delta$ for $0 < \varepsilon < \tilde{\varepsilon}_n$ and for $x \in [2^{-n}, 1]$. Moreover, for each $m > n$, $|f(x, \varepsilon)| < 2^{-m} < 2^{-n} \leqslant \delta$ for $0 < \varepsilon < \tilde{\varepsilon}_m$ and $x \in [2^{-m}, 1]$.

Therefore, as $n \to \infty$, $\tilde{\varepsilon}_n \to 0$ and hence $\varepsilon \to 0+$ and $f(x, \varepsilon) \to 0$ *uniformly* in the interval $[\xi(\varepsilon), 1]$, which is included in $[2^{-n}, 1]$.

As a consequence of this theorem, one can state the invertibility of the order in a double limit (see also [**5**]). Indeed, in general

$$\lim_{\varepsilon \to 0+} \Big(\lim_{x \to 0+} f(x, \varepsilon) \Big) \neq \lim_{x \to 0+} \Big(\lim_{\varepsilon \to 0+} f(x, \varepsilon) \Big), \tag{2.2}$$

whenever x and ε go to 0 *independently*. However, if $f(x, \varepsilon)$ admits an asymptotic representation in powers of ε, as $\varepsilon \to 0+$, then

$$\lim_{x \to 0+} \Big(\lim_{\varepsilon \to 0+} f(x, \varepsilon) \Big) = \lim_{x \to 0+} f(x, 0). \tag{2.3}$$

Under hypothesis (2.1), moreover,

$$\lim_{\varepsilon \to 0+} \Big(\lim_{\substack{x \to 0+ \\ x \geqslant \xi(\varepsilon)}} f(x, \varepsilon) \Big) = \lim_{\varepsilon \to 0+} f(\xi(\varepsilon), \varepsilon) = \lim_{x \to 0+} f(x, 0), \tag{2.4}$$

where $\xi(\varepsilon)$ is the function constructed above.

From (2.3) and (2.4) we obtain the invertibility result.

This completes our discussion of the idea of matching asymptotic expansions, of Kaplun's extension theorem and of its connection with the invertibility of double limits. Something similar to the "extension" procedure is encountered when "expanding domains" are introduced in Turning Point Theory. These expanding domains are analogous to the "outer regions" of singular perturbation theory: the uniform validity occurs here in sets having the form $c\varepsilon^{\rho} \leqslant x \leqslant x_1$, c, ρ being real positive constants (see, e.g., [**11**]). In particular, interchanging the order of the limits as above is permitted, provided that x goes to zero "slowly" enough, with respect to ε, i.e., with $x^{-1}\varepsilon^{\rho}$ bounded (or with $x \to +\infty$ with $x^{-1} = O(\varepsilon^{-\rho})$).

For an excellent survey on the literature concerning Perturbation Theory, see R. E. O'Malley [7].

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INFINITE-DIMENSIONAL BANACH SPACES MUST HAVE UNCOUNTABLE BASIS—AN ELEMENTARY PROOF

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The fact that any infinite-dimensional Banach space must have algebraic dimension (i.e., the cardinality of its Hamel basis) $\geqslant c$ is well known. The proof, which depends on the Baire category theorem ([1], [3], [4]) or the Hahn-Banach theorem ([2]), is often given as an exercise in many texts on functional analysis. This note gives an elementary proof in which the only tool is the completeness of finite-dimensional normed vector spaces.

PROPOSITION. *Every normed vector space with a countably infinite Hamel basis is not complete.*

Proof. Suppose $(E, \| \ \|)$ is a normed vector space and $(e_1, e_2, \ldots)$ a Hamel basis of E (i.e., every vector of E can be written as a linear combination of finitely many e_i's) with $\|e_i\| = 1$ for all $i \in \mathbb{N}$. Let

$$r_i = \inf\{\|x + e_i\|: x \in S_{i-1}\},$$

where S_{i-1} denotes the subspace spanned by $\{e_1, e_2, \ldots, e_{i-1}\}$ for all $i = 2, 3, \ldots$.

Since $0 \in S_{i-1}$, it follows that $r_i \leqslant \|0 + e_i\| = 1$ for all $i \geqslant 2$. Since S_{i-1} is finite-dimensional, it is complete, and hence closed in E. Since $e_i \notin S_{i-1}$, we have $r_i > 0$ for all $i \geqslant 2$. Define

$$t_1 = 1, \quad t_2 = \frac{1}{3}, \quad t_{i+1} = \frac{r_i t_i}{3} \quad \text{for all } i \geqslant 2.$$

Then

$$0 < t_{i+k} \leqslant \frac{1}{3^k} r_i t_i \leqslant \frac{1}{3^{i+k-1}} \quad \text{for all } i \geqslant 2,\ k \in \mathbb{N}.$$

Define $u_n = \sum_{i=1}^{n} t_i e_i$; then $\{u_n\}$ is a Cauchy sequence in E. But for any $u = \sum_{i=1}^{m-1} a_i e_i \in E$ (we may always choose $m \geqslant 2$),

$$\begin{aligned}
\|u_n - u\| &= \left\| \sum_{i=1}^{m-1} (t_i - a_i) e_i + t_m e_m + \sum_{j=m+1}^{n} t_j e_j \right\| \\
&\geqslant \left\| \sum_{i=1}^{m-1} (t_i - a_i) e_i + t_m e_m \right\| - \left\| \sum_{j=m+1}^{n} t_j e_j \right\| \\
&\geqslant t_m \left\| \sum_{i=1}^{m-1} \frac{t_i - a_i}{t_m} e_i + e_m \right\| - \sum_{j=m+1}^{n} t_j \\
&\geqslant t_m r_m - \sum_{j=1}^{n-m} \frac{r_m t_m}{3^j} \\
&\geqslant \frac{r_m t_m}{2} \quad \text{for all } n > m.
\end{aligned}$$

So $\|u_n - u\|$ does not tend to zero, which implies $\{u_n\}$ cannot converge to u. Hence E is not complete.

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THE MATRIX EQUATION $AX - XB = C$

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The matrix equation $AX - XB = C$ is quite well known and classical (see [1]). It plays an important role in the study of linear systems of ordinary differential equations and their stability properties and is, in a certain sense, a generalized form of the Lyapunov matrix equation. At the college level it often arises in courses in differential equations and control theory. The proofs that solutions exist which we have found in standard texts on these subjects ([1], [3], [4]) all seem to involve knowledge of the Jordan canonical form.

This note presents an elementary proof of a necessary and sufficient condition for the equation to have a unique solution for each C. The only totally nonelementary fact that we use is that a pair of commuting linear transformations can be simultaneously triangularized over the field of complex numbers (over an algebraically closed field). This is not much harder to show than the fact that a single transformation has a triangular form (see, for example [2], p. 206).

Suppose A is an $n \times n$ matrix and B an $m \times m$ matrix over the algebraically closed field F. We will use the idea, by now standard (see, for example, [2], p. 84, p. 115) of associating with the equation a transformation $\mathscr{T}$ defined on the vector space V of $n \times m$ matrices over F in the following way. Let $\mathscr{A}$: $V \to V$ be defined by $\mathscr{A}(X) = AX$ and $\mathscr{B}$: $V \to V$ be defined by $\mathscr{B}(X) = XB$. Now let $\mathscr{T} = \mathscr{A} - \mathscr{B}$. It is clear that $AX - XB = C$ has a unique solution for each $C \in V$ if and only if $\mathscr{T}$ is invertible. Thus we want to determine when $\mathscr{T}$ is invertible, or

equivalently, when zero is not an eigenvalue of $\mathscr{T}$. In fact, we will characterize the set of eigenvalues of $\mathscr{T}$.

We will denote the set of eigenvalues of a transformation T by $\sigma(T)$ and will not differentiate between a linear transformation and a matrix representation of that transformation.

THEOREM. *The equation $AX - XB = C$ has a unique solution for each $n \times m$ matrix C if and only if A and B have no common eigenvalues.*

We will compute $\sigma(\mathscr{A})$ and $\sigma(\mathscr{B})$. We will then use this information and the fact that $\mathscr{A}$ and $\mathscr{B}$ commute (verify!), to compute $\sigma(\mathscr{T})$.

LEMMA. $\sigma(\mathscr{A}) = \sigma(A)$, $\sigma(\mathscr{B}) = \sigma(B)$.

Proof. $\lambda \notin \sigma(A)$ if and only if $\lambda I_n - A$ is invertible and this holds if and only if for all nonzero $X \in V$, $(\lambda I_n - A)X \neq 0$. But this is equivalent to the statement that for all such X, $(\lambda I_n - \mathscr{A})(X) \neq 0$. Thus $\sigma(\mathscr{A}) = \sigma(A)$. A similar argument holds for B and $\mathscr{B}$.

The theorem itself is an immediate consequence of the following description of $\sigma(T)$.

PROPOSITION. $\sigma(\mathscr{T}) = \{\lambda - \mu : \lambda \in \sigma(A), \mu \in \sigma(B)\}$.

Proof. We first show that if $\alpha = \lambda - \mu$, with $\alpha \in \sigma(A)$, $\beta \in \sigma(B)$, then α is an eigenvalue of $\mathscr{T}$; in fact, we produce an eigenvector for α. Let f be an eigenvector of A corresponding to λ and written as a column vector $f = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$. Let g be an eigenvector of B corresponding to μ and written as a row vector $g = (g_1 \dots g_m)$. If $X = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (g_1 \dots g_m)$, then

$$\begin{aligned} \mathscr{T}(X) &= \mathscr{A}(X) - \mathscr{B}(X) \\ &= A \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (g_1 \dots g_m) - \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (g_1 \dots g_m) B \\ &= \lambda \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (g_1 \dots g_m) - \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} (g_1 \dots g_m) \mu \\ &= (\lambda - \mu) X. \end{aligned}$$

On the other hand, every eigenvalue of $\mathscr{T}$ must be of this form. If we triangularize $\mathscr{A}$ and $\mathscr{B}$ simultaneously, we obtain that $\mathscr{T}$ is an upper triangular matrix with elements on the diagonal of the form $\lambda - \mu$ with $\lambda \in \sigma(\mathscr{A}) = \sigma(A)$ and $\mu \in \sigma(\mathscr{B}) = \sigma(B)$. But these are all the eigenvalues of $\mathscr{T}$ and the proof is complete.

Note that the Theorem is false for fields that are not algebraically closed; for example, take $A = B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ over the real field. Then A has no eigenvalues but $\mathscr{T}(X) = AX - XA$ is not invertible. In particular, $\mathscr{T}(I_2) = 0$. One can, in this case, reformulate the condition of disjoint eigenvalues for A and B in terms of no common factors for the minimal polynomials of A and B and then avoid any conditions on the field.

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that the infimum in (5) is attained at $n_0 = [\sqrt{2m}]$, i.e., the largest integer not exceeding $\sqrt{2m}$. If

$$m = (1 - \vartheta)\frac{k^2}{2} + \vartheta\frac{(k+1)^2}{2},$$

with $0 \leqslant \vartheta < 1$ and $k \in \{0,1,2,\dots\}$, then $n_0 = k$ and $g(n_0) = k + \vartheta$.

Consequently, as a function of m, $g(n_0)$ is the piecewise linear function interpolating $H_c(m) = \sqrt{2m}$ at the points $\frac{1}{2}k^2$. Therefore, our proof will be completed if a sequence $(a_n) \in S_m$ exists for which $|a_1 - a_0| = g(n_0)$. One easily verifies, with help of Formula (3), that the following sequence has the desired property:

$$a_n = \begin{cases} -\frac{1}{2}n^2 + \frac{1}{2}n + g(n_0)(n - \frac{1}{2}) & (n = 0,1,\dots,n_0 + 1), \\ \frac{1}{2}n^2 - \frac{1}{2}n + g(n_0)(n - \frac{1}{2}) & (n = 0,-1,\dots,-n_0), \\ m & (n > n_0 + 1), \\ -m & (n < -n_0). \end{cases}$$

REMARK. If $0 \leqslant m \leqslant \frac{1}{2}$, then $H_d(m) = 2m$. This immediately follows from the condition $|a_n| \leqslant m$ $(n \in \mathbb{Z})$. In this case the restriction $|\delta^2 a_n| \leqslant 1$ has no effect.

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Digital data is encoded in increasingly sophisticated ways. It is encoded not only for simple storage and transmission. It is compressed, it is encrypted, and errors are automatically detected and corrected. A computer scientist should be aware of the capabilities of various coding schemes and the algebraic principles behind them. A mathematician should be aware of devising these codes and investigating their properties as an active research area within applied algebra. For both the computer science student and the mathematics student the topic provides an application of linear algebra, Galois theory, and discrete mathematics to real world problems.

The following is a discussion of a special topics course in algebraic coding theory at the University of Michigan-Dearborn. The subject is not usually taught at the undergraduate level, but a course in coding theory can be a valuable (and surprisingly popular) addition to the undergraduate curriculum.

Background. The essential background for such a course includes elementary linear algebra and certain topics from discrete mathematics. The students should be familiar with the notions of vector space, subspace, basis, etc. They should be familiar with matrices and simple operations on them. They should also be familiar with simple counting (permutations and combinations), proofs by induction, and some graph theory (trees, Eulerian and Hamiltonian paths). Abstract algebra is helpful, but the required material can be developed as needed. The content and pace of the course can be adjusted to the background of the students. My choices and possible alternatives are discussed below.

The course was taught in the winter and spring terms of 1983 and was cross-listed under three headings: mathematics, computer mathematics, and computer science. Most students elected it under one of the latter two headings, and many were computer science majors who had just finished linear algebra and discrete mathematics. With these students in mind I took a concrete approach, emphasized practical examples, and presented preliminary material on vector spaces over finite fields (see the discussion of the Gray code and the ISBN code).

Sources. Many of the early and central papers in algebraic coding theory require no more background than the minimum indicated above. Undergraduates seldom learn their mathematics by reading primary literature, but this is a field in which they can do just that. The particular papers I chose are mentioned below, but there is substantial flexibility in this area. The initial survey of coding theory was documented by handouts which I prepared as well as by the appropriate papers. I found no single suitable text for this segment of the course. The topic of coding for automatic error control is well presented at an undergraduate level in the book by Vera Pless [1]. The explanations are clear, examples are plentiful, only elementary linear algebra is assumed, and just enough is left for the student to discover. The text material was supplemented by original papers.

Course Structure. The course was divided into two sections, each of approximately twenty class sessions. The first was an introduction to the history, scope, and algebraic tools of coding theory. The second was a deeper look at coding for error detection and correction.

The choice of introductory topics was governed by four considerations:

1. The topics should illustrate the scope of algebraic coding theory.
2. They should be applications of familiar ideas from discrete mathematics.
3. They should give the students the experience of learning a topic by reading the original paper.
4. They should provide an opportunity to review linear algebra and extend previously mastered material to vector spaces over finite fields.

Clearly not every topic could fulfill every requirement, and the final choice was a compromise in which the first requirement weighed most heavily.

The early history of coding for data transmission was illustrated by a brief discussion of Morse code and Baudot code (International Telegraph Code I) as well as various early weighted codes. The tradeoff between compactness and error resistance is already evident in these early codes.

Morse code is a familiar code in which compactness is achieved by assigning shorter coded forms to more frequently used letters. Thus Morse code provides a natural introduction to optimal prefix codes (Huffman codes) and the whole topic of data compression. (A prefix code is one in which no coded form is a prefix of (the beginning of) another coded form. An optimal prefix code is one which minimizes what Huffman called the average message length (the expected length for the coded form of one item). $L_{\text{av}} = \Sigma_i l_i \cdot p_i$, where l_i is the length of the coded form for the ith

item and p_i is the probability that the ith item will occur.) The modern approach to Huffman codes is to present them in terms of coding trees, and we used that approach when reading his original paper [2]. The paper is lucidly written, but many of the results can be stated more compactly in terms of trees. Much of data compression is concerned with data other than text (instrument readings, digital pictures, the sounds of human speech, etc.). Codes for this type of compression were mentioned only briefly, and the student was referred to the excellent book on this topic by Davisson and Gray [3].

We then moved from data compression to special purpose codes, from the minimum redundancy Huffman code to the minimum distance Gray code. (A Gray code on bit strings of length n is a list of all the 2^n bit strings of this length in which each bit string differs from its successor in exactly one entry, and the last differs from the first in exactly one entry.) The students were intrigued to discover that this practical code (developed for use in analog to digital converters) is elegant as well as useful. Gray's original recursive construction [4] is simplicity itself and shows that an entire class of graphs (unit cubes in n dimensions) possess easily constructible Hamiltonian cycles. Also, the transformation from a binary number to the corresponding Gray code introduces the students to linear transformations on bit strings (that is, on vectors over the binary number field Z_2) and to the peculiarities of linear algebra over this simplest of finite fields.

The students were also intrigued to discover that the solution to the Towers of Hanoi (that paradigm of "useless" computer science problems) produces this practical analog to digital converter code. (The key is to represent the state of the disks after each move by a bit string in which each bit represents one disk and is set to 0 or 1 according to whether the total number of moves for that disk has been even or odd. The initial state is represented by a string of 0's. For more detail see [5].)

For mathematical balance we then mentioned chain codes. (A chain code on bit strings of length n is a list of some or all of those strings in which the last $n-1$ bits of each string are the first $n-1$ bits of its successor. The first is the successor of the last.) These equally practical analog to digital converter codes can be produced by constructing not Hamiltonian cycles but rather Eulerian cycles in suitable graphs [6, p. 99].

The material for the second half of the course divides naturally into two topics:

1. coding for control of errors made by people and
2. coding for control of errors made by machines.

Human blunders in transcribing numbers or text are often automatically detected (though not automatically corrected) by weighted mod p codes, where p is prime. (In a numeric code each digit of the number is assigned a weight. Correct n digit numbers are those for which $\sum_{i=1}^{n} w_i \cdot d_i \equiv 0 \bmod p$, where d_i represents the ith digit, and w_i represents its weight. In general, $n-1$ of the digits in a number can be chosen freely and the remaining digit chosen so that the check equation is satisfied. The scheme can be extended to text if each letter is assigned a numerical value. The error detecting properties of a scheme are determined by the choice of the prime p and the weights w_i.) Weight schemes have been devised which detect not only single errors but also transpositions and other common typographical errors. The International Standard Book Number code [7] is a weighted mod 11 scheme for purely numeric information and provided our introduction to such codes. Closer to home, every student identification number at the University of Michigan-Dearborn is a social security number followed by a check digit which is computed according to a modified weighted mod 11 scheme. The students were interested to discover how and why that extra digit had been added to their social security numbers.

For a detailed frequency analysis of various typographical errors, I referred the students to J. Verhoeff [8]. Verhoeff discusses numerous weighting schemes for mod p codes as well as several purely decimal error detecting codes.

The weighted mod p codes are linear codes (group codes) over the integers mod p and provide an introduction to vector spaces over Z_p. (A linear code over a field F is a vector space over F.) Modular arithmetic may be reviewed at this point, especially if the operations in Z_2 were presented not as multiplication and addition mod 2 but as the logical operations of conjunction and exclusive disjunction.

We approached coding for control of machine errors by means of the binary Hamming single error correcting code. Hamming's classic paper presents the code concretely in terms of interleaved partial parity check equations [9]. I think the students benefit from having this code as a familiar example when they study the theory of linear codes. As mentioned, we followed the text by Vera Pless in our study of linear codes. I did however use the paper by Peterson and Brown [10] to introduce cyclic codes. (A cyclic code is one in which a cyclic shift of a valid codeword is also a valid codeword. These codes are polynomial codes. That is, each bit string is considered as a polynomial (for instance, 1101 is considered as the polynomial $x^3 + x^2 + 1$), and valid strings are those which are divisible by a fixed polynomial called the generator polynomial for the code.)

The applications of these codes were presented in a daily feature we came to call "coding in the news." Coding for error control is frequently mentioned in computer science and engineering periodicals. The codes tend to be mentioned in two contexts:

1. Data communication, especially in computer networks and along satellite links. The codes are usually cyclic codes and appear in descriptions of specific communication protocols.

2. Fault-tolerant computing. The codes are usually Hamming type codes and are used for memory scrubbing.

Electronics magazine is a particularly good source for these articles, and I duplicated a number of them for the students. Before long the students were bringing in their own contributions to "coding in the news."

Comments. If time permits, an initial survey of coding theory might include cryptography. In a three hour semester course I found that time did not permit, and I decided to concentrate on aspects of coding with which students were less likely to be familiar. The National Bureau of Standards data encryption standard is a widely used key driven code for which the algorithm is public (see [11]). This code might be discussed as might public key cryptography. Many cryptographic schemes from the pre-computer age are ingenious, but the codes are so easily broken by computer that they are no longer widely used.

In a course designed strictly for mathematics majors one might approach coding more abstractly. Each topic need not be introduced by a practical example, and "coding in the news" could be dropped entirely. If abstract algebra were a prerequisite, an entire course might be taught from Vera Pless's book. The study of multiple error correcting codes involves the study of polynomials over finite fields. If the students have had some Galois theory, one might devote half the semester to multiple error correcting codes.

One ought to be prepared to do some preliminary advertising for a course in coding theory. The initial response when I first proposed the course was one of puzzlement. Most students had never heard of algebraic coding theory. There was concern that the course would not attract enough enrollment to run. So I gave a colloquium talk on the Hamming code, I circulated a short explanation of the course, and I answered student questions. And the course did run. Instead of the usual dozen students we expect in a special topics course, the course attracted some fifty students (either enrolled or officially wait-listed) the first term. Some thirty five could be accepted in the winter term, and the course is being repeated in our short spring term, again to a full house. There is even interest in a sequel. The course is a challenge, but this fascinating and important topic can be taught successfully to undergraduates.

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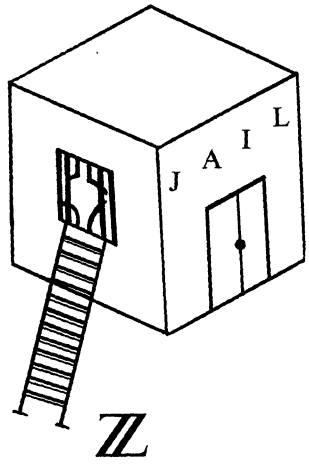
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MORE REBUSES

According to the American Heritage Dictionary, a *rebus* is a riddle composed of words or syllables depicted by symbols of pictures that suggest the words or syllables they represent. Can you decipher the ones below?

A few additional mathematical rebuses will appear in this department from time to time. If you can think of others, please send them to the editor.

1.



—Jim Griffith

2. R O O
N T
G I S

—Sharon Ozersky

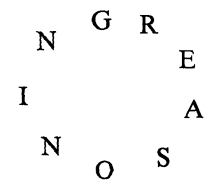


—Janet Tremain

4. ALGEBRA = ARBEGLA

—Dale Bachman

5. space space space

6.


—Carl Morris

7. ROUPG

Answers on p. 521.

133. MISCELLANEA

...a remarkable change is at present passing over the mathematical work done in the universities and colleges of this country. Courses that a short time ago were offered in only a few of our leading universities are now not uncommon in institutions of college rank. They are often given by men who have taken advanced degrees in mathematics at Göttingen, Berlin, Paris, or other leading universities abroad, and they are awakening a great interest in the modern field. A recent investigation (1903) showed that 67 students in ten American institutions were taking courses in the theory of functions, 11 in the theory of elliptic functions, 94 in projective geometry, 26 in the theory of invariants, 45 in the theory of groups, and 46 in the modern advanced theory of equations, courses which only a few years ago were rarely given in this country.

—David Eugene Smith, *History of Modern Mathematics*, Wiley, New York, 1906, pp. 75–76.

ANSWER TO PHOTO ON PAGE 496

H. A. Schwarz, of lemma and inequality fame.

word "almost" cannot be omitted from (a).

Also solved by Preben Alsholm, Michael B. Gregory & Gerri M. Dunnigan, John H. Riley, Jr., Anton R. Schep, John C. Tripp, Alfonso Villani (Italy), and the proposer.

On Primitive Roots

6420 [1983, 60]. *Proposed by Barry Powell, Kirkland, WA.*

For any odd prime p, let g_p and G_p be the least positive and greatest negative primitive root of p respectively. Prove that for any positive integer M, there exist infinitely many primes p for which $M < g_p < (p - M)$ and $(-p + M) < G_p < -M$.

Solution by Keith Kearnes (student), Riverside CA. The proposition follows from an easy lemma:

LEMMA. *For any positive integer M, there exist infinitely many primes p for which $(-1/p) = (q_i/p) = 1$ where q_i $(i = 1, 2, \ldots, n)$ runs through the primes $\leqslant M$.*

Proof. Suppose that the prime $p = 1 + 8q_2q_3 \cdots q_nk$ for some integer k. Then, by Euler's criterion, $(-1/p) = (-1)^{(p-1)/2} = 1$. Further,

$$\left(\frac{q_1}{p}\right) = \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = 1.$$

By the Law of Quadratic Reciprocity

$$\left(\frac{p}{q_i}\right)\left(\frac{q_i}{p}\right) = (-1)^{(p-1)(q_i-1)/4} = 1.$$

But $(p/q_i) = (1/q_i) = 1$ so that $(q_i/p) = 1$, and p has the required properties. That there are infinitely many primes of this form is a consequence of Dirichlet's theorem.

Let p be a prime satisfying the conditions of the lemma. Then all integers r such that $-M \leqslant r + tp \leqslant M$ (where t is any integer) are quadratic residues modulo p, and therefore not primitive roots of p. Since there is a primitive root between 0 and p, it follows that $M < g_p < p - M$ and, since $G_p = -g_p$, that $-p + M < G_p < -M$.

Also solved by L. E. Mattics, Robert E. Shafer, University of South Alabama Problem Group, and the proposer.

Mattics showed that M can be replaced by $c \log\log p$ with $0 < c < 1$. He used a theorem of Birkhoff and Vandiver (*On Integral Divisors of $a^n - b^n$*, Ann. of Math. (2) vol. 5 (1904) pp. 173–180) which ensures that $2^{8q_2 \cdots q_n} - 1$ is divisible by a prime $p \equiv 1 \pmod{8q_2 \cdots q_n}$ and hence that $q_n > cn \log q_n > c \log\log p$ for n sufficiently large.

ANSWERS TO REBUSES ON PAGES 513–514

1. Free group
2. Square root sign
3. Improper integral
4. Commutative algebra
5. 3-space
6. Circular reasoning
7. Group of odd order.

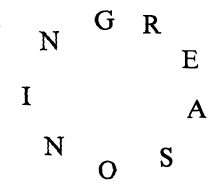


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Answers on p. 521.

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ANSWER TO PHOTO ON PAGE 496

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PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS),
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*Send all **proposed** problems, typed and in duplicate if possible, to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by February 28, 1985. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3054. *Proposed by Vania D. Mascioni (student), Swiss Federal Institute of Technology, Zurich, Switzerland.*

If α_1, α_2, α_3 are the angles of a triangle with sides a_1, a_2, a_3, inradius r, area A and semi-perimeter s, prove the following inequalities:

(a) $$\prod_{i=1}^{3} \alpha_i a_i \geqslant \left(\frac{2\pi}{3}\right)^2 rA,$$

(b) $$\prod_{i=1}^{3} (\pi - \alpha_i) a_i \geqslant \left(\frac{4\pi\sqrt{3}}{9}\right)^3 sA.$$

When does equality hold?

E 3055. *Proposed by Mark F. Kruelle (student), Wright State University.*

Find all solutions of the Riccati equation $u' = u^2 + (a/x) - b$, $a, b \neq 0$, which are real rational functions of x.

E 3056. *Proposed by Bruce Reznick, University of Illinois at Urbana-Champaign.*

For which integers $n > 1$ and real numbers r does the curve $y = x^n + rx$ contain the vertices of a rectangle?

E 3057. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

Prove that for all $n \geqslant 3$,

$$\sum_{k=1}^{n} k^3 \binom{n}{k} D_{n-k} = 5n!,$$

where $D_m = m! \sum_{r=0}^{m} (-1)^r \frac{1}{r!}$ denotes the derangement number (of $1, 2, \ldots, m$).

E 3058. *Proposed by Allen J. Schwenk, U.S. Naval Academy.*

(a) We all know that a permutation which is an n-cycle, for example $(1\ 2\ 3 \ldots n)$, cannot be written as a product of fewer than $n - 1$ transpositions. Prove it.

(b) In how many ways can $(1\ 2\ 3 \ldots n)$ be written as a product of precisely $n - 1$ transpositions?

SOLUTIONS OF ELEMENTARY PROBLEMS

Compact Sets of Unit Width

E 2834 [1980, 404; 1983, 286]. *Proposed by James W. Fickett, Texas A & M University.*

The width of a compact subset S of $\mathbb{R}^3$ is the least $d \geqslant 0$ such that S lies between some two parallel planes a distance d apart. (i) Prove or disprove: There is a constant $\gamma > 0$ such that any compact set of unit width contains a four-point subset of width at least γ. (ii)* Generalize to $\mathbb{R}^n$.

Solution to (i) *by Irl C. Bivens, Davidson College.* One such constant that meets the requirements of the problem is $\gamma = \sqrt{21}/21$. First, choose two points A and B in S which maximize dist(A, B). Next, select a point C in S such that dist$(C,$ line $AB)$ is a maximum. Finally, choose a point D in S which maximizes dist$(D,$ plane $ABC)$. Each of these choices is possible by the compactness of S. Since S is of width one, we must have

$$2r = \text{dist}(A, B) \geqslant 1, \qquad s = \text{dist}(C, \text{line } AB) \geqslant 1/2,$$
$$\text{and } t = \text{dist}(D, \text{plane } ABC) \geqslant 1/2.$$

It is routine to show that the width of $T = \{A, B, C, D\}$ is realized either by the distance of one point of T to the plane determined by the other three points or by the distance between a pair of parallel planes, each containing two points of T. By our construction dist$(D,$ plane $ABC) \geqslant 1/2$. There are thus six remaining cases which must be considered. Listed below are a couple of elementary computational results we will need.

(1) If $k > 0$, then in the first quadrant $x^2y^2/(kx^2 + y^2)$ is an increasing function of x and y.

(2) Let P, Q, R, and S denote four noncoplanar points in $\mathbb{R}^3$ with K_{PQ} and K_{RS} the unique pair of parallel planes containing P, Q and R, S, respectively. Then

$$\text{(a)} \qquad \text{dist}^2(P, \text{plane } QRS) = \frac{\text{dist}^2(P, \text{line } RS)\ \text{dist}^2(Q, \text{plane } PRS)}{\text{dist}^2(Q, \text{line } RS)}$$

and

$$\text{(b)} \qquad \text{dist}^2(K_{PQ}, K_{RS}) = \frac{\text{dist}^2(P, \text{line } RS)\text{dist}^2(Q, \text{plane } PRS)}{\text{dist}^2(Q, L)}$$

where L denotes the line through P parallel to line RS.

In the diagram of Fig. 1, $EFGH$ denotes a rectangle in plane ABC with L the line through A parallel to line BC. If D' denotes the point in plane ABC which is closest to D, then our selection of A, B, C, and D implies D' belongs to the interior of rectangle $EFGH$. Thus

$$\text{dist}^2(D, \text{line } FG) = \text{dist}^2(D', \text{line } FG) + t^2 \leqslant 4s^2 + t^2.$$

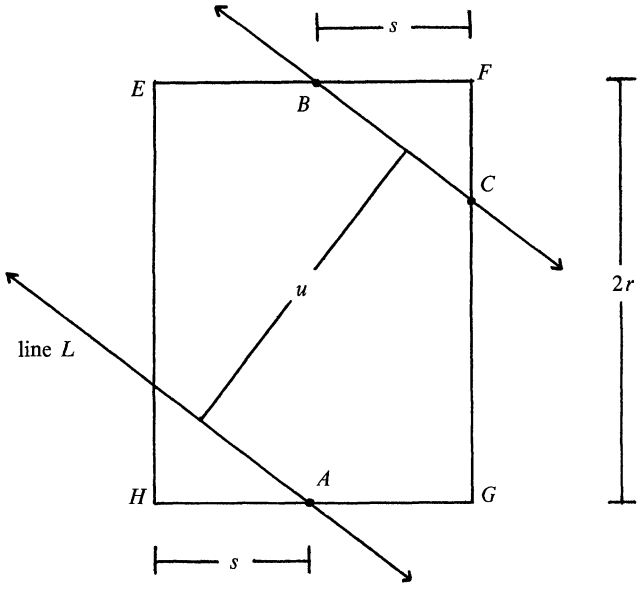


FIG. 1

Since $r, s \geqslant 1/2$ it follows that

$$u = \operatorname{dist}(A, \text{line } BC) = \operatorname{dist}(L, \text{line } BC) \geqslant \sqrt{5}/5.$$

Furthermore, an easy argument shows that every point inside rectangle $EFGH$ lies at most distance $2u$ from either L or line BC. Consequently

$$\operatorname{dist}^2(D, L) = \operatorname{dist}^2(D', L) + t^2 \leqslant 4u^2 + t^2$$

and

$$\operatorname{dist}^2(D, \text{line } BC) = \operatorname{dist}^2(D', \text{line } BC) + t^2 \leqslant 4u^2 + t^2.$$

Now

$$\operatorname{dist}^2(C, \text{plane } ABD) = \frac{s^2t^2}{\operatorname{dist}^2(D, \text{line } AB)} \geqslant \frac{s^2t^2}{s^2} = t^2 \geqslant 1/4,$$

$$\operatorname{dist}^2(K_{CD}, K_{AB}) = \frac{s^2t^2}{\operatorname{dist}^2(D, \text{line } FG)} \geqslant \frac{s^2t^2}{4s^2 + t^2} \geqslant 1/20, \text{ since } s, t \geqslant 1/2,$$

$$\operatorname{dist}^2(A, \text{plane } BCD) = \frac{u^2t^2}{\operatorname{dist}^2(D, \text{line } BC)} \geqslant \frac{u^2t^2}{4u^2 + t^2} \geqslant 1/21, \text{ since } u \geqslant \sqrt{5}/5, t \geqslant 1/2,$$

and

$$\operatorname{dist}^2(K_{AD}, K_{BC}) = \frac{u^2t^2}{\operatorname{dist}^2(D, L)} \geqslant \frac{u^2t^2}{4u^2 + t^2} \geqslant 1/21.$$

Likewise $\operatorname{dist}^2(B, \text{plane } ACD)$, $\operatorname{dist}^2(K_{AC}, K_{BD}) \geqslant 1/21$ and the result follows.

Editor's note. It would be of interest to determine the largest possible γ with the required property.

Simple Closed Curves and Two Consecutive Steiner Symmetrizations

E 2856 [1980, 755]. *Proposed by F. W. Luttmann, Sonoma State University.*

Does there exist a simple closed curve, other than the circle, such that two consecutive Steiner symmetrizations with respect to two orthogonal lines always produce a circle?

Solution by Peter Loomis, Lockheed Corporation, Sunnyvale, CA. The answer is no. Suppose K is a compact convex set in the plane with the property that every pair of consecutive Steiner symmetrizations in orthogonal lines turns it into a circle. We show that K is a circle. Let some direction be chosen, and let the width of K in that direction be w. If K is symmetrized with respect to a line in this direction, its width in this direction is unchanged. If this new body is now symmetrized in the orthogonal direction, the width in the original direction is again unchanged because there is a section of length w in this direction. But a circle is obtained by hypothesis, and the width of this circle is its diameter. The width w is therefore equal to the diameter of this circle. Since the circles obtained by each pair of orthogonal symmetrizations are all of the same area as K and hence of the same diameter, K's width in every direction is the same, namely equal to the diameter of the circle of equal area. So K is a curve of constant width. It is well known that such a curve has as its perimeter that of the circle of the same width. But if K has both the perimeter and the area of a circle, it must *be* a circle by the isoperimetric inequality.

An Analytic Characterization of Egyptian Fractions

E 2934 [1982, 212]. *Proposed by R. P. Boas, Northwestern University.*

Let f be a real-valued continuous function on $[0,1]$ and let h be a number between 0 and 1. Suppose the average of f over each subinterval of $(0,1)$ of length h is less than 1. Can the average of f over $[0,1]$ be greater than 1?

Solution by Edward T. Ordman, Memphis State University. This can happen if $h \neq 1/n$ for any positive integer n; for small enough ε, greater than 0, $f(x) = 1 - \varepsilon + \sin(2\pi x/h)$ is an example. Note that $\int \sin(2\pi x/h)\,dx$ is 0 over any interval of length exactly h and positive over the interval $[0,1]$ since 1 is not a multiple of h. Let δ be the value of the integral over $[0,1]$ and pick a positive ε less than δ. Then $f(x)$ has average value $1 - \varepsilon$ over any subinterval of length h and average value $1 - \varepsilon + \delta > 1$ over $[0,1]$.

If $h = 1/n$ for some positive integer n, then the average value of f cannot exceed 1. To see this, cover $[0,1]$ "approximately" with a collection of subintervals $[\alpha, h + \alpha]$, $[h, 2h], \ldots,$ $[(n-2)h, (n-1)h]$, $[1 - h - \alpha, 1 - \alpha]$, where $nh = 1$. The integral of f over each of these subintervals is less than h, so the integral of f over $[\alpha, 1 - \alpha]$ is less than $nh = 1$. Since f is continuous on $[0,1]$, we may take the limit as α approaches 0 and find that the integral of f over $[0,1]$ does not exceed 1.

Also solved by 57 other readers and the proposer. Several solutions were incomplete.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor G. L. Alexanderson, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by February 28, 1985. The solver's full post-office address should be on each sheet.

6469. *Proposed by A. Wilansky, Lehigh University.*

1. Suppose that a countable set E of real sequences has the property that $E^\beta = L = \{x: \Sigma|x_n| < \infty\}$. Show that E has a finite subset with the same property. (Notation: $E^\beta = \{x: \Sigma x_i y_i$ converges $\forall y \in E\}$.)

2. Does such a set E exist?

6470. *Proposed by M. L. Glasser, Clarkson College.*

Evaluate the definite integral

$$I = \int_0^\infty x^{1/2}\exp\left\{-a^2x(x-b)^2/(x-1)^2\right\}dx$$

for $b \geqslant 1$.

6471. *Proposed by Justin Peters, Iowa State University.*

Let R be a ring and α an injective endomorphism of R; i.e., α: $R \to R$ is a 1-1 ring homomorphism, not necessarily onto. Show that there is a ring S containing R as a subring and an automorphism β of S such that $\beta(x) = \alpha(x)$ for all $x \in R$.

6472. *Proposed by Robin Harte, University College, Cork, Ireland.*

Prove that if T: $H \to H$ is a bounded linear operator on Hilbert space for which $T^2 = 0$, then the following two conditions are equivalent:

(a) $I = UT + TV$ for some bounded linear U and V on H;

(b) $T + T^*$ is invertible.

6473. *Proposed by Mowaffaq Hajja, Yarmouk University, Irbid, Jordan.*

Let k be a field and let R be defined by

$$R = \left\{f(x) \in k[x]\colon f \text{ is monic and } \mathrm{GCD}(f, f') = 1\right\}.$$

Define the binary operations $\odot$ and $\oplus$ on R by

$$f \odot g = \mathrm{GCD}(f, g),$$
$$f \oplus g = \mathrm{LCM}(f, g)/\mathrm{GCD}(f, g).$$

Prove that $(R, \oplus, \odot)$ is a ring.

SOLUTIONS OF ADVANCED PROBLEMS

A Sum of Sines

6418 [1983, 60]. *Proposed by George Benke, Georgetown University.*

Prove that

$$\sum_{n=1}^{2N-1} \frac{\sin\frac{\pi n^2}{2N}}{\sin\frac{\pi n}{2N}} = N.$$

Solution by Otto G. Ruehr, Michigan Technological University. Denote the sum by S_N. Using the well-known (and easily verifiable) identities,

$$\sin 2j\theta/\sin\theta = 2\sum_{s=1}^{j}\cos(2s-1)\theta \text{ and } \sin(2j-1)\theta/\sin\theta = 1 + 2\sum_{s=1}^{j-1}\cos 2s\theta \text{ with } \theta = \pi n/2N,$$

we break S_N into even and odd terms to get

$$S_N = \sum_{j=1}^{N}\left\{1 + 2\sum_{s=1}^{j-1}\cos\frac{s(2j-1)\pi}{N}\right\} + 2\sum_{j=1}^{N-1}\sum_{s=1}^{j}\cos\frac{(2s-1)j\pi}{N}.$$

After interchanging order of summation in the first term, and interchanging j with s in the second, we find that

$$S_N = N + 2\sum_{j=1}^{N}\sum_{s=1}^{N-1}\cos\frac{s(2j-1)\pi}{N}.$$

Finally, since

$$2\sum_{s=1}^{N-1}\cos s\theta = \sin N\theta\cot\theta/2 - (1+\cos N\theta) = 0 \text{ for } N\theta = \pi(2j-1),$$

we are done.

Also solved by S. J. Bernau, Robert Breusch, C. Cosgrove & M. L. Glasser, Robert Heller, E. G. Kwon (Korea), L. E. Mattics, Pei Yuan Wu (Republic of China), Bruno Remillard (Canada), St. Olaf College Problem Solving Group, Matt Wyneken, and the proposer.

Additive Functions on $\mathbb{R}$

6419 [1983, 60]. *Proposed by Wim Vervaat, Cornell University.*

Let f be a real-valued Lebesgue measurable function on $\mathbb{R}$.

(a) Suppose that $f(qx) = qf(x)$ for all $q \in \mathbb{Z}, x \in \mathbb{R}$. Prove that $f(x) = bx$ almost everywhere on $\mathbb{R}$ for some real b.

(b) If $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$, then there is a real b such that $f(x) = bx$ for all $x \in \mathbb{R}$. Many proofs are known for this result. Prove it this time as a corollary of (a).

Solution by Karl Stromberg, Kansas State University.

(a) For rational $r = p/q$ ($p, q \in \mathbb{Z}, q \neq 0$), we have $qf(rx) = f(px) = pf(x)$ and so $f(rx) = rf(x)$ for all real x. Define g and h on $\mathbb{R}$ by $g(0) = 0$, $g(x) = \tan^{-1}(f(x)/x)$ for $x \neq 0$, and

$$h(x) = \int_0^x g(t)\,dt.$$

For any rational $r \neq 0$ we use the fact that $g(rx) = g(x)$ for all real x to obtain

$$h(r) = \int_0^r g(t)\,dt = \int_0^1 g(x)\,r\,dx = ar$$

where $a = h(1)$. Since h is continuous on $\mathbb{R}$, it follows that $h(x) = ax$ for all real x. Thus, for almost all x in $\mathbb{R}$ we have

$$g(x) = h'(x) = a, f(x)/x = \tan(g(x)) = \tan a, \text{ and } f(x) = bx, \text{ where } b = \tan a.$$

(Notice that $|a| < \pi/2$ because $|g(t)| < \pi/2$ for $0 \leq t \leq 1$.)

(b) Induction shows that $f(qx) = qf(x)$ for all positive integers q. Also,

$$f(0) = f(0+0) = 2f(0)$$

so $f(0) = 0$, and

$$f(-qx) + f(qx) = f(0) = 0$$

so

$$f((-q)x) = -f(qx) = (-q)f(x) \text{ for } q > 0.$$

Thus, (a) provides $b \in \mathbb{R}$ such that the set $H = \{x \in \mathbb{R}: f(x) = bx\}$ satisfies $\lambda(\mathbb{R} \setminus H) = 0$ where λ is Lebesgue measure on $\mathbb{R}$. If $c \in \mathbb{R} \setminus H$, then the set $c + H = \{c + x: x \in H\}$ is a subset of $\mathbb{R} \setminus H$ so $0 = \lambda(c+H) = \lambda(H) = \infty$. We conclude that $H = \mathbb{R}$.

REMARK. The example $f(x) = x$ for rational x and $f(x) = 2x$ for irrational x shows that the

word "almost" cannot be omitted from (a).

Also solved by Preben Alsholm, Michael B. Gregory & Gerri M. Dunnigan, John H. Riley, Jr., Anton R. Schep, John C. Tripp, Alfonso Villani (Italy), and the proposer.

On Primitive Roots

6420 [1983, 60]. *Proposed by Barry Powell, Kirkland, WA.*

For any odd prime p, let g_p and G_p be the least positive and greatest negative primitive root of p respectively. Prove that for any positive integer M, there exist infinitely many primes p for which $M < g_p < (p - M)$ and $(-p + M) < G_p < -M$.

Solution by Keith Kearnes (student), Riverside CA. The proposition follows from an easy lemma:

LEMMA. *For any positive integer M, there exist infinitely many primes p for which $(-1/p) = (q_i/p) = 1$ where q_i $(i = 1, 2, \ldots, n)$ runs through the primes $\leqslant M$.*

Proof. Suppose that the prime $p = 1 + 8q_2q_3 \cdots q_nk$ for some integer k. Then, by Euler's criterion, $(-1/p) = (-1)^{(p-1)/2} = 1$. Further,

$$\left(\frac{q_1}{p}\right) = \left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8} = 1.$$

By the Law of Quadratic Reciprocity

$$\left(\frac{p}{q_i}\right)\left(\frac{q_i}{p}\right) = (-1)^{(p-1)(q_i-1)/4} = 1.$$

But $(p/q_i) = (1/q_i) = 1$ so that $(q_i/p) = 1$, and p has the required properties. That there are infinitely many primes of this form is a consequence of Dirichlet's theorem.

Let p be a prime satisfying the conditions of the lemma. Then all integers r such that $-M \leqslant r + tp \leqslant M$ (where t is any integer) are quadratic residues modulo p, and therefore not primitive roots of p. Since there is a primitive root between 0 and p, it follows that $M < g_p < p - M$ and, since $G_p = -g_p$, that $-p + M < G_p < -M$.

Also solved by L. E. Mattics, Robert E. Shafer, University of South Alabama Problem Group, and the proposer.

Mattics showed that M can be replaced by $c \log\log p$ with $0 < c < 1$. He used a theorem of Birkhoff and Vandiver (*On Integral Divisors of $a^n - b^n$*, Ann. of Math. (2) vol. 5 (1904) pp. 173–180) which ensures that $2^{8q_2 \cdots q_n} - 1$ is divisible by a prime $p \equiv 1 \pmod{8q_2 \cdots q_n}$ and hence that $q_n > cn \log q_n > c \log\log p$ for n sufficiently large.

ANSWERS TO REBUSES ON PAGES 513–514

1. Free group
2. Square root sign
3. Improper integral
4. Commutative algebra
5. 3-space
6. Circular reasoning
7. Group of odd order.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Introduction to Coding Theory. By J. H. van Lint. Graduate Texts in Mathematics, Vol. 86, Springer-Verlag, New York, 1982. ix + 171 pp. $24.00.

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Let me begin this review of van Lint's excellent exposition of major topics in coding theory with a brief introduction to the subject. The MONTHLY has published three survey articles; the two by Levinson [4] and Pless [**10**] are general, and the one by Sloane [**12**] discusses connections with classical invariant theory for linear groups. There is an older popular article by Peterson in *Scientific American* [**9**], and an informative introduction to the combinatorial aspects of coding theory by Assmus and Mattson [**1**] that also covers much of the basic material.

Coding theory applies to situations such as this one: suppose the readings from a remote gauge are to be conveyed to an observer over a communication line or *channel*. The readings can be viewed as messages sent to the observer by electronic equipment connected to the gauge. A collection of standard signals (of equal duration) is available and a separate sequence of them is assigned to each possible reading. These sequences can be displayed as *words* made up of letters chosen from an *alphabet* A used to label the signals (this terminology is common, even if A consists of conventional digits). After the gauge registers a particular reading, the corresponding word is sent over the line. That is, the electronic equipment transmits the sequence of signals the word prescribes. The observer receives the word and records the reading specified.

This routine will be upset if there is noise on the line, leading to possible misinterpretation of a received signal. The model for the effect of noise used in the book works from the following assumptions: first, the channel is *memoryless*, the passage of a signal having no influence on later signals. Second, each time a letter is sent one is received (there are no *erasures*). Third, for any two different letters a and b, there is a fixed probability that b is received when a is sent, independent of a and b. The most likely letter to be received when a is sent, though, is a itself.

The list of words corresponding to possible readings is called a *code* and its members *codewords*. Normally the codewords all have the same number of letters; the code is then a *block code* and that common number is its block *length*. On receiving a word, the observer *decodes* it by finding the codeword whose transmission would give the highest probability of receiving the word actually received. (If there is more than one candidate, the observer might select one at random, or simply declare a *decoding failure*.) This scheme of *maximum likelihood* decoding amounts to finding the codeword nearest the received word in the *Hamming metric*, the distance between two words being the number of positions in which they have different letters.

In the abstract, then, a (block) code of length n over the finite alphabet A is a nonempty subset C of A^n, the set of n-tuples or words of length n with entries in A. Suppose the minimum distance between distinct members of C is d, and e is the integer part of $(d-1)/2$. Then in the decoding method above, correct decoding will occur if at most e letters in the transmitted word are received incorrectly; briefly, C is an *e-error correcting code*. If there are M readings or messages to be encoded, and the alphabet A has q letters, the length of the code will have to be at least $\log_q M$ since there are q^l words of a given length l. For a code of length n with M words the number $(\log_q M)/n$ is called the *rate* of the code. It is a measure of how much information a codeword conveys, or what portion of the letters in a codeword is devoted to meaningful content as distinguished from provision for error correction. (M need not be a power of q.)

Consider the probability of correct decoding when a codeword is sent; the minimum such

probability among codewords is a numerical measure of the reliability of the code as a whole. With M messages to be communicated, intuition suggests that using longer words allows codes with higher reliability. For at the least, one could encode the messages with words of some convenient length and then make long words by simply repeating each original word over and over. With sufficient repetition the correct letter in a given position of the original word should appear more frequently than any other in the corresponding repeated positions of the long word received. Of course the rate goes down as the length goes up. One way to try to compensate for that is to accumulate consecutive messages and send them in groups, say of size g. If the original M messages were encoded by words by length n, the M^g possible amalgamated messages could be encoded by words of length gn without changing the rate. But now it is much less plausible that the increased length offers a chance of boosting reliability, in light of the vast increase in the number of messages. Yet this is indeed the case, with one important qualification: associated to the channel is a quantity called its *capacity*, depending only on the probabilities involved in the channel (and positive for the model at hand). The *channel coding theorem* guarantees that if R is a rate strictly less than this capacity, and if r is a given reliability ($r < 1$), then for suitably large n, there is a code of length n whose rate is at least R and whose reliability is at least r.

This fundamental theorem of C. E. Shannon appeared in 1948, marking the beginning of coding theory. Its proofs are not constructive, and coding theory developed in the systematic search for useful codes. The most common alphabet is the binary one composed of 0 and 1, reflecting on-off choices in circuits. It seems natural to regard the words created as vectors over the field of two elements. At the very outset that additional structure helped in creating codes. The resulting *algebraic* coding theory involves such subjects as group theory, polynomial algebra, and field theory, primarily in dealing with the problems of linear algebra that arise. Now it is not just the error-correcting capability of a code that measures its quality; a practical code should have reasonable encoding and decoding procedures. These must circumvent the apparent need for a search through the list of messages (usually numerical data) in the encoding process and for one through the code in decoding received words. Generally, codes constructed with additional algebraic structure will come with algorithms for these procedures. The algebraic viewpoint has also produced an understanding of the relations among parameters of a code, for example, its length, its size, and the distribution of the distances between codewords. This is especially true for codes belonging to broad classes, such as cyclic codes, whose codewords remain codewords when their letters are cycled. The paper of Sloane cited [**12**] describes recent major results in the study of parameter relations.

In its present state, coding theory is an excellent topic to have in a mathematics curriculum. One reason is that it blends two rather disparate subjects, abstract algebra and probability. Each of these can be pursued separately in its relation to coding theory, algebra in group theoretic and combinatorial connections, and probability in information and communication theory. When coding theory is presented in the engineering curriculum, it is usually in its relation to communication theory as a whole; it can open a door into that theory for a mathematics student. The main reason we introduced a course on coding theory at the University of Virginia, however, is that the algebraic ideas involved appear so concretely. For example, the alphabet A is usually a finite field, as in the binary case. Finite fields then become more than special instances of fields in general; they are forced to be reckoned with, so to speak. The course is a semester one at the sophomore-junior level, with the recent book by Pless [**11**] as a text. One attractive feature of coding theory, though, is that it can be presented at almost any level. It certainly helps if students have a background in linear algebra, although coding theory is a rich source of material for talks to high school and even junior high school students. We have given upper division and graduate courses in it (combining it with other topics in algebraic and conventional combinatorics). We offer the lower-level course every semester, and in testimony to the interdisciplinary nature of the subject, the group of faculty wishing to teach it no longer contains algebraists only.

There is only a modest collection of texts on coding theory, even with those primarily on

information theory counted in. Of course they overlap, but each is distinctive enough that selection is not difficult. The one by McEliece [8] contains a helpful guide to the others (Pless's book and a revised edition [5] of an earlier one by Lin have appeared since it was written). This new book by van Lint, involving both a revision of his well-known Springer notes [6] and a translation of a Dutch text, meshes effectively with the others and continues the distinctiveness of its earlier version. It is a carefully and compactly written introduction to modern coding theory. If a colleague, especially one contemplating teaching our course, were to ask for a survey longer than a paper, I would recommend van Lint's book. It presents its survey not by merely listing results, but by giving detailed expositions of key aspects of most major areas. (The relations to combinatorics are only touched on because van Lint's book with Cameron [2] covers them.) Each chapter contains references and suggestions for further study, so that this book also incorporates a guide to the literature. As a text for advanced undergraduate and graduate students, the book is enhanced by a first-rate collection of problems whose solutions and discussions occupy a ninth of it.

The *ambient space* A^n for codes of length n over the alphabet A will be a vector space over A when A is a field. Codes that are subspaces of A^n are called *linear* codes. Although most of the book deals with linear codes, the last two chapters provide introductions to arithmetic and convolutional codes. (Arithmetic codes are for checking and correcting ordinary arithmetical operations; and convolutional codes, which encode data streams, have in effect words of infinite length. The book includes some recent work on automorphisms of convolutional codes that holds promise for deeper understanding of their structure. About half the book by Lin and Costello [5] is on convolutional codes.) There are several chapters on special classes of linear codes, emphasizing a variety of algebraic ideas. The one on cyclic codes displays the author's ability to focus on significant details. The vector space A^n can be regarded as the group algebra of a cyclic group of order n, and the author shows how properties of cyclic codes relate to idempotents in that algebra. A clear presentation of the decoding algorithm for the important and much-used cyclic codes called BCH codes gives the reader an understanding of its principles and a background for more technical expositions.

The excellent chapter on bounds involving block length, code and alphabet size, and error-correcting capability covers all major inequalities. About ten years ago Delsarte developed a linear programming method for obtaining bounds, based on a study of association schemes [3]. This chapter succinctly explains how that technique has recently led to the best-known bounds. A combinatorial topic related to bounds is that of perfect codes. If for each codeword in an e-error correcting code, one takes the "sphere" of radius e centered at that word, consisting of the words within distance e of it, these spheres must be disjoint. The code is *perfect* when every word of the ambient space is in one of these spheres. The author and his colleagues have contributed much to studying the packing problems involved [7], and a chapter deals with these codes and variations known as uniformly packed codes. Early in the development of coding theory Golay discovered two now-famous perfect codes related to Mathieu groups; another chapter on special codes toward the beginning of the book contains descriptions of them. The material on block codes concludes with a chapter on a high point in algebraic coding theory, Justesen's construction of asymptotically good codes.

Many people come to coding theory from an algebraic background, and their outlook will influence the way they see the structure of the subject. For example, parts of the theory can be considered as applications of group representation theory, and several mathematicians have approached these parts that way. The presentation in this book is an inspiration to mull over the ideas, examine them from all angles, and add one's insights. I recommend it highly.

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Bits 'n Bytes about Computing: A Computer Literacy Primer. By Rachelle S. Heller and C. Dianne Martin. Computer Science Press, Rockville, Maryland, 1982. x + 174 pp. $17.95.

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Computer Literacy! A catchy phrase, which has caught on as few others have in this jargon-ridden field. Like other catch phrases, its meaning has changed over time. The earliest use of the phrase was intended to apply to skill in reading and writing computer programs. After all, what does "literacy" mean?

Computer literacy now means almost the exact opposite—knowing some general facts about the computer without necessarily being able to use (program) it. It is too bad, since just about anyone who has English literacy (and can read the newspaper) has computer literacy. But far fewer can actually make the beast do what they want it to do—yet!

Partial blame for this dilution of meaning falls on "... experts in various areas of the societal impact of computers," who, in 1978, specified that "Computer Literacy" involves (page 2):

(1) how computers work;
(2) step-by-step technique for problem solving;
(3) social implications; and
(4) computer applications.

Item (1) excites me about as much as "how automobiles work." Surely important, but would one glamorize that topic with the phrase "automotive literacy"? Item (2), "step-by-step technique" perhaps means programming, but that is not said. What do items (3) and (4) mean, except the obvious? Should we inform our students that "computers are used in word processing" when glancing into the school office will reveal that fact. At little more than a hundred dollars, some of our students might even own word processors. In contrast, programming is the skill that allows one to expand the role of the computer and to "control" it, rather than be controlled by it.

Back down from the soap box, if one accepts the four-part definition of computer literacy quoted above, then this book serves a role. It is not a text, but is intended for use by teachers, presumably at the elementary and early secondary levels, as a resource for projects and ideas. It contains cultural, historical, and background information on computing. The book honestly admits that programming is important, but leaves that task to others. The chapter titles accurately reflect the flavor of the book:

What is Computer Literacy?
What Do People Think About Computers?
Where Did Computers Come From?
How do Computers Work?
What Do Computers Do for Us?

Where Do We Fit into the Computer Picture?
How Will Computers Affect Our Lives?

Related reading lists identify books accessible to young students, books with titles like "How to Talk like a Martian" and "My Friend the Computer." Unfortunately, the readings suggested for teachers are thin in number and substance and, with few exceptions, date from the early or mid-1970's.

A good point—the book includes brief descriptions of jobs in computing. But despite the boom in computers, it is estimated that only about one percent of each graduating class will eventually find employment as programmers or analysts.

Although not discussing programming in detail, the book suggests in numerous ways that the important function of computing is arithmetic. Several examples using flowcharts (more about this later) are about arithmetic problems. Extensive play is given to number representations used in primitive cultures. (This reviewer has never seen why talking about knots in string has anything to do with understanding computers. History of arithmetic, yes; primitive cultures, yes; computing, no.) More to the point, the stress on binary numbers as fundamental to the general understanding of computers is misplaced, if for no other reason that, in a few years, most computers will work solely with decimal numbers.

A more serious problem with this stress on arithmetic is that by far the most interesting and crucial applications are based on numbers only indirectly or not at all. Character processing and text manipulation are obvious examples. General data processing applications use arithmetic only trivially. The interesting applications involve sorting, searching, pattern recognition (as in speech recognition), and artificial intelligence. Airline reservation systems and electronic telephone exchanges involve symbol manipulation, not arithmetic.

Concerning techniques for representing programs, I agree with the use of "structured English" (pages 63–64), but not with flowcharts (pages 65–66). Just about every writer in the field today agrees that flowcharts are worthless. To give either teachers or their students the idea that is what computing is about is a mistake.

An advantage of the history of this field is that most of the originators are still alive, and they and others have begun to document their work using the methods of historians. (See the references.) Writers of books like this one, and teachers themselves, should be encouraged to consult such historical sources. For instance, the first computer bug was already dead when found, and was not "creeping around the relays" (as stated on page 38) [**1**]. Another mild historical error occurs on page 109, where it is stated: "It is interesting to note that MARK I, the first general-purpose digital computer developed in 1944, was designed to meet a military need; the production of ballistic weapons tables for the new weapons developed during World War II." Actually, it was the ENIAC that was developed to compute ballistic tables, although that machine was not put into service until 1946, after the war [**2**]. It is true that MARK I went into regular service in 1944, but it was completed and demonstrated in 1943.

It doesn't take a genius to realize that the computing field is growing with blinding speed. Any book attempting to cope with such growth risks being obsolete before it is published. This book is no exception. Although published recently (in 1982), it makes no more than a passing reference to the explosion in personal computers. In fact, it may soon be cheaper to buy a real computer than to purchase the materials needed to construct a model of one! My recommendation to any teacher of this material at any level is to buy (or borrow, or get your school to buy) a real computer and learn how to use (program) it. Genuine computer literacy will then be achieved.

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It is abundantly clear to anyone who has examined the contents of recently published (say, within the last five years) intermediate level textbooks in mathematics that the in vogue word is "applications." This term does not currently refer to the traditional areas of physics and engineering to which mathematics owes much of its development, but mainly refers to the "softer" disciplines such as biology, sociology, economics, and political science. One even encounters calculus and differential equations books in which the back of the front cover and the flyleaf contain extensive lists of all the applications and applied problems found with the text. The topics read like somebody's idea of a semiscientific trivia quiz, but clearly the author(s) want to reinforce the point that the book is "applied."

It is difficult to explain this phenomenon, but the reviewer suspects it is in part due to the emphasis on applied mathematics and its relation to the job market within the last ten years, coupled with an increased interest in mathematical models in the life sciences, especially population biology, ecology, and physiology. The trend in applications of mathematics has been on modelling in the life and social sciences where plausible hypotheses rather than physical laws are often the guidelines. This is usually followed by some computer simulations or adjustments of parameters until a fit is found with some previously conducted experiment or past historical events. Given this training and some brashness, one could call oneself an "applied mathematician."

The above description was not intended to criticize certain areas of investigation such as operations research, game theory, or applied probability, where many contributions to the social and economic sciences have been made. Rather, it is intended to point out that what passes for applied mathematics today is far removed from what it was twenty years ago, and it is certainly built on a sandier foundation. But at the same time areas are being opened up and questions are being asked in branches of mathematics previously considered "pure"; this cannot be regarded in any way but beneficial.

The line of thinking described above was certainly adopted by the editors of the series *Modules in Applied Mathematics*. "Model" is the key word, and the range of topics in the four volumes, dealing (in order) with differential equation models, political and related models, discrete and system models, and life science models, is impressive. For instance, if you want to consider detecting an art forgery, planning a war, designing a highway system, building an arctic bungalow that won't sink into the tundra, or just studying the growth of competing or single populations, the volume under review, *Differential Equation Models*, will provide a model for your ruminations. In many cases it will also provide some reasonable justification for the hypotheses employed, some problems and/or classroom projects, and in all cases, notes for an instructor on objectives, prerequisites, and time required for classroom presentation.

It is difficult to criticize a book like this since there is an element of "chacun à son goût" when it comes to models in the nonphysical sciences. There is no real reason to assume that a differential equation describing some phenomenon has a quadratic nonlinearity, other than the fact that the analysis can be accomplished and the ensuing graphs or phase plane portraits look plausible or match some selected experiments or historical data. But the book's penchant for logistic equations, or quadratic nonlinearities for systems of differential equations is stretched beyond credibility when they are applied to the spread of technological innovations (M. Braun, Ch. 6), or conventional and guerilla warfare (C. Coleman, Ch. 8).

Six of the chapters are excerpted from Martin Braun's excellent and influential book on differential equations (Springer-Verlag, New York, 1978). The first two chapters in the book are by Beverly Henderson West and they are the best discussion this reviewer has seen on the transmutation of a word problem to a differential equation, and on the sketching of direction

fields. Especially commendable in the latter was the use of the second derivative, calculated directly from the equation, to determine concavity—something rarely seen in introductory texts. Chapters 15 through 19 by Braun and Courtney Coleman are a very well guided tour of the questions related to the existence of limit cycles for two dimensional quadratic systems. It begins with an analysis by Braun of the Lotka-Volterra equations for predator-prey systems and ends with Coleman enthusiastically discussing his favorite problem—Hilbert's 16th problem.

Chapters 10 through 14 consider traffic problems, and the reviewer enjoyed the discussion (Ch. 13) by Robert L. Baker Jr. on car-following models. Using a variation of a well-known linear model by L. A. Pipes, the author gradually introduces delays, then discusses stability, and ends with some nonlinear variations. In the remaining chapters, written by Donald A. Drew, the writing was very technical, the models sometimes had nothing to do with differential equations, and there was a lack of calculations, examples, and problems to guide the student or instructor on how the models are employed. Despite the author's plea for perseverance, one wonders whether the material was ever tested in the classroom.

Chapters 20 through 23 deal with partial differential equation models, and in the reviewer's opinion, the topics are best suited for a student who has had at least a first course in partial differential equations, and possibly some numerical analysis. The discussion is definitely at a more advanced level than in the previous chapters, and the problems modelled are not elementary ones but they are certainly interesting. Chapter 23 by T. A. Porsching uses a network model and the nonlinear Gauss-Siedel method to analyze questions in steam generator flow, so it is more accessible than its predecessors. The only drawback is that no problems are given to guide a reader or instructor not familiar with this alternate approach to partial differential equation problems.

Incidentally, the reviewer was present at the conference at the University of Bath in 1972, where P. C. Parks presented his paper on shaking a string to a standstill (discussed by Robert L. Borrelli in Ch. 21). Parks ended his presentation by taking a very long (est. 50 ft.) heavy piece of rope, and he and an assistant demonstrated with much jerky exertion the validity of the analysis. For instructors who intend to discuss this chapter this "hands-on" example of applied mathematics in action is highly recommended if they are in a large classroom or are prepared to go outdoors.

In the final analysis there is much to recommend about this book since every reader will surely find some topics or models worth pursuing. There are little jewels hidden here and there for every treasure hunter (the reviewer's favorite was the example of driving a car blindfolded with a person alongside giving instructions—used by James C. Frauenthal in Chap. 3 to explain instability in difference equations), and any instructor reading through the book will discover something to enrich his course in differential equations or applied mathematics. The reviewer can only urge that instructor not to forget those topics from yesterday's "hard" applied mathematics, e.g., resonance, perturbation methods, the pendulum and van der Pol equation, etc., whose analysis provided the tools needed for today's applications.

Introductory Problem Courses in Analysis and Topology. By Edwin E. Moise. Springer-Verlag, New York, 1982. viii + 94 pp.

CARL C. COWEN
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R. L. Moore and his "Method" are famous in the mathematical community, although it seems most of us know of the Moore Method indirectly, having never participated in a course in which it was used. In spite of this, it is easy to elicit strong opinions about the Method from mathematicians.

In this essay, the words "problem course" and "Moore Method" will mean the technique of teaching mathematics in which the teacher provides the students with a list of definitions and theorems, the students individually prove the theorems without using books or other aids, and, in class, successively present their proofs to their fellow students. Moise's book consists of two lists of definitions and theorems (and problems) for use in Moore Method courses.

The Method, as Moore employed it and some of his students describe it [**5**], [**7**], [**9**], has developed a mystique in which it is impossible to separate Moore's personality from his Method. This is most strongly put by R. L. Wilder [**9**, p. 418] eulogizing Moore, "[the Moore Method] was a unique method employed by a unique man in a unique situation." It is not useful to take that point of view here. Moore's aim apparently was to develop research ability in his students, eventually leading them to a Ph.D. A factor that surely contributed to Moore's success was his careful selection of the students for his courses, an option not open to most of us. I support the contention in Moise's preface that participation in one or two problem courses would be desirable for every undergraduate mathematics major: it seems inconceivable that a student could complete a problem course and not have a very good understanding of the subject.

The status of solving problems in the learning of mathematics is not in dispute—untold numbers of us declare to our students and our readers that the only way to learn mathematics is to solve problems. However, I am less optimistic than Moise seems to be that it is *possible* for all undergraduate majors to succeed in a problem course.

Moise, pleading lack of space, declines to write an essay on problem courses in his preface. This is unfortunate because his insights would be highly interesting and very valuable. But he does make a few observations. Discussing the advantages for nonbrilliant students: the slow pace of a problem course "means a competent student is able to keep track and finds at the end that he understands the course completely. This is a valuable experience, and for many students it is new." Discussing basic real variables theory: "[it] needs to be absorbed so completely that it forms, forever after, a part of the student's intuition. Some of this material is exciting, but some of it is dull. Personally I can tolerate almost any sort of spade-work, if there is a good reason to do it, and if I am doing it myself; but to watch... ."

Several things are necessary for a successful Moore Method course. The most obvious is a good list of theorems. The theorems need to be arranged in such a way that there are small steps between them, and at times, the theorems need to be stated so as to provide hints for the proofs of the next. Elegance in theorems and definitions must be secondary to clarity and ease of use. Moise has done a masterful job of this. In addition, Moise has divided the material into two parts: the theorems "which are guaranteed to be true" and the problems which may be true or false, the idea being to challenge the students with a more realistic "research" situation. The following sequence of problems from the "Topology" portion, Chapter 4, page 67, illustrates both the stepping-stone idea and the "research" approach. (A *countable set* has been defined as a set that is either empty or is the image of a function whose domain is the positive integers.)

Prove or disprove.

1. The set of all integers is countable.

⋮

4. Let A be the set of ordered pairs of integers.

Then A is countable.

5. Let B be the set of all finite sequences of integers.

Then B is countable.

6. Let C be the set of all (infinite) sequences of integers.

Then C is countable.

It is an excellent idea to include problems of this sort in addition to the theorems of the course

because all mathematics students must eventually learn to distinguish for themselves the True from the False.

Moise's book is not the only book of this general form, but it seems to be the most usable, directly, in a Moore Method course. Whyburn and Duda [8] (Whyburn, like Moise, was a Moore student) is not an undergraduate text despite jacket proclamations, and, amazingly, has the proof of each theorem a page or two following the statement. (Moise's book contains only three proofs: a proof of the Chain Rule so students won't make up false ones, and two proofs as illustrations in the chapter "The Use of Choice in Existence Proofs.") Clark [2] proves the major theorems and leaves examples and other theorems to the student.

Desirable though it may be, this approach is not feasible for all classes. There is a certain mathematical maturity required; although the students needn't be able to write flawless proofs, they must have at least a vague notion of what a proof is and the role proofs and examples have in mathematics. Most importantly though, the students must have a certain enthusiasm and drive. Students who refuse to think more than ten minutes about any problem are not appropriate here. However, from my (limited) experience, uniformity of background is not as difficult a problem as some would suggest [5], [7]; if the instructor does not let only the most advanced speak, and invites those who know a proof solely from earlier work to disqualify themselves for that proof, the class will quickly become fairly uniform.

The main advantage of students learning by proving the theorems themselves is obvious, but so is the main disadvantage: not as much material is covered. Any lecturer can easily do more than a Moore Method student (indeed, as we all know, it is even possible for lecturers to leave their own students in the dust). This has implications for the choice of material: only the most important theorems can be covered; the interesting corollaries and secondary topics that make the fabric of a subject whole must be bypassed. Balancing the two objectives of deep understanding and wide coverage is a matter of personal judgment. Comparing Moise's course in "Analysis" with a typical introductory text [1] reveals that little of importance has been left out, and indeed, Moise even includes necessary and sufficient conditions for Riemann integrability of a bounded function. The situation in the "Topology" course is somewhat different. Moise devotes considerable time to topics I would prefer to minimize in a course called "Topology" (although I want the students to learn them *somewhere*); for example, cardinality, the Schroeder-Bernstein Theorem, the Axiom of Choice, and linearly ordered spaces. He is then forced to skip other topics (included, for example, in [6]) like quotient and product spaces, separation axioms, (nonsequential) convergence, and completion and compactification. That is, Moise strikes what is for me an acceptable balance of depth and breadth in the "Analysis" portion, but not in the "Topology" portion.

The Moore Method is an idea with many fruitful aspects. Let us not throw out the whole idea because it has some difficult points, rather, let us search for a wider application of the good aspects. Moise has written an excellent book; it should make it easier for the problem course approach to find a larger place in the undergraduate curriculum.

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LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405 (after December 15, 1984: Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053).

Editor:

R. Fuster and A. Marquina gave an interesting equivalence for the property of being a Q-algebra (one in which the set of invertible elements is open) (this MONTHLY, 91 (1984) 49–51). W. Zelazko's "Metric generalizations of Banach algebras" discusses Q-algebras in a general setting (*Selected Topics in Topological Algebras*, Aarhus University Lecture Notes #31 (1971); see, for example, Corollary 1.8). In this MONTHLY, 73 (1966) Problem 5260, it was pointed out that any inverse closed subalgebra of a Banach algebra is a Q-algebra. Examples are the set of continuous functions which are constant on some neighborhood of a fixed point and $1 \oplus I$, where I is an ideal.

This raises the question: is there a normed Q-algebra which is not inverse closed in its completion?

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134. MISCELLANEA

Soddy goes to Limerick or The kiss precise revisited

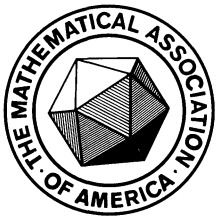
RICHARD K. GUY

If the pairs of four circles each touch
Then their curvatures' values are such
That their products in pairs
When compared with their squares
Have a total that's just half as much.

Editorial note. Two other poetic statements of the same result are in the literature. The author of the first is Soddy, who gave it the title "The kiss precise"; the author of the second is Ray Bobo, this MONTHLY, 87 (1980) 715. Bobo accompanied his version by the following note:

> "Frederick Soddy (1877–1956) won the 1921 Nobel Prize for Chemistry for his conceptualization of isotopes. His mathematical verses appear in volumes 137 and 138 of *Nature* and are discussed by Soddy and others in volume 139. By the way, the radial reciprocals of $n + 2$ mutually touching n-spheres have the property that the square of their sum is n times the sum of their squares. The result on four circles was known to Descartes in 1643."

The n-dimensional generalization is due to H.S.M. Coxeter, Loxodromic sequences of tangent spheres, *Aequationes Math.*, 1 (1968) 104–121.



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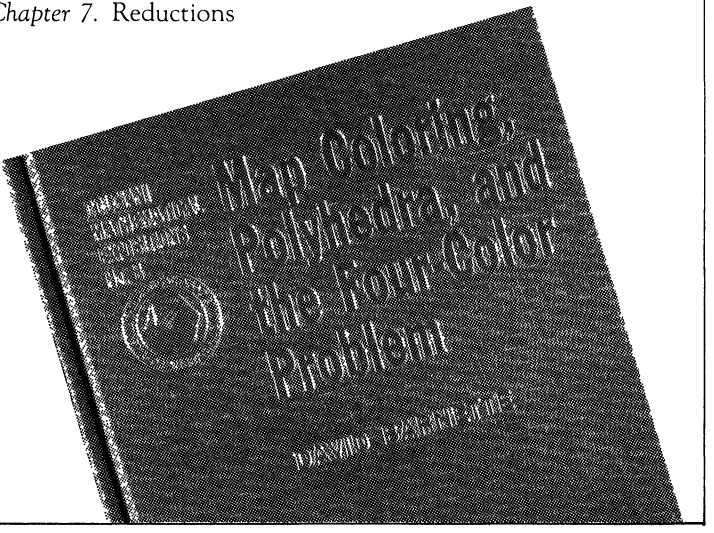
For over a hundred years professional mathematicians and an army of amateurs struggled with a disarmingly simple conjecture. Its statement was so clear, its meaning so easy to visualize that it attracted the interest of the great and the small. "Every map can be colored in four colors in such a way that no two countries sharing a common border have the same color."

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The object of the book is not to present the Appel-Haken proof, but to discuss some of the powerful and diverse mathematical ideas—including much of modern combinatoric mathematics—that were developed during the hundred year assault on the four color problem.

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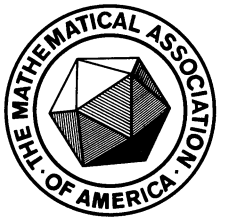
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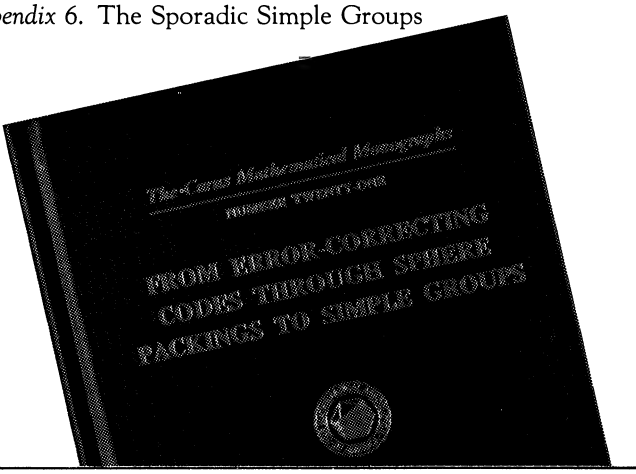
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EULER*

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Until the latter part of the seventeenth century, mathematics had sometimes bestowed high reputation upon its adepts but had seldom provided them with the means to social advancement and honorable employment. VIÈTE had made his living as a lawyer, FERMAT as a magistrate; even in Fermat's days, endowed chairs for mathematics were few and far between. In Italy, the University of Bologna ("lo studio di Bologna", as it was commonly called), famous throughout Europe, had indeed counted Scipione del FERRO among its professors in the early sixteenth century; but CARDANO had been active as a physician; BOMBELLI was an engineer, and so was Simon STEVIN in the Netherlands. NAPIER, the inventor of logarithms, was a Scottish laird, living in his castle of Merchiston after coming back from the travels of his early youth. Neighboring disciplines did not fare better. COPERNICUS was an ecclesiastical dignitary. Kepler's teacher MAESTLIN had been a professor in Tübingen, but KEPLER plied his trade as an astrologer and maker of horoscopes. GALILEO's genius, coupled with his domineering personality, earned him, first a professorship in Padova, then an enviable position as a protégé of the Grand-Duke of Tuscany, which saved him from the worst consequences of his disastrous conflict with the Church of Rome; his pupil TORRICELLI succeeded him as "philosopher and mathematician" to the Grand-Duke, while CAVALIERI combined the Bologna chair with the priorate of the Gesuati convent in the same city.

Among Fermat's scientific correspondents, few held professorial rank. ROBERVAL, at the Collège de France (then styled Collège Royal), occupied the chair founded in 1572 from a legacy of the scientist and philosopher Pierre de la Ramée. The Savilian chair at Oxford, created for H. BRIGGS in 1620, was held by WALLIS from 1649 until his death in 1703; but his talented younger friend and collaborator William BROUNCKER, second Viscount, was a nobleman whose career as commissioner of the Navy, and whose amours, are abundantly documented in Pepys's diary. It was only in 1663 that Isaac BARROW became the first Lucasian professor in Cambridge, a position which he relinquished to NEWTON in 1669 to become preacher to Charles II and achieve high reputation as a divine. In the Netherlands, while Descartes's friend and commentator F. SCHOOTEN was a professor in Leiden, René de SLUSE, a mathematician in high esteem among his contemporaries and an attractive personality, was a canon in Liège. DESCARTES, as he tells us, felt himself, by the grace of God ("*graces à Dieu*": *Discours de la Méthode*, *Desc.* VI.9), above the need of gainful employment; so were his friends Constantin HUYGENS and Constantin's son, the great Christian HUYGENS. LEIBNIZ was in the employ of the Hanoverian court; all his life he preserved his love for mathematics, but his friends marveled sometimes that his occupations left him enough leisure to cultivate them.

Whatever their position, the attitude of such men towards mathematics was often what we can describe as a thoroughly professional one. Whether through the printed word or through their correspondence, they took pains to give proper diffusion to their ideas and results and to keep abreast of contemporary progress; for this they relied largely upon a private network of informants. When they traveled, they looked up foreign scientists. At home they were visited by scientifically inclined travelers, busy bees intent on disseminating the pollen picked up here and there. They eagerly sought correspondents with interests similar to their own; letters passed from hand to hand until they had reached whoever might feel concerned. A private library of reasonable size was almost a necessity. Booksellers had standing orders to supply customers with

*These historical paragraphs, as well as the biography of Euler and the proof of a theorem about sums of squares that follow, are excerpts from the author's book *Number Theory* (Birkhäuser Boston, 1984); they appear there on pp. 159–169 and 292–295. The only difference between the book's version and this one is that the detailed references are omitted here. *Editor.*

the latest publications within each one's chosen field. This system, or lack of system, worked fairly well; indeed it subsists down to the present day, supplementing more formalized modes of communication, and its value is undiminished. Nevertheless, even the seventeenth century must have found it increasingly inadequate.

By the time of Euler's birth in 1707, a radical change had taken place; its first signs had become apparent even before Fermat's death. The *Journal des Sçavans* was started in January 1665, just in time to carry an obituary on Fermat ("*ce grand homme*", as he is called there). Louis XIV's far-sighted minister Colbert had attracted Huygens to Paris in 1666, and the astronomer CASSINI in 1669, awarding to each a royal pension of the kind hitherto reserved to literati. In 1635 Richelieu had founded the *Académie françoise*; the more practical-minded Colbert, realizing the value of scientific research (pure no less than applied) for the prosperity of the realm, set up the *Académie des sciences* in 1666 around a nucleus consisting largely of Fermat's former correspondents; Fermat's great friend and former colleague Carcavi was entrusted with its administration and became its first secretary. In England, some degree of political stability had been restored in 1660 by the recall of Charles II; in 1662 the group of amateurs ("*virtuosi*") who had for some time held regular meetings in Gresham College received the charter which made of them the *Royal Society*, with Brouncker as its first president; in 1665 they started the publication of the *Philosophical Transactions*, which has been continued down to the present day. In 1698 the French academy followed suit with a series of yearly volumes, variously entitled *Histoire* and *Mémoires de l'Académie des Sciences*. In 1682 Leibniz was instrumental in creating, not yet an academy, but at least a major scientific journal, the *Acta Eruditorum* of Leipzig, to whose early issues he contributed the articles by which he was giving birth to the infinitesimal calculus.

Soon universities and academies were competing for scientific talent and sparing neither effort nor expense in order to attract it. Jacob BERNOULLI had become a professor in his native city of Basel in 1687; as long as he lived, this left little prospect to his younger brother and bitter rival Johann of finding academic employment there; at first he had to teach Leibniz's infinitesimal calculus to a French nobleman, the Marquis de l'HÔPITAL, even agreeing to a remarkable contract whereby the latter acquired an option upon all of BERNOULLI's mathematical discoveries. In 1695, however, Joh. Bernoulli became a professor in Groningen, eventually improving his position there by skilfully playing Utrecht against Groningen; finally he settled down in Basel in 1705 after his brother's death. No wonder, then, that in 1741 we find him congratulating Euler on the financial aspects of his Berlin appointment and suggesting at the same time that he would be willing (for a moderate yearly stipend, "*pro modico subsidio annuo*") to enrich the memoirs of the Berlin Academy with regular contributions of his own. In short, scientific life, by the turn of the century, had acquired a structure not too different from what we witness to-day.

Euler's father, Paul EULER, was a parish priest established in Riehen near Basel; he had studied theology at the university of Basel, while at the same time attending the lectures of Jacob Bernoulli; he had planned a similar career for his son, but placed no obstacle in the way of young Leonhard's inclinations when they became manifest. Clearly, by that time, a bright future was in store for any young man with exceptional scientific talent.

In 1707, when EULER was born, Jacob Bernoulli was dead, and Johann had succeeded him; Johann's two sons Nicolas (born in 1695) and Daniel (born in 1700) were following the family tradition, except that, in contrast to their father and uncle, they loved each other dearly, as they took pains to make known. Euler became their close friend and Johann's favorite disciple; in his old age, he liked to recall how he had visited his teacher every Saturday and laid before him the difficulties he had encountered during the week, and how hard he had worked so as not to bother him with unnecessary questions.

Three monarchs came to play a decisive role in Euler's career: Peter the Great, Frederic the Great, and the Great Catherine. Peter, a truly great czar perhaps, died in 1725; but he had had

time to found Saint Petersburg, erect some of its most impressive buildings, and, most important of all for our story, to make plans for an Academy of Sciences modelled on what he had seen in the West; those plans were faithfully carried out by his widow. In 1725 the two younger BERNOULLIS, Nicolas and Daniel, were called there. Nicolas died the next year, apparently of appendicitis. About the same time an offer went to Euler to join the Petersburg Academy. He was not quite twenty years old; he had just won a prize for an essay on ship-building, never having seen a sea-going ship in his life. He had no early prospects at home. He accepted with alacrity.

From Basel he sailed down the Rhine to Mainz, then traveled to Lübeck, mostly on foot, visiting Christian Wolff on the way; this was a philosopher and follower of Leibniz, banished from Berlin (as he told Euler) by a king with little understanding for philosophy; his hobby-horse was Leibniz's theory of monads, and Euler was clearly not impressed. From Lübeck a ship took the young mathematician to Petersburg.

In those days academies were well-endowed research institutions, provided with ample funds and good libraries. Their members enjoyed considerable freedom; their primary duty was to contribute substantially to the academy's publications and keep high its prestige in the international scientific world. At the same time they were the scientific advisers to the monarch and to state authorities, always on hand for such tasks, congenial or not, as the latter might find fit to assign to them; had it not been so, no state would have undergone the high expense of maintaining such institutions, as Euler once acknowledged to Catherine. In 1758, at the height of his fame, Euler (who had acquired in Petersburg a good command of the Russian language) did not find it beneath him, nor inconsistent with his continuing close relations with the Petersburg Academy, to translate for king Frederic some dispatches seized during military operations against the Russian army.

In 1727, however, the political situation had changed by the time Euler reached Petersburg. Under a new czar all academic appointments were in abeyance. On the strength of his prize essay, Euler was commissioned into the Russian navy, but not for long. Soon he was a salaried member of the academy, at first with the junior rank of "adjunct". When his friend Daniel left for Basel in 1733, he was appointed in his place; thus he could afford to marry, naturally into the local Swiss colony, and to buy for himself a comfortable house. His bride was the daughter of the painter GSELL; in due course she was to give birth to thirteen children, out of whom only three sons survived Euler; little is recorded of her otherwise. The eldest son Johann Albert, born in 1734, was to become one of his father's collaborators, and later a leading member of the academy.

Once Euler was thus well established in Petersburg, his productivity exceeded all expectations, in spite of the comparative isolation in which he had been left by Daniel Bernoulli's departure. It was hardly interrupted by a severe illness in 1735 and the subsequent loss of his right eye. He had beyond doubt become the most valuable member of the Academy, and his reputation had been growing by leaps and bounds, when two events in the higher spheres of European politics brought about a major change in his peaceful life. In Petersburg the death of the czarina in 1740, a regency, and the ensuing turmoils, seemed to threaten the very existence of the Academy. Just at this juncture Frederic II succeeded his father (the same king who had so cavalierly thrown Chr. Wolff out of Berlin) on the throne of Prussia; he immediately took steps directed towards the establishment of an academy under his patronage, for which he sought out the most famous names in European science; naturally Euler was on the list. A munificent offer from Frederic, coupled with a fast deteriorating situation in Petersburg, brought Euler to Berlin in July 1741, after a sea-voyage of three weeks on the Baltic during which he alone among his family (or so he claimed) had been free from sea-sickness. In the following year, to his great satisfaction, he was able to purchase an excellent house, well situated, and, by special royal order, exempt from requisition. There he lived for the next 24 years, with apparently the sole interruption of seasonal visits to the country estate he acquired in 1752 in Charlottenburg, and of a family trip to Frankfurt in 1750 to meet his widowed mother who was coming from Basel to live in Berlin with him; his father, who had been disappointed in his hope of getting Euler's visit in Basel had died in 1745.

With Euler's change of residence one might have expected that the steady flow of his publications would be diverted from Petersburg to Berlin; but far from it! He was not only allowed to keep his membership in the Petersburg Academy, but his pension from Petersburg was continued, and he was intent upon giving his former colleagues value for their money. Well might his great-grandson P.-H. Fuss describe Euler's Berlin period as "twenty-five years of prodigious activity". More than 100 memoirs sent to Petersburg, 127 published in Berlin on all possible topics in pure and applied mathematics were the products of those years, side by side with major treatises on analysis, but also on artillery, ship-building, lunar theory; not to mention the prize-winning essays sent to the Paris academy (whose prizes brought substantial cash rewards in addition to high reputation); to which one has to add the *Letters to a German Princess* (one of the most successful popular books on science ever written) and even a defense of christianity (*Rettung der göttlichen Offenbarung*...) which did nothing to ingratiate its author with the would-be philosopher-king Frederic. At the same time Euler was conducting an increasingly heavy correspondence, scientific, personal and also official since the administrative burdens of the academy tended to fall more and more upon his shoulders.

As years went by, Euler and Frederic became disenchanted with each other. The king was not unaware of the lustre that Euler was throwing upon his academy, but French literati stood far higher in his favor. He was seeking to attract d'ALEMBERT to Berlin and was expected to put him above Euler as head of the Academy. Euler was spared this blow to his self-esteem; d'Alembert, forewarned perhaps by Voltaire's unpleasant experience with Frederic in 1753, enjoyed basking in the king's favor for the time of a short visit but valued his freedom far too highly to alienate it more durably. Nevertheless, as early as 1763, Euler's thoughts started turning again towards Russia.

Fortunately another political upheaval had just taken place there. In 1762 the czar's German wife had seized power as Catherine II after ridding herself and Russia of her husband. One of her first projects was to restore the Petersburg academy to its former glory. This was almost synonymous with bringing Euler back. Negotiations dragged on for three years. Finally, in 1766, the Russian ambassador in Berlin was instructed to request Euler to write his own contract. Frederic, realizing too late the magnitude of this loss, had tried to put obstacles in the way; he soon found that he could not afford to displease the imperial lady. In the same year Euler was back in Petersburg, after a triumphal journey through Poland where Catherine's former lover, king Stanislas, treated him almost like a fellow-sovereign.

By then Euler was losing his eyesight. He had lost the use of his right eye during his first stay in Petersburg. About the time when he left Berlin, or shortly thereafter, a cataract developed in his left eye. In 1770, in answer to a letter from Lagrange on number theory, he described his condition as follows: "*Je me suis fait lire toutes les opérations que vous avez faites sur la formule* $101 = pp - 13qq$ *et je suis entièrement convaincu de leur solidité; mais, étant hors d'état de lire ou d'écrire moi-même, je dois vous avouer que mon imagination n'a pas été capable de saisir le fondement de toutes les déductions que vous avez été obligé de faire et encore moins de fixer dans mon esprit la signification de toutes les lettres que vous y avez introduites. Il est bien vrai que de semblables recherches ont fait autrefois mes délices et m'ont coûté bien du tems; mais à présent je ne saurois plus entreprendre que celles que je suis capable de dévellopper dans ma tête et souvent je suis obligé de recourir à un ami pour exécuter les calculs que mon imagination projette*" ["I have had all your calculations read to me, concerning the equation $101 = p^2 - 13q^2$, and I am fully persuaded of their validity; but, as I am unable to read or write, I must confess that my imagination could not follow the reasons for all the steps you have had to take, nor keep in mind the meaning of all your symbols. It is true that such investigations have formerly been a delight to me and that I have spent much time on them; but now I can only undertake what I can carry out in my head, and often I have to depend upon some friend to do the calculations which I have planned"].

An operation was attempted in 1771 and was successful at first, but the eye soon became infected, and total or near-total blindness ensued. Except for this misfortune, and for a fire which destroyed his house in 1771 among many others in Petersburg, he lived on in comfort, greatly

honored and respected; neither old age nor blindness could induce him to take a well-deserved rest. He had assistants, one of whom was his own son; others were sent to him from Basel, with the co-operation of his old friend Daniel Bernoulli; to one of them, N. FUSS, who had come to Petersburg in 1773 and later married a granddaughter of Euler's, we owe a vivid description of Euler's method of work in the last decade of his life. Hundreds of memoirs were written during that period; enough, as Euler had predicted, to fill up the academy publications for many years to come. He died suddenly on 18 September 1783, having preserved excellent general health and his full mental powers until that very day.

AN ELEMENTARY PROOF FOR SUMS OF SQUARES

In 1751, Euler, having proved that every integer is a sum of (at most) four squares "*in fractis*", i.e. in rational numbers, wrote as follows:

"*In Analysi quidem Diophantea pro certo assumi solet nullum numerum integrum in quatuor quadrata fracta dispertiri posse, nisi eius resolutio in quatuor quadrata integra vel pauciora constet... Verum nusquam adhuc eiusmodi demonstrationem inveni...*" ["In diophantine analysis one usually takes for granted that no integer can be split into four rational squares unless it has an expression as a sum of four integral squares or less... but so far I have nowhere found a proof of this..."]. This is indeed a question which must occur naturally, concerning sums of 2, 3, or 4 squares, to any reader of Diophantus, and Fermat had already given it some thought, apparently without success, at the beginning of his career as a number-theorist. Euler himself had raised it, more particularly concerning sums of two squares, in 1745, and again repeatedly, concerning sums of four squares, in his correspondence with Goldbach.

Here we shall reproduce a proof due to L. Aubry (*Sphinx-Œdipe* 7 (1912), pp. 81–84) which applies equally well to sums of 2, 3, or 4 squares and a few other quadratic forms. We first describe it in geometric terms for sums of 3 squares.

Points in $\mathbf{R}^3$ will be called *rational* (resp. *integral*) if they have rational (resp. integral) coordinates. To each point $a = (x, y, z)$ there is an integral point $a' = (x', y', z')$ at a euclidean distance less than 1 from a; for instance one can take for x', y', z' the integers respectively closest to x, to y and to z, in which case the distance from a to a' is $\leqslant \sqrt{3}/2$.

Now assume that an integer N is a sum of three rational squares; this is the same as to say that there is a rational point a_0 on the sphere S given by $N = x^2 + y^2 + z^2$. Let a_0' be the integral point (or one of the integral points, if there are two or more) closest to a_0; its distance from a_0 is < 1. Call a_1 the second intersection of S with the straight line joining a_0 to a_0'; it is rational. Let a_1' be an integral point closest to a_1; let a_2 be the intersection of S with the line joining a_1 to a_1'; etc. It will now be shown that, for some n, a_n is an integral point.

Put $a_0 = (x, y, z)$; assume that it is not an integral point, and call m the lowest common denominator for x, y, z; write $x = n/m$, $y = p/m$, $z = q/m$; we have

$$Nm^2 = n^2 + p^2 + q^2.$$

Let $a_0' = (n', p', q')$ be an integral point closest to a_0, and put

$$r = n - mn',\ s = p - mp',\ t = q - mq',$$

$$N' = n'^2 + p'^2 + q'^2,\ M = 2(nn' + pp' + qq').$$

The squared distance between a_0 and a_0' is then

$$\frac{1}{m^2}(r^2 + s^2 + t^2) = N + N' - \frac{M}{m}$$

which can be written as m'/m, where m' is an integer; as it is < 1, we have $0 < m' < m$, and at the same time:

$$r^2 + s^2 + t^2 = mm',\ M = m(N + N') - m'.$$

Now the line joining a_0 to a_0' consists of the points

$$(n' + \lambda r, p' + \lambda s, q' + \lambda t);$$

here the point a_1 is given by

$$\begin{aligned} 0 &= (n' + \lambda r)^2 + (p' + \lambda s)^2 + (q' + \lambda t)^2 - N \\ &= mm'\lambda^2 + (M - 2mN')\lambda + N' - N \\ &= (m\lambda - 1)(m'\lambda + N - N'). \end{aligned}$$

The root $\lambda = 1/m$ corresponds to the point a_0, so that the other root $\lambda = (N' - N)/m'$ corresponds to a_1. Thus m' is a common denominator for the coordinates of a_1; as it is $< m$, this shows that the lowest common denominators for the coordinates of a_0, a_1, a_2, etc. make up a decreasing sequence of positive integers, which proves our assertion.

The only property of the quadratic form

$$F(X, Y, Z) = X^2 + Y^2 + Z^2$$

which has been used in this proof is that, to every nonintegral point (x, y, z), there is an integral point (x', y', z') such that

$$0 < |F(x - x', y - y', z - z')| < 1.$$

This applies equally well, for instance, to the forms $X^2 + Y^2$, $X^2 \pm 2Y^2$, $X^2 - 3Y^2$. Now, modifying the notation in an obvious manner, we will show how the same proof can be applied to some quadratic forms $F(x)$ in $\mathbf{R}^n$, taking integral values at all integral vectors x and such that, to every nonintegral vector x in $\mathbf{R}^n$, there is an integral vector x' for which $0 < |F(x - x')| \leqslant 1$; this will be the case for instance for $X^2 + 3Y^2$, $X^2 + Y^2 + 2Z^2$, $X^2 + Y^2 + Z^2 + T^2$. Define the bilinear form $B(x, y)$ by

$$F(\lambda x + \mu y) = \lambda^2 F(x) + \lambda\mu B(x, y) + \mu^2 F(y).$$

As before, take a rational point a_0 such that $F(a_0) = N$, and an integral point a_0' such that

$$0 < |F(a_0 - a_0')| \leqslant 1.$$

Let m be the smallest integer such that $n = ma_0$ is an integral point; put $a_0' = n'$, $r = n - mn'$, $N' = F(n')$, $M = B(n, n')$. We have

$$F(a_0 - a_0') = F(m^{-1}r) = F(m^{-1}n - n') = N + N' - \frac{M}{m};$$

writing this as m'/m, we have $0 < m' \leqslant m$, and m' is an integer.

The line joining a_0 to a_0' consists of the points $n' + \lambda r$, and its intersection a_1 with the hypersurface $F(x) = N$ is given by

$$0 = F(n' + \lambda r) - N = mm'\lambda^2 + (M - 2mN')\lambda + N' - N = (m\lambda - 1)(m'\lambda + N - N').$$

As before, a_1 corresponds to the root $\lambda = (N' - N)/m'$, so that $m'a_1$ is an integral point, and m' is a common denominator for the coordinates of a_1; however, we have now to take into account the case $|m'| = m$, i.e. $|F(m^{-1}r)| = 1$. Taking for F one of the three forms listed above, we may assume that we have taken for a_0' the point, or one of the points, whose coordinates are the integers closest to those of a_0, so that the coordinates of $m^{-1}r$ are $\leqslant \frac{1}{2}$ in absolute value; then, for $F(m^{-1}r)$ to be 1, they must all be $\pm \frac{1}{2}$, and $2a_0$ must be integral, so that $m = 2$. In that case there are 2^n possible choices for a_0' and this choice can be made so that $N - N'$ is even; since at the same time m' must be 1 or 2, a_1 is then integral.

The above proof would have been easily understood by Euler; perhaps, with a little more effort, it would have been understood by Fermat, whose algebraic skills still fell somewhat short of the required level. That it was discovered so late may serve as an encouragement to those who seek elementary proofs for supposedly sophisticated results.

TRIGONOMETRY IN LORENTZIAN GEOMETRY

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Euclidean geometry is the geometry in an affine space arising from the existence of a positive-definite inner product among its vectors. When such an inner product is replaced by an inner product of signature $(+,\ldots,+,-)$, what results is called Lorentzian geometry. It is too well known that Lorentzian geometry of 4 dimensions is a most appropriate mathematical language for the special theory of relativity.* See [3], [4], [6], for example.

One must say, however, that the most elementary part of Lorentzian geometry—Lorentzian plane geometry—has yet to become part of the household mathematical knowledge. Particularly, trigonometry—that old science on lengths and angles pertaining to triangles—is little known in Lorentzian geometry. Some of the material we need is scattered in [2, pp. 237–244], [3, p. 144], [4, p. 26], [6, p. 74], but it seems that [7] is the only book available in English that treats Lorentzian trigonometry.

In this article we shall give a rapid introduction to Lorentzian trigonometry starting with a basic lemma on the so-called reversed Cauchy-Schwarz inequality. By employing a linear-algebraic approach we can provide proofs that are often left as exercises in [7]. Also in contrast to [7], we adopt the currently standard terminology and speak of timelike and spacelike lines or null lines instead of lines of the first and second kind or special lines. Proposition 1 and Theorem 3 are not in [7], although [7] has other material that we don't discuss here. We also mention that much of the common material can be found in the paper [5]. Finally, we note that the notion of angle can be refined and generalized so as to lead to a formulation and proof of the Gauss-Bonnet theorem for 2-dimensional Lorentzian manifolds [1].

1. Preliminaries. Let L^2 be the vector space R^2 provided with Lorentzian inner product

$$\langle x,y\rangle = x_1y_1 - x_2y_2 \quad \text{for } x=(x_1,x_2), y=(y_1,y_2). \tag{1}$$

We denote by G the *proper Lorentz group* $\mathrm{SO}^+(1,1)$ consisting of all matrices of the form

$$A(u) = \begin{bmatrix} \mathrm{ch}\,u & \mathrm{sh}\,u \\ \mathrm{sh}\,u & \mathrm{ch}\,u \end{bmatrix}, u \in R, \tag{2}$$

where we are writing ch and sh for hyperbolic functions cosh and sinh. The reader will certainly compare (2) to a rotation matrix

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Katsumi Nomizu received his M.S. from Osaka University and his Ph.D. from the University of Chicago in 1953. His main area of research has been differential geometry. His publication list includes *Lie Groups and Differential Geometry* (1956), *Foundations of Differential Geometry* (with S. Kobayashi) vol. I (1963) and vol. II (1969), *Fundamentals of Linear Algebra* (1966), and *Introduction to Contemporary Differential Geometry* (in Japanese; 1981). He started his teaching at Nagoya University, has been at Brown University since 1960, and has held various visiting appointments in Brazil, France, West Germany, Italy, and Japan.

*It was indeed H. Minkowski who introduced this geometry immediately following Einstein's special theory of relativity. H. A. Lorenz pioneered the notion of so-called Lorentz transformations before Einstein. We are following here the current usage of Lorentzian inner product, Lorentzian manifold, etc., partly due to the fact that the name of Minkowski geometry sometimes refers to the geometry of convexity arising from a norm rather than a positive-definite inner product.

$$R(u) = \begin{bmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{bmatrix}, u \in R,$$

which preserves the Euclidean inner product $\langle x, y \rangle = x_1 y_1 + x_2 y_2$.

In L^2, a vector x is said to be *timelike* if $\langle x, x \rangle < 0$, *spacelike* if $\langle x, x \rangle > 0$, and *null* if $\langle x, x \rangle = 0$. The *norm* $\|x\|$ is defined to be $\sqrt{|\langle x, x \rangle|}$. We also consider the time orientation as follows. Let $e = (0, 1)$. A timelike vector $x = (x_1, x_2)$ is *future-pointing* (resp., *past-pointing*) if $\langle x, e \rangle < 0$ (resp., $\langle x, e \rangle > 0$). So a vector $x = (x_1, x_2)$ is timelike and future-pointing if and only if $x_1^2 - x_2^2 < 0$ and $x_2 > 0$; in other words, if and only if $|x_1| < x_2$. We note that the group G is in fact the group of all linear transformations of L^2 which preserve inner product, orientation and time-orientation.

The following lemma is essential.

LEMMA. *Let x and y be future-pointing timelike vectors in L^2. Then*

(i) $\langle x, y \rangle \leqslant 0$;

(ii) *$x + y$ is a future-pointing timelike vector*;

(iii) $-\langle x, y \rangle \geqslant \|x\| \|y\|$; *the equality holds if and only if $y = cx$ for some $c > 0$*;

(iv) $\|x + y\| \geqslant \|x\| + \|y\|$; *the equality holds if and only if $y = cx$ for some $c > 0$.*

The proof is easy if we note that $x = (x_1, x_2)$ is timelike and future-pointing if and only if $|x_1| < x_2$. The lemma holds in a Lorentzian vector space of an arbitrary dimension, and (iii) is the so-called *reversed Cauchy-Schwarz inequality*. For two vectors $x = (x_1, x_2, \ldots, x_n, x_{n+1})$ and $y = (y_1, y_2, \ldots, y_n, y_{n+1})$, the Lorentzian inner product is

$$\langle x, y \rangle = \sum_{k=1}^{n} x_k y_k - x_{n+1} y_{n+1};$$

the vector x is timelike and future pointing if and only if $\sqrt{\sum x_k^2} < x_{n+1}$. (iii) can be proved using the usual Cauchy-Schwarz inequality

$$\left| \sum_k x_k y_k \right| \leqslant \sqrt{\sum_k x_k^2} \sqrt{\sum_k y_k^2}.$$

We shall now define the notion of angle. Let x and y be two future-pointing timelike unit vectors. We say that $u \in R$ is the (*oriented*) *angle from x to y* if $A(u)x = y$. In this case, the (*oriented*) *angle from y to x* is obviously $-u$. The (*unoriented*) *angle between x and y* is defined to be $|u|$. We have easily

$$\operatorname{ch} u = -\langle x, y \rangle, \tag{3}$$

where the right-hand side is greater than 1 by the lemma.

When x and y are future-pointing timelike vectors, the angle u (oriented or unoriented) for x and y is that for $x/\|x\|$ and $y/\|y\|$. We have

$$\operatorname{ch} u = -\langle x, y \rangle / \|x\| \|y\|. \tag{4}$$

For two timelike lines (through the zero vector), the angle between them is, by definition, the angle between two future-pointing timelike unit vectors which lie on the respective lines.

REMARK. For two spacelike lines (through the zero vector) we can define the angle in a similar fashion. We shall need this notion later (in Theorem 4).

We are now passing to the Lorentzian plane, namely, the affine plane associated to the Lorentzian vector space L^2. For any two points A and B of the plane, there is a vector $\overrightarrow{AB} \in L^2$; for any three points A, B and C, we have $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$.

We may recall here that for the Euclidean plane a motion is a combination of a rotation and a translation. Relative to a rectangular coordinate system, a motion can be expressed by a matrix operation

$$\begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix} \to \begin{pmatrix} \cos t & -\sin t & a \\ \sin t & \cos t & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ 1 \end{pmatrix}.$$

For the Lorentzian plane, the group

$$\tilde{G} = \left\{ \begin{pmatrix} \operatorname{ch} u & \operatorname{sh} u & a \\ \operatorname{sh} u & \operatorname{ch} u & b \\ 0 & 0 & 1 \end{pmatrix}; \quad u, a, b \in R \right\} \tag{5}$$

acts as the group of motions in the same way. Lorentzian plane geometry is thus the study of properties invariant by the action of the group $\tilde{G}$.

Let A, B, C be three noncollinear points such that $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are future-pointing timelike vectors and $\overrightarrow{BC}$ is a spacelike vector such that $\langle \overrightarrow{AB}, \overrightarrow{BC} \rangle = 0$. So B is obtained from C by the orthogonal projection on the line AB. Then

$$\operatorname{ch} u = \| \overrightarrow{AB} \| / \| \overrightarrow{AC} \|, \qquad \operatorname{sh} u = \| \overrightarrow{BC} \| / \| \overrightarrow{AC} \|, \tag{6}$$

where u is the angle between the lines AB and AC. This follows from (4).

We now introduce a class of triangles that are important in Lorentzian plane geometry. By a (timelike) *pure triangle* we mean a triangle with vertices A, B, C (after appropriate naming) such that $\overrightarrow{AB}$ and $\overrightarrow{BC}$ are future-pointing timelike vectors (and so is $\overrightarrow{AC}$, as a consequence of the lemma). In what follows we assume that the vertices of a pure triangle ABC are named in this manner, and B is called the *middle vertex*. The angle $\hat{A}$ at A is the angle between the lines AB and AC, and the angle $\hat{C}$ at C is that between the lines BC and AC. Finally, the angle $\hat{B}$ at the middle vertex B is the angle between the lines AB and BC, that is, between the vectors AB and BC, thus looking more like the exterior angle to Euclid. See Fig. 1.

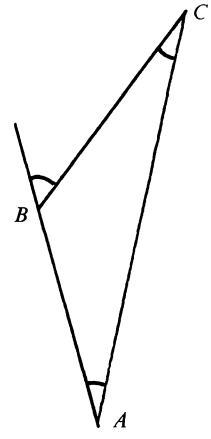


FIG. 1

We now prove

PROPOSITION 1. *Let ABC be a pure triangle. Then the angle at the middle vertex is equal to the sum of the other two angles: $\hat{B} = \hat{A} + \hat{C}$.*

Proof. Let $x = \overrightarrow{AB}/\|\overrightarrow{AB}\|$, $y = \overrightarrow{BC}/\|\overrightarrow{BC}\|$, and $z = \overrightarrow{AC}/\|\overrightarrow{AC}\|$, and let α, β, and γ be the (oriented) angles from x to z, from x to y, and from z to y, respectively. Since z lies between x and y (more precisely, if $x = (\operatorname{sh} u, \operatorname{ch} u)$, $y = (\operatorname{sh} v, \operatorname{ch} v)$, $z = (\operatorname{sh} w, \operatorname{ch} w)$, then w lies between u and v), we have clearly $\beta = \alpha + \gamma$. Since α and γ are both positive or both negative, we get $|\beta| = |\alpha| + |\gamma|$.

The notion of area of a parallelogram makes sense in Lorentzian plane geometry. Given two vectors $x = (x_1, x_2), y = (y_1, y_2)$, the area of the parallelogram spanned by x and y is equal to $|x_1 y_2 - x_2 y_1|$, which is invariant by the group G. If A, B, C are three noncollinear points, the area of the triangle ABC is equal to one half of the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$. This area is invariant by the group $\tilde{G}$.

We have

PROPOSITION 2. *The area of a pure triangle ABC is given by*

$$S = (bc \operatorname{sh} \hat{A})/2, \tag{7}$$

where $b = \|\overrightarrow{AC}\|, c = \|\overrightarrow{AB}\|$. (See Fig. 2.)

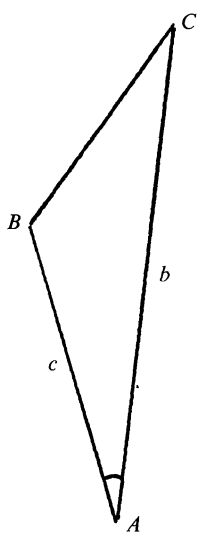


FIG. 2

Proof. We may assume that $A = (0, 0), B = c(\operatorname{sh} u, \operatorname{ch} u)$, and $C = b(\operatorname{sh} v, \operatorname{ch} v)$. Then the area is

$$S = bc|\operatorname{sh} u \operatorname{ch} v - \operatorname{ch} u \operatorname{sh} v|/2 = bc(\operatorname{sh}|u - v|)/2,$$

where $|u - v|$ is equal to the angle $\hat{A}$.

2. Lorentzian circles. The analogue of a circle is given as follows. Let P be a point in the Lorentz plane and let $r > 0$. The curve $(Q; \langle PQ, PQ \rangle = r^2)$ has two branches and each of them is called a (timelike) *Lorentzian circle* with center P and radius r. If $P = (p_1, p_2)$, then the equation is

$$(x_1 - p_1)^2 - (x_2 - p_2)^2 = r^2.$$

From the point of view of differential geometry, each branch of this hyperbola can be considered as a timelike curve (that is, the tangent vectors are timelike) with constant curvature. Namely, one branch can be parametrized by $x(t) = (r \operatorname{ch} t + p_1, r \operatorname{sh} t + p_2)$, and the tangent vector is given by $dx/dt = (r \operatorname{sh} t, r \operatorname{ch} t)$ so that $\langle dx/dt, dx/dt \rangle = -r^2$.

We see that for any point Q on the Lorentzian circle, $\overrightarrow{PQ}$ is perpendicular to the tangent line T_Q at Q, of course, relative to the Lorentzian inner product. The reader will certainly recall the familiar fact in Euclidean geometry that any radius vector of a circle is orthogonal to the tangent at that point.

We now have

THEOREM 3. *A triangle inscribed in a Lorentzian circle is a pure triangle. Conversely, a pure triangle can be inscribed in a unique Lorentzian circle.*

Proof. Let $A = r(\operatorname{ch} a, \operatorname{sh} a)$, $B = r(\operatorname{ch} b, \operatorname{sh} b)$, $C = r(\operatorname{ch} c, \operatorname{sh} c)$ be three points on $x^2 - y^2 = r^2$, where $a < b < c$. Then

$$\langle \overrightarrow{AB}, \overrightarrow{AB} \rangle = r^2\left[(\operatorname{ch} b - \operatorname{ch} a)^2 - (\operatorname{sh} b - \operatorname{ch} a)^2\right]$$
$$= 2r^2[1 - \operatorname{ch}(b - a)] < 0$$

since $a \neq b$. Thus $\overrightarrow{AB}$ is timelike. It is easy to check that it is future-pointing. The same holds for $\overrightarrow{BC}$.

Now let ABC be a pure triangle and let $A = (a_1, a_2)$, $B = (b_1, b_2)$ and $C = (c_1, c_2)$. We want to show that there exist p_1, p_2 and $r \neq 0$ such that we have

$$\begin{aligned} (a_1 - p_1)^2 - (a_2 - p_2)^2 &= r^2, \\ (b_1 - p_1)^2 - (b_2 - p_2)^2 &= r^2, \\ (c_1 - p_1)^2 - (c_2 - p_2)^2 &= r^2. \end{aligned} \tag{8}$$

From these we get two linear equations for p_1 and p_2

$$\begin{aligned} 2(b_1 - a_1)p_1 + 2(a_2 - b_2)p_2 &= b_1^2 - a_1^2 + a_2^2 - b_2^2, \\ 2(c_1 - b_1)p_1 + 2(b_2 - c_2)p_2 &= c_1^2 - b_1^2 + b_2^2 - c_2^2. \end{aligned} \tag{9}$$

The system (9) has unique solutions p_1, p_2, because $(b_1 - a_1)(b_2 - c_2) \neq (a_2 - b_2)(c_1 - b_1)$. We can now determine r from the first equation of (8); the other two equations are then satisfied. It remains to check that $r \neq 0$.

Suppose $r = 0$. Then we must have

$$a_1 - p_1 = \pm(a_2 - p_2), b_1 - p_1 = \pm(b_2 - p_2), c_1 - p_1 = \pm(c_2 - p_2).$$

Here we must have at least two plus signs or two minus signs. If, for instance, $a_1 - p_1 = a_2 - p_2, b_1 - p_1 = b_2 - p_2$, then we get $b_1 - a_1 = b_2 - a_2$ which implies $\langle \overrightarrow{AB}, \overrightarrow{AB} \rangle = 0$—a contradiction. The other cases are similarly eliminated. Thus $r \neq 0$.

We shall explore some more properties pertaining to Lorentzian circles which are analogues of well-known facts in Euclidean geometry.

Let A and B be two points on the Lorentzian circle $x^2 - y^2 = r^2$, with $x > 0$. Then the lines OA and OB are spacelike and the angle between them can be defined as indicated in the Remark following (4). We shall call it the *central angle* for the chord AB. If $A = r(\operatorname{ch} a, \operatorname{sh} a)$, $B = r(\operatorname{ch} b, \operatorname{sh} b)$, then the central angle is equal to $|a - b|$.

Let T_B be the tangent line to the Lorentzian circle at B. Then we can talk about the angle between T_B and the chord AB. We shall prove

THEOREM 4. *Let A and B be two points on the Lorentzian circle $x^2 - y^2 = r^2, x > 0$. Then the angle between the tangent line T_B and the chord AB is equal to one half of the central angle for the chord AB.* (See Fig. 3.)

Proof. By applying a transformation of G we may assume that $A = r(\operatorname{ch} a, \operatorname{sh} a)$, $B = (r, 0)$. The angle α between T_B and AB is given by

$$\operatorname{ch}\alpha = -\langle x, y \rangle / \|y\|, \quad \text{where } x = (0,1), y = (\operatorname{ch} a - 1, \operatorname{sh} a).$$

Thus

$$\operatorname{ch}\alpha = \operatorname{sh} a \Big/ \sqrt{-(\operatorname{ch} a - 1)^2 + \operatorname{sh}^2 a} = \operatorname{sh} a \Big/ \sqrt{2(\operatorname{ch} a - 1)}\,.$$

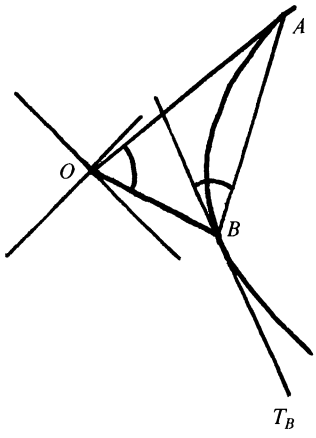


FIG. 3

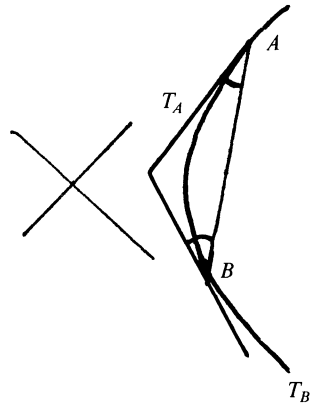


FIG. 4

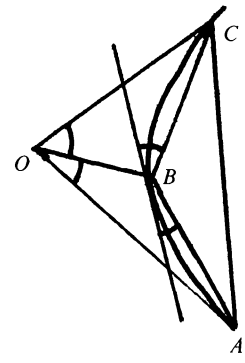


FIG. 5

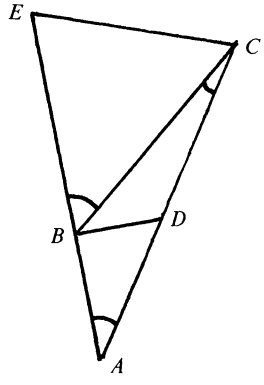


FIG. 6

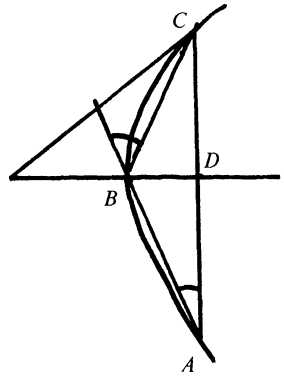


FIG. 7

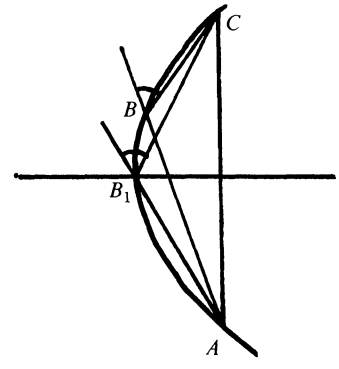


FIG. 8

Using $\operatorname{ch} a - 1 = 2\operatorname{sh}^2(a/2)$ and $\operatorname{sh} a = 2\operatorname{sh}(a/2)\operatorname{ch}(a/2)$, we get $\operatorname{ch}\alpha = \operatorname{ch}(a/2)$, that is, $\alpha = |a|/2$. We know that the central angle for the chord is $|a|$.

COROLLARY 1. *Let A and B be two points on a Lorentzian circle. Then the chord AB makes the same angle with the tangents T_A and T_B.* (See Fig. 4.)

COROLLARY 2. *Let ABC be a pure triangle inscribed in a Lorentzian circle. Then the angle at the middle vertex B is equal to one half of the central angle for the chord AC. Thus, when B varies on the Lorentzian circle between A and C, the angle at B of ΔABC is constant.*

Proof (see Fig. 5). Let α and γ be the angles which the tangent line T_B makes with the chord AB and BC, respectively. Then the angle at B of ΔABC is $\alpha + \gamma$. We know from Theorem 4 that the central angle for the chord AB is 2α and that for BC is 2γ. Thus the central angle for the chord AC is $2(\alpha + \gamma)$.

3. Hyperbolic sine and cosine laws. We prove

THEOREM 5. *Let ABC be a pure triangle. Then*

$$(\operatorname{sh}\hat{A})/a = (\operatorname{sh}\hat{B})/b = (\operatorname{sh}\hat{C})/c \qquad \text{(Hyperbolic sine law)}$$

where $a = \|\overrightarrow{BC}\|$, $b = \|\overrightarrow{CA}\|$, $c = \|\overrightarrow{AB}\|$ and $\hat{A}, \hat{B}, \hat{C}$ denote the angles at A, B, C, respectively.

Proof (see Fig. 8). Let D be a point on AC such that $\overrightarrow{BD}$ is spacelike and $\langle \overrightarrow{BD}, \overrightarrow{AC} \rangle = 0$.

From (6), we obtain $\operatorname{sh}\hat{A} = \|\overrightarrow{BD}\|/c$ and $\operatorname{sh}\hat{C} = \|\overrightarrow{BD}\|/a$ and hence $(\operatorname{sh}\hat{A})/a = (\operatorname{sh}\hat{C})/c$. Let E be a point on the line AB such that $\overrightarrow{CE}$ is spacelike and $\langle \overrightarrow{CE}, \overrightarrow{AB} \rangle = 0$. Then $\operatorname{sh}\hat{B} = \|\overrightarrow{CE}\|/a$ and $\operatorname{sh}\hat{A} = \|\overrightarrow{CE}\|/b$ and hence $(\operatorname{sh}\hat{B})/b = (\operatorname{sh}\hat{A})/a$.

THEOREM 6. *The common ratio in Theorem 5 is equal to* $\frac{1}{2r}$, *where r is the radius of the Lorentzian circle determined by* ΔABC *as in Theorem 3.*

Proof. We first prove the theorem in the special case where B is on the x-axis and A and C are symmetric relative to the x-axis (see Fig. 7). For ΔABC, let α denote the equal angles $\hat{A}$ and $\hat{C}$. By Proposition 1, we have $\hat{B} = 2\alpha$. By Corollary 2 to Theorem 4, the central angle for the chord BC is 2α. If D is the intersection of AC with the x-axis, then Formula (6), which is applicable to ΔODC, gives

$$\|\overrightarrow{CD}\| = r\operatorname{sh}(2\alpha) = 2r\operatorname{ch}\alpha\operatorname{sh}\alpha.$$

The same formula applied to ΔCBD gives

$$\|\overrightarrow{CD}\| = \|\overrightarrow{CB}\|\operatorname{ch}\alpha = a\operatorname{ch}\alpha.$$

Hence $2r\operatorname{sh}\alpha = a$, that is, $(\operatorname{sh}\hat{A})/a = \frac{1}{2r}$.

In the general case, we may apply a transformation belonging to G so that A and C become symmetric relative to the x-axis. If B_1 is the intersection of the Lorentzian circle with the x-axis (see Fig. 8), then from Corollary 2 to Theorem 4 we know that the angle $\hat{B}$ of ΔABC is equal to the angle $\hat{B}_1$ of ΔAB_1C. Thus $(\operatorname{sh}\hat{B})/\|\overrightarrow{AC}\| = (\operatorname{sh}\hat{B}_1)/\|\overrightarrow{AC}\|$. Since the latter is equal to $1/(2r)$ from the special case we have already proved, we conclude that $(\operatorname{sh}\hat{B})/b = 1/(2r)$.

COROLLARY. *The area of a pure triangle* ΔABC *inscribed in a Lorentzian circle of radius r is given by* $\frac{abc}{4r}$, *as well as by* $b^2\operatorname{sh}\hat{A}\operatorname{sh}\hat{C}/2\operatorname{sh}\hat{B}$.

Proof. This follows from Proposition 2 and Theorem 6.

We shall conclude with

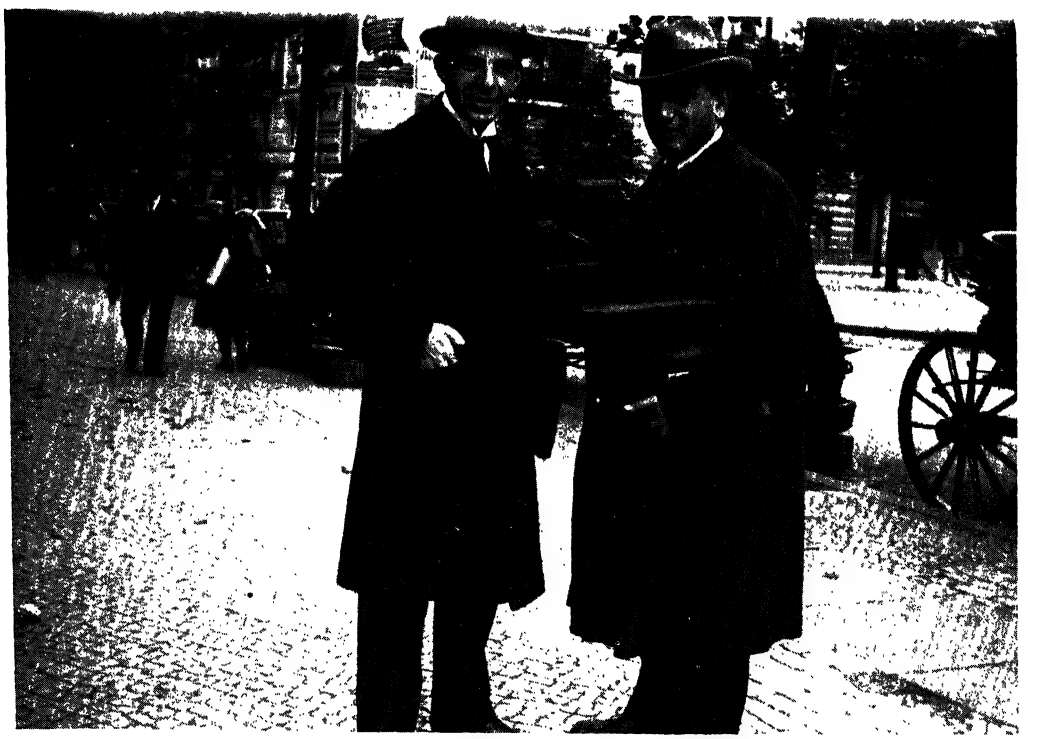
THEOREM 7. *For a pure triangle ABC we have*

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\operatorname{ch}\hat{A}, \\ c^2 &= a^2 + b^2 - 2ab\operatorname{ch}\hat{C}, \qquad \text{(hyperbolic cosine law)} \\ b^2 &= a^2 + c^2 + 2ac\operatorname{ch}\hat{B}. \end{aligned}$$

Proof. Easy and omitted. Note the sign for the last term in the formula corresponding to the middle vertex B.

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What's the problem? (See p. 572.)

$$c(G) \geqslant \frac{1}{\phi(n)} \sum_{s \in U_n} (s-1, n). \tag{4}$$

The first part of the theorem is now immediate from Formulae 3 and 4.

A consideration of Formulae 3 and 4 indicates that the number of cyclic subgroups of a finite group, of order n, can only equal $d(n)$ if, for each s such that $(s, n) = 1$, the number of solutions of the equation $x^{(s-1,n)} = 1$ in G is exactly $(s-1, n)$. To prove the second part of the theorem, it suffices to show that if a group G has this property, then G is cyclic. It is necessary to ask therefore for a description of the values of $(s-1, n)$ as s runs over all invertible residues modulo n. The answer is simply stated.

LEMMA 1. *Let* $D_n = \{(s-1, n); s \in U_n\}$. *If* n *is even, then* $D_n = \{d; d|n, d \text{ is even}\}$. *If* n *is odd, then* $D_n = \{d; d|n\}$.

The proof of this result is an elementary exercise in number theory, using the Chinese Remainder Theorem. We next quote a well-known result of group theory; for proof see [3].

LEMMA 2. *Let* G *be a group of order* n. *Suppose that for each* d *dividing* n *the number of solutions of the equation* $x^d = 1$ *in* G *is less than or equal to* d. *Then* G *is cyclic.*

From Lemmas 1 and 2 and the remarks preceding them, it follows that if G is a group of odd order n such that the number of cyclic subgroups of G is $d(n)$, then G is cyclic. Therefore let G be a group of even order n. Suppose that the number of cyclic subgroups of G is $d(n)$. By Lemma 1 this amounts to the assumption that for each even divisor, d, of n, the number of solutions of the equation $x^d = 1$, in G, is exactly d. In particular, the number of solutions of the equation $x^2 = 1$ is 2. Therefore G has a unique involution. Let this involution be called z. Since it is fixed under conjugation, z is necessarily central. We consider m, an odd divisor of n. Let $y^m = 1$. Then both $y^{2m} = 1$ and $(zy)^{2m} = 1$. Thus to each solution of the equation $x^m = 1$ there correspond two distinct solutions of the equation $x^{2m} = 1$. Since the number of solutions of the equation $x^{2m} = 1$ is only $2m$, then the number of solutions of the equation $x^m = 1$ cannot exceed m. It is therefore true of any divisor, d, of n, that the number of solutions of $x^d = 1$, in G, does not exceed d. Lemma 2 now suffices to prove that G is cyclic and so the second part of the theorem is proved.

Note that Formula 3 gives an unusual relation between the divisor function and the Euler totient function. It may be generalized as follows. Let $f(x)$ be any polynomial in $Z[x]$. Let U_n once again denote the invertible residues modulo n. Then

$$\sum_{s \in U_n} (f(s), n) = \phi(n) \sum_{d|n} |\{r \in U_d; f(r) \equiv 0 \bmod d\}|. \tag{5}$$

In conclusion the author wishes to thank his supervisor at Exeter University, Dr. R. Odoni, for his assistance in the preparation of this note. While this note was written, the author was in receipt of a grant from the Science and Engineering Research Council.

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ANSWER TO PHOTO ON PAGE 550

From left to right, Gábor Szegő and George Pólya, authors of the path-breaking and still outstanding problem book of the century. The picture was taken in 1925, when the subjects were in Berlin for the signing of the contract for their book.

RIEMANN-STIELTJES AND LEBESGUE-STIELTJES INTEGRABILITY

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1. Introduction. The purpose of this paper is to give characterizations of the class of Riemann-Stieltjes integrable functions as a subset of the class of Lebesgue-Stieltjes integrable functions. We shall consider several definitions of the Riemann-Stieltjes integral, including a definition that appears to be proposed here for the first time.

An "inherent defect" [3, p. 420] of a standard Riemann-Stieltjes integral [**10**, Chapter 6] is the well-known necessary condition that the function to be integrated cannot share points of discontinuity from the same side with the integrator function. This unsatisfactory condition can be dropped by using either the definition of the Riemann-Stieltjes integral recently proposed by Ross [**8**], [**9**], or the definition proposed here. For these definitions, moreover, Lebesgue's classical necessary and sufficient condition for Riemann-Stieltjes integrability—continuity μ-almost everywhere—can be improved to another simple criterion: continuity μ-almost everywhere on the set of points of μ-measure zero.

The definition of the Riemann-Stieltjes integral proposed here can be used for integration over arbitrary elementary sets, i.e., finite unions of (not necessarily closed) bounded intervals. It is stated in terms of a "premeasure", which is the restriction of a measure to the ring of elementary sets, and which can be obtained immediately from the integrator function. Theorem E below shows precisely when this premeasure suffices for integration, and when the full power of a measure on a σ-algebra is needed.

Although in this paper we consider only the real line, the definition proposed here allows Riemann-Stieltjes integrals in more general settings, for example $\mathbb{R}^n$. Results on $\mathbb{R}^n$ will be published elsewhere.

2. Construction of a measure. We shall first briefly recapitulate a way of associating a positive measure with a monotonically increasing function $\alpha\colon \mathbb{R} \to \mathbb{R}$ (cf. Rudin [**10**, Chapter 11]). Define the function μ on bounded intervals, including sets containing just one point, by

$$\begin{aligned} \mu([a,b)) &= \alpha(b-) - \alpha(a-) \\ \mu([a,b]) &= \alpha(b+) - \alpha(a-) \\ \mu((a,b]) &= \alpha(b+) - \alpha(a+) \\ \mu((a,b)) &= \alpha(b-) - \alpha(a+). \end{aligned} \tag{1}$$

The symbols $\alpha(x+)$ and $\alpha(x-)$ denote the right-hand limit and the left-hand limit of α at x, respectively. The set A is called an *elementary set* if A is the union of a finite number of bounded intervals $B_1, \ldots, B_m$. If these intervals are pairwise disjoint, define

$$\mu(A) = \mu(B_1) + \cdots + \mu(B_m). \tag{2}$$

The elementary sets form a ring of sets, which will be denoted by $\mathscr{E}$. The finitely additive function μ defined by (1) and (2) on $\mathscr{E}$, which will be called a *premeasure*, plays an important role in the definition of the Riemann-Stieltjes integral described in Section 6. Note that different monotonic functions may give rise to the same premeasure: if α is replaced by any function $\beta\colon \mathbb{R} \to \mathbb{R}$ such that $\alpha(x-) \leqslant \beta(x) \leqslant \alpha(x+)$ for all $x \in \mathbb{R}$, then the same premeasure μ results.

It is not difficult to show that the nonnegative additive set function defined by (1) and (2) is

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regular, in the sense that to every elementary set A and to every $\varepsilon > 0$ there exists an open elementary set G and a closed elementary set F such that $F \subset A \subset G$ and $\mu(G) - \varepsilon \leqslant \mu(A) \leqslant \mu(F) + \varepsilon$. Using a general theorem [**10**, Theorem 11.10], the premeasure μ can be extended to a measure on a σ-algebra $\mathcal{M}$. This measure will also be denoted by μ. The σ-algebra $\mathcal{M}$ contains the set of all Borel sets, i.e., the smallest σ-algebra containing the usual topology of $\mathbb{R}$. Furthermore, the measure thus induced by the function α is complete: every subset of a set of μ-measure zero is μ-measurable, and has μ-measure zero.

3. The Riemann integral. If in the construction of the measure μ in Section 2 we choose $\alpha(x) \equiv x$, the identity function on $\mathbb{R}$, then μ is the usual Lebesgue measure. For this case, many texts on analysis or measure theory contain the following theorem of Lebesgue (see, e.g., Rudin [**10**, p. 323] or Hewitt and Stromberg [**4**, p. 183]):

THEOREM A. 1. *If f is Riemann integrable on $[a, b]$, then f is Lebesgue integrable on $[a, b]$ and the integrals coincide.*

2. *Suppose f is bounded on $[a, b]$. Then f is Riemann integrable on $[a, b]$ if and only if f is continuous almost everywhere on $[a, b]$.*

Almost everywhere (a.e.) refers here, of course, to Lebesgue measure. In the following we shall, for several definitions of the Riemann-Stieltjes integral, state and prove generalizations of Theorem A to Stieltjes integrals.

4. Standard definitions of the Riemann-Stieltjes integral. If the function f is Riemann-Stieltjes integrable with respect to the function α, for the definition of the Riemann-Stieltjes integral under consideration, we shall write $f \in \mathcal{R}(\alpha)$. If f is Lebesgue integrable with respect to the measure μ, we write $f \in \mathcal{L}(\mu)$. The Riemann-Stieltjes integral of f with respect to α over $[a, b]$ will be written as $\int_a^b f\,d\alpha$, and the associated Lebesgue-Stieltjes integral of f over $[a, b]$ as $\int_a^b f\,d\mu$. We shall consider only monotonically increasing integrator functions α: $\mathbb{R} \to \mathbb{R}$. We assume, only in this section, that $\alpha(a -) = \alpha(a)$ and $\alpha(b +) = \alpha(b)$.

Here is the simplest statement concerning Stieltjes integrals that contains Theorem A as a special case.

THEOREM B. 1. *If $f \in \mathcal{R}(\alpha)$ on $[a, b]$, then $f \in \mathcal{L}(\mu)$ on $[a, b]$ and $\int_a^b f\,d\mu = \int_a^b f\,d\alpha$.*

2. *Suppose f is bounded on $[a, b]$. Then $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if f is continuous* a.e. $[\mu]$ *on $[a, b]$.*

This proposition is true when the Riemann-Stieltjes integral is defined in a standard way as the limit of a sequence of sums [**7**], [**11**]. An unsatisfactory aspect of this Riemann-Stieltjes integral follows immediately from Theorem B: the function to be integrated cannot share points of discontinuity with the integrator function.

This problem is partly solved when a more general standard definition of the Riemann-Stieltjes integral is used, namely the definition in terms of upper and lower integrals [**10**, Chapter 6], to which the name of Darboux is often connected. We proceed to review this definition, which will be referred to as the standard Darboux definition.

Let the function f: $[a, b] \to \mathbb{R}$ be bounded. Define, for an arbitrary partition

$$P = \{ a = x_0 < x_1 < \cdots < x_q = b \} \tag{3}$$

of $[a, b]$, the numbers ($j = 1, \ldots, q$)

$$M_j = \sup_{x \in [x_{j-1}, x_j]} f(x) \quad \text{and} \quad m_j = \inf_{x \in [x_{j-1}, x_j]} f(x). \tag{4}$$

The upper and lower Riemann-Stieltjes sums associated with the partition P are defined as

$$U(P, f, \alpha) = \sum_{j=1}^{q} M_j \Delta\alpha_j \quad \text{and} \quad L(P, f, \alpha) = \sum_{j=1}^{q} m_j \Delta\alpha_j, \tag{5}$$

where $\Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1})$. The upper and lower Riemann-Stieltjes integrals of f with respect to α over $[a, b]$ are defined as

$$\text{(6)} \qquad \int_a^{-b} f\,d\alpha = \inf_P U(P, f, \alpha) \qquad \text{and} \qquad \int_{-a}^{b} f\,d\alpha = \sup_P L(P, f, \alpha).$$

The infimum and supremum are taken here over all partitions (3). If the upper and lower Riemann-Stieltjes integrals (6) are equal, this common value defines the Riemann-Stieltjes integral of f with respect to α over $[a, b]$, denoted by $\int_a^b f\,d\alpha$.

In connection with the problem of Riemann-Stieltjes integrability, sets of points of μ-measure zero are of special importance. Accordingly, define the set

$$\text{(7)} \qquad Z_\mu([a, b]) = \{x \in [a, b] \mid \mu(\{x\}) = 0\}.$$

When the standard Darboux definition is used for the Riemann-Stieltjes integral, the generalization of Theorem A to Stieltjes integrals takes the following form.

THEOREM C. 1. *If $f \in \mathscr{R}(\alpha)$ on $[a, b]$, then $f \in \mathscr{L}(\mu)$ on $[a, b]$ and $\int_a^b f\,d\mu = \int_a^b f\,d\alpha$.*

2. *Suppose f is bounded on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if f is continuous* a.e. $[\mu]$ *on $Z_\mu([a, b])$ and if, for all $x \in [a, b]$, f and α are not simultaneously discontinuous from the left or from the right in x.*

In Section 7 we shall give an elementary proof of Theorem C. Alternative characterizations of Riemann-Stieltjes integrability in the standard Darboux sense have been given in [2], [5], [6]. In words, Theorem C shows that with the standard Darboux definition, Lebesgue's classical criterion for Riemann-Stieltjes integrability—continuity μ-almost everywhere—is improved to continuity μ-almost everywhere on the set of points of μ-measure zero. In addition, however, the functions f and α cannot share points of discontinuity from the same side. As we shall see in the next sections, this inconvenience can be dropped by using a modified definition of the Riemann-Stieltjes integral.

5. Ross' definition of the Riemann-Stieltjes integral. Actually, Ross [8], [9], recently proposed two, equivalent, new definitions of the Riemann-Stieltjes integral. One of these definitions is a generalized limit-of-sums definition, whereas the other definition given by Ross is a generalization of the standard Darboux definition studied in the previous section. We shall use Ross' Darboux-type definition, for which only equations (4) and (5) in the standard Darboux definition have to be changed. In equations (4), the closed intervals $[x_{j-1}, x_j]$ are replaced by the open intervals (x_{j-1}, x_j):

$$\text{(8)} \qquad M_j = \sup_{x \in (x_{j-1}, x_j)} f(x) \qquad \text{and} \qquad m_j = \inf_{x \in (x_{j-1}, x_j)} f(x).$$

The upper and lower Riemann-Stieltjes sums are now defined as

$$\text{(9)} \qquad \begin{aligned} U(P, f, \alpha) &= \sum_{j=0}^{q} f(x_j)\big(\alpha(x_j +) - \alpha(x_j -)\big) + \sum_{j=1}^{q} M_j\big(\alpha(x_j -) - \alpha(x_{j-1} +)\big), \\ L(P, f, \alpha) &= \sum_{j=0}^{q} f(x_j)\big(\alpha(x_j +) - \alpha(x_j -)\big) + \sum_{j=1}^{q} m_j\big(\alpha(x_j -) - \alpha(x_{j-1} +)\big). \end{aligned}$$

The following theorem is the generalization of Theorem A to Stieltjes integrals, when Ross' definition is used for the Riemann-Stieltjes integral.

THEOREM D. 1. *If $f \in \mathscr{R}(\alpha)$ on $[a, b]$, then $f \in \mathscr{L}(\mu)$ on $[a, b]$ and $\int_a^b f\,d\mu = \int_a^b f\,d\alpha$.*

2. *Suppose f is bounded on $[a, b]$. Then $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if f is continuous* a.e. $[\mu]$ *on Z_μ $([a, b])$.*

Recall that $Z_\mu([a, b])$ is the set of points in $[a, b]$ of μ-measure zero. See Section 8 for a proof of Theorem D.

$$\int_A f d\mu = \inf_{P \subset \mathcal{M}} U(P, f, \mu) = \sup_{P \subset \mathcal{M}} L(P, f, \mu), \tag{14}$$

where $U(P, f, \mu)$ *and* $L(P, f, \mu)$ *are defined as in* (11), *and where* $\mathcal{M}$ *is the* σ*-algebra forming the domain of the measure* μ.

This lemma can be proved in a simple way using Lebesgue's dominated convergence theorem. The details of the proof are omitted here.

Suppose that, maintaining the natural form (11) for the upper and lower sums, we want to define a Darboux-type integral that is a special case of the Lebesgue-Stieltjes integral, but which gives strictly more integrable functions than a Riemann-Stieltjes integral described by Theorem E. The lemma shows that this is not possible without enlarging the class of sets $\mathcal{E}$ in (13) with sets from $\mathcal{M} - \mathcal{E}$. This departure from the ring of elementary sets to the full σ-algebra is, of course, completely beyond the classical pre-Lebesgue conception of the integral. In this sense, Theorem E may be said to describe the largest possible class of Riemann-Stieltjes integrable functions on the real line.

The definition of the Riemann-Stieltjes integral proposed here is also applicable to other situations than the real line. For example, when the domain of a measure under consideration has a special subset consisting of "elementary sets", the restriction of the measure to this subset might be similarly used to define a "Riemann-Stieltjes integral".

7. Proof of Theorem C. The idea of the proof of Theorem C given here is the same as that of a standard proof of Theorem A [**10**, p. 323]. The details are, however, somewhat more complicated. A large part of this proof will also be used in the proofs of Theorems D and E in the next sections. The proof of Theorem C given here is more complicated than the proofs in the next sections, since only in this proof we have to keep track of the three different ways in which the monotonic integrator function α can be discontinuous (i.e., from the left, from the right, or both).

Since the proof is rather long, it will be divided into several steps. The first four steps are directed towards the auxiliary necessary and sufficient condition for $f \in \mathcal{R}(\alpha)$ on $[a, b]$, expressed in (29). Theorem C will be proved in the last four steps.

Step 1. Define the sets

$$\begin{aligned} D_+ &= \{ x \in [a,b] \mid \alpha(x+) > \alpha(x) \}, \\ D_- &= \{ x \in [a,b] \mid \alpha(x-) < \alpha(x) \}, \\ D &= D_+ \cup D_-, \\ C &= [a,b] - D. \end{aligned} \tag{15}$$

Since the function α is monotonically increasing, and since D is precisely the set of points of discontinuity of α, D is at most countable. The set C, being the set of points of continuity of α, coincides with the set $Z_\mu([a, b])$ used in the statement of Theorem C.

Suppose that the function f is bounded on $[a, b]$. It is quite easy to verify that there exists a sequence of partitions

$$P_k = \{ a = x_0 < x_1 < \cdots < x_q = b \} \tag{16}$$

of $[a, b]$, such that

$$P_k \subset P_{k+1} \quad \forall k \in \mathbb{N} \qquad (P_{k+1} \text{ is a refinement of } P_k) \tag{17}$$

$$x_j - x_{j-1} < 1/k \quad \forall j = 1, \ldots, q \quad \forall k \in \mathbb{N} \tag{18}$$

$$D \subset \bigcup_{k=1}^{\infty} P_k \tag{19}$$

$$\lim_{k \to \infty} L(P_k, f, \alpha) = \int_{-a}^{b} f d\alpha \qquad \text{and} \qquad \lim_{k \to \infty} U(P_k, f, \alpha) = \int_{a}^{-b} f d\alpha. \tag{20}$$

Assumption (19), which will only be used in the last part of the proof, is possible because D is at most countable. For simplicity of notation we do not indicate explicitly that the points x_j depend on the partition number k.

Step 2. We shall associate with every partition P_k a collection of disjoint Borel subsets $E_1, \ldots, E_r$ of $[a, b]$, in such a way that the union of these sets is $[a, b]$. Most of these sets are the *intervals*

$$B_j = \big[(x_{j-1}, x_j)\big] \qquad j = 1, \ldots, q. \tag{21}$$

Here, the symbol [(denotes one of [or (, and the symbol)] denotes either) or] , to be assigned as follows. If $j = 1$ and $j = q$, we have the intervals $[x_0, x_1)]$ and $[(x_{q-1}, x_q]$, respectively. If $j = 1, \ldots, q-1$, then the point $x_j \in P_k$ belongs either to the interval on its left, to the interval on its right, or to none of these intervals. These three cases occur respectively when

$$x_j \in D'_+, \qquad x_j \in D_+ \cap D'_-, \qquad x_j \in D_+ \cap D_-.$$

The complement in $[a, b]$ of a set $V \subset [a, b]$ is denoted by V'. In the last case ($x_j \in D_+ \cap D_-$), add the set $\{x_j\}$ to these intervals. The union of the disjoint Borel sets thus obtained is $[a, b]$.

Define Borel measurable simple functions L_k and U_k on $[a, b]$:

$$\begin{aligned} &L_k(x) = m_j \quad \text{and} \quad U_k(x) = M_j \quad \text{for} \quad x \in B_j \\ &L_k(x_j) = m_j \Delta_\alpha^-(x_j) + m_{j+1} \Delta_\alpha^+(x_j) \quad \text{and} \\ &U_k(x_j) = M_j \Delta_\alpha^-(x_j) + M_{j+1} \Delta_\alpha^+(x_j) \quad \text{for} \quad x_j \in D_+ \cap D_-, \end{aligned} \tag{22}$$

where

$$\Delta_\alpha^-(x) = \frac{\alpha(x) - \alpha(x-)}{\alpha(x+) - \alpha(x-)} \qquad \text{and} \qquad \Delta_\alpha^+(x) = \frac{\alpha(x+) - \alpha(x)}{\alpha(x+) - \alpha(x-)}.$$

Step 3. The Lebesgue-Stieltjes integrals over $[a, b]$ of these functions satisfy

$$\int_a^b L_k \, d\mu = L(P_k, f, \alpha) \quad \text{and} \quad \int_a^b U_k \, d\mu = U(P_k, f, \alpha). \tag{23}$$

This is most easily seen by considering the "contributions from each x_j" to these integrals separately: if we put $m_0 = m_{q+1} = 0$, then the contribution from each $x_j \in P_k$ to the integral $\int_a^b L_k \, d\mu = \sum_{i=1}^r L_{ki} \mu(E_i)$ is $m_j \alpha(x_j) - m_{j+1} \alpha(x_j)$.

Since $\alpha(a-) = \alpha(a)$ and $\alpha(b+) = \alpha(b)$, this is trivial for $j = 0$ and $j = q$. To prove this for $j = 1, \ldots, q-1$, suppose first $x_j \in D'_+$. Then x_j contributes

$$m_j \alpha(x_j +) - m_{j+1} \alpha(x_j +) = m_j \alpha(x_j) - m_{j+1} \alpha(x_j)$$

to the integral $\int_a^b L_k \, d\mu$. The case $x_j \in D_+ \cap D'_-$ can be treated similarly. If $x_j \in D_+ \cap D_-$, then x_j contributes

$$\begin{aligned} &m_j \alpha(x_j -) - m_{j+1} \alpha(x_j +) + \big[m_j \Delta_\alpha^-(x_j) + m_{j+1} \Delta_\alpha^+(x_j) \big] \big[\alpha(x_j +) - \alpha(x_j -) \big] \\ &\quad = m_j \alpha(x_j) - m_{j+1} \alpha(x_j). \end{aligned}$$

Hence

$$\int_a^b L_k \, d\mu = \sum_{j=1}^q m_j \Delta \alpha_j = L(P_k, f, \alpha),$$

and it can be shown similarly that

$$\int_a^b U_k \, d\mu = \sum_{j=1}^q M_j \Delta \alpha_j = U(P_k, f, \alpha).$$

Step 4. Defining $m = \inf_{x \in [a,b]} f(x)$ and $M = \sup_{x \in [a,b]} f(x)$, it is clear that

$$m \leqslant L_k(x) \leqslant L_{k+1}(x) \leqslant f(x) \leqslant U_{k+1}(x) \leqslant U_k(x) \leqslant M \tag{24}$$

for all $k \in \mathbb{N}$ and for all $x \in [a, b]$. Hence we can define the functions

$$L(x) = \lim_{k \to \infty} L_k(x) \qquad \text{and} \qquad U(x) = \lim_{k \to \infty} U_k(x) \tag{25}$$

for which

$$m \leqslant L(x) \leqslant f(x) \leqslant U(x) \leqslant M \quad \forall x \in [a, b]. \tag{26}$$

The functions L_k, U_k, and L and U are μ-measurable and bounded, hence Lebesgue-Stieltjes integrable on $[a, b]$. Lebesgue's dominated convergence theorem shows that

$$\int_a^b L\, d\mu = \lim_{k \to \infty} \int_a^b L_k\, d\mu \qquad \text{and} \qquad \int_a^b U\, d\mu = \lim_{k \to \infty} \int_a^b U_k\, d\mu. \tag{27}$$

From (20), (23), and (27) it follows that

$$\int_a^b L\, d\mu = \int_{-a}^{b} f\, d\alpha \qquad \text{and} \qquad \int_a^b U\, d\mu = \int_a^{-b} f\, d\alpha. \tag{28}$$

Hence $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if and only if $\int_a^b (U - L)\, d\mu = 0$. Since $U(x) \geqslant L(x)$ for all $x \in [a, b]$, this happens if and only if $U(x) = L(x)$ a.e. $[\mu]$ on $[a, b]$. Hence

$$f \in \mathscr{R}(\alpha) \text{ on } [a, b] \Leftrightarrow U(x) = L(x) \text{ a.e. } [\mu] \text{ on } [a, b]. \tag{29}$$

Step 5. Suppose now that $f \in \mathscr{R}(\alpha)$ on $[a, b]$. From (26) and (29), it follows that $L(x) = f(x)$ a.e. $[\mu]$ on $[a, b]$. Using the completeness of μ, the Lebesgue-Stieltjes integrability of L, and (28), the Lebesgue-Stieltjes integrability of f follows, as well as $\int_a^b f\, d\mu = \int_a^b f\, d\alpha$. This proves the first part of the theorem.

Step 6. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$, then f and α are nowhere in $[a, b]$ simultaneously discontinuous from the left or from the right. See [1, p. 160] for the proof.

Step 7. We shall now show that $f \in \mathscr{R}(\alpha)$ on $[a, b]$ implies that f is continuous a.e. $[\mu]$ on $C = Z_\mu([a, b])$. If $f \in \mathscr{R}(\alpha)$ on $[a, b]$, then by (29), $U(x) = L(x)$ a.e. $[\mu]$ on $[a, b]$. If $x \in C$ is chosen such that $U(x) = L(x)$ and such that x is an interior point of a set B_j for all P_k, then f is continuous in x.

To prove this assertion, suppose it is not true. Then there exists an $\varepsilon > 0$ and a sequence $y_m \in [a, b]$ such that $|y_m - x| < 1/m$ and $|f(y_m) - f(x)| \geqslant \varepsilon$ for all $m \in \mathbb{N}$. For $k \in \mathbb{N}$ arbitrary, there is a j such that x is an interior point of B_j. Then for m large enough, $y_m \in B_j$. Hence

$$U_k(x) - L_k(x) = M_j - m_j \geqslant \sup_{y, z \in B_j} |f(y) - f(z)| \geqslant |f(y_m) - f(x)| \geqslant \varepsilon,$$

and also $U(x) - L(x) \geqslant \varepsilon$, in contradiction with $U(x) = L(x)$.

It follows that if f is discontinuous in $x \in C$, then $x \in Z_1 \cup Z_2$, where

$$Z_1 = \{ x \in C | U(x) \neq L(x) \},$$

$$Z_2 = \{ x \in C | \exists k \in \mathbb{N} \colon \forall j \colon x \notin \operatorname{Int}(B_j) \}.$$

Here, $\operatorname{Int}(B_j)$ denotes the interior of the set B_j. By $U(x) = L(x)$ a.e. $[\mu]$ on $[a, b]$, $\mu(Z_1) = 0$. It is easily seen that Z_2 is a countable union of one-point sets of μ-measure zero, hence $\mu(Z_2) = 0$. This proves that f is continuous a.e. $[\mu]$ on $C = Z_\mu([a, b])$.

Step 8. It remains only to prove that $f \in \mathscr{R}(\alpha)$ on $[a, b]$ if f is continuous a.e. $[\mu]$ on $C = Z_\mu([a, b])$, and if f and α are not simultaneously discontinuous from the same side in any $x \in [a, b]$.

If f is continuous in $x \in C$, then $U(x) = L(x)$. To show this, choose $\varepsilon > 0$ arbitrarily. There is a $K \in \mathbb{N}$ such that $|f(y) - f(z)| < \varepsilon/2$ for all $y, z \in \left(x - \frac{1}{K}, x + \frac{1}{K} \right) \cap [a, b]$. Choose for

P_K that j for which $x \in B_j \subset \left(x - \frac{1}{K}, x + \frac{1}{K}\right)$, where (18) is used in the last step. It follows that for all $k \geqslant K$,

$$U_k(x) - L_k(x) \leqslant U_K(x) - L_K(x) = M_j - m_j = \sup_{y,z \in [x_{j-1}, x_j]} |f(y) - f(z)| \leqslant \varepsilon/2 < \varepsilon. \tag{30}$$

This proves $U(x) - L(x) = \lim_{k \to \infty} (U_k(x) - L_k(x)) = 0$. Hence $U(x) = L(x)$ a.e. $[\mu]$ on C.

If we can prove that $U(x) = L(x)$ for all $x \in D$, then it follows that $U(x) = L(x)$ a.e. $[\mu]$ on $[a, b]$, so that by (29), $f \in \mathscr{R}(\alpha)$ on $[a, b]$. To prove that $U(x) = L(x)$ for all $x \in D$, note that by (17) and (19), $x \in D$ implies $x \in P_k$ for k large enough. Suppose first $x \in D'_+ \cap D_-$, so that $f(x-) = f(x)$. For $\varepsilon > 0$ arbitrary, there is a $\delta > 0$ such that $|f(y) - f(z)| < \varepsilon/2$ for all $y, z \in (x - \delta, x]$. Choose $K \in \mathbb{N}$ such that $1/K < \delta$ and $x = x_j \in P_K$. Then (30) holds again for all $k \geqslant K$, so that $U(x) = L(x)$. It can be shown in the same way that $U(x) = L(x)$ for $x \in D_+ \cap D'_-$. (Note that in this case, $U_K(x) - L_K(x) = M_{j+1} - m_{j+1}$.) Suppose next $x \in D_+ \cap D_-$. Then $f(x-) = f(x) = f(x+)$. For $\varepsilon > 0$ arbitrary there exists a $\delta > 0$ such that $|f(y) - f(z)| < \varepsilon/2$ for all $y, z \in (x - \delta, x + \delta)$. Choose $K \in \mathbb{N}$ such that $1/K < \delta$ and $x = x_j \in P_K$. Then it follows in the same way as above that $M_j - m_j < \varepsilon$ and $M_{j+1} - m_{j+1} < \varepsilon$, so that $M_j - m_j \to 0$ and $M_{j+1} - m_{j+1} \to 0$ as $k \to \infty$. This implies

$$U_k(x) - L_k(x) = (M_j - m_j)\Delta_\alpha^-(x) + (M_{j+1} - m_{j+1})\Delta_\alpha^+(x) \to 0$$

as $k \to \infty$, so that $U(x) = L(x)$ for all $x \in D$.

The proof is now complete.

8. Proof of Theorem D. The proof of Theorem D described here is just a simplification of the proof of Theorem C given above. Only the main changes to be made will be described. Instead of the four sets (15), here we need only the two sets

$$D = \{x \in [a, b] | \alpha(x+) > \alpha(x-)\}, \qquad C = [a, b] - D. \tag{31}$$

Furthermore, the auxiliary functions (22) can now be replaced by the much simpler functions

$$L_k(x) = m_j \quad \text{and} \quad U_k(x) = M_j \quad \text{for} \quad x \in B_j, \qquad L_k(x_j) = U_k(x_j) = f(x_j) \quad \text{for} \quad x_j \in P_k, \tag{32}$$

where, instead of (21), the sets B_j are now simply defined as $B_j = (x_{j-1}, x_j)$. Apart from the simplifications that result from these changes, the proof of Theorem D can be taken almost verbatim from the proof of Theorem C.

By (1) and (9), Step 3 in the proof of Theorem C now becomes trivial. Furthermore, in Step 8 of the proof of Theorem C, the fact that $U(x) = L(x)$ for all $x \in D$ can now be shown simply as follows. By (17) and (19), $x \in D$ implies the existence of a $K \in \mathbb{N}$ such that $x \in P_k$ for all $k \geqslant K$. Hence by (32), $U_k(x) = L_k(x) = f(x)$ for all $k \geqslant K$, so that $U(x) = L(x)$. This completes the proof.

9. Proof of Theorem E. The definition of the partitions P, which pervaded the proofs of Theorems C and D, has now changed. In spite of this, the proof of Theorem D described above for Ross' definition can be applied almost unchanged to the definition of the Riemann-Stieltjes integral proposed here. Of course, the interval $[a, b]$ should be replaced throughout the proof by the elementary set A. The assumptions (17) to (20), concerning the sequence of partitions

$$P_k = \{B_1, \ldots, B_q\} \subset \mathscr{E} \tag{33}$$

of A into nonempty sets, need reformulation:

(34) For all $k \in \mathbb{N}$, P_{k+1} is a subpartition of P_k

(35) $$\operatorname{diam}(B_j) < 1/k \qquad \forall j = 1, \ldots, q \qquad \forall k \in \mathbb{N}$$

(36) $$x \in D \Rightarrow \{x\} \in \bigcup_{k=1}^{\infty} P_k$$

(37) $$\lim_{k \to \infty} L(P_k, f, \mu) = \int_A f \, d\alpha \qquad \text{and} \qquad \lim_{k \to \infty} U(P_k, f, \mu) = \int_A f \, d\alpha.$$

Assumption (34) means that every element of P_{k+1} is a subset of some element of P_k. In (35), $\operatorname{diam}(B_j)$ is defined as $\sup_{x, y \in B_j} |x - y|$. In addition to these changes, definition (32) of the functions L_k and U_k should be replaced by

(38) $$L_k(x) = m_j \quad \text{and} \quad U_k(x) = M_j \qquad \text{for} \quad x \in B_j \in P_k.$$

Apart from a few minor changes, this completes the proof.

It is a pleasure to thank Dr. J. D. Stegeman for some valuable discussions concerning the manuscript.

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encounterd in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics. The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

A PENTAGONAL POT-POURRI OF PERPLEXING PROBLEMS, PRIMARILY PROBABILISTIC

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Five items which might otherwise have fallen between the several stools of Elementary, Advanced and Unsolved Problems. They may not all be unsolved, but solutions weren't found by proposers, editors or referees.

Chung Kai-Lai, Department of Mathematics, Stanford University, Stanford, CA 94305, asks us to

Solve This Catalan Type Recurrence.

Find an explicit formula for f_n $(n \geqslant 1)$, defined by the convolutionary recurrence

$$p_n = \sum_{k=1}^{n} f_k p_{n-k}$$

where

$$p_n = \binom{bn}{an} a^{na}(b-a)^{n(b-a)}/b^{nb} \quad (n \geqslant 0)$$

(i.e., $p_0 = 1$) and a, b are coprime integers, $0 < a < b$, $(a, b) = 1$.

This looks to be pure combinatorial number theory: what is the probabilistic connexion? Let X_n $(n \geqslant 1)$ be a Bernoullian random walk with probabilities $P(X_n = b - a) = a/b$, $P(X_n = -a) = (b-a)/b$ (step $b - a$ to the right with probability a/b, otherwise step a to the left). If $S_n = \Sigma_{k=1}^{n} X_n$ is the distance from the origin after n steps, then $f_n = P(S_{bn} = 0, S_k \neq 0$ for $1 \leqslant k < bn)$, the probability that we are back at the origin, for the first time, after exactly bn steps.

The answer for $a = 1$ is

$$f_n = \frac{(bn-2)!}{(n-1)!(bn-n)!}(b-1)^{bn-n+1}/b^{bn-1}.$$

See K. L. Chung [**1**]; the solution, by H. W. Gould [**8**], appeared ten years later.

J. F. Crook, School of Business Administration, Winthrop College, Rock Hill, SC 29730 and I. J. Good, Department of Statistics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061, ask a question which arises in a Bayesian analysis of contingency tables [**2**], [**6**]:

Is This Matrix Function Unimodal?

The $m \times n$ matrix $A = (a_{ij})$ with nonnegative integer entries has margins (row and column sums) $\Sigma_{j=1}^{n} a_{ij} = r_i$ $(1 \leqslant i \leqslant m)$ and $\Sigma_{i=1}^{m} a_{ij} = c_j$ $(1 \leqslant j \leqslant n)$ and total $\Sigma_{i=1}^{m} r_i = \Sigma_{j=1}^{n} c_j = N$, where we assume $N > 1$. Define

$$f(A, x) = \prod_{i=1}^{m} \prod_{j=1}^{n} \prod_{k=0}^{a_{ij}-1} \frac{k+x}{k+1}$$

(the innermost product is 1 when $a_{ij} = 0$) and define $g(x)$ as $f(A, x)$ divided by the average of $f(B, x)$ taken over all nonnegative integer matrices, B, which have the same given margins as A. [The enumeration of such matrices is a classical combinatorial problem that is only partly solved.] Prove that $g(x)$ has at most one local maximum for $x > 0$.

Crook and Good have looked at numerous examples, never finding a counterexample. In particular the result is true in the following 2×4 and 3×3 cases: $\{r_i; c_j\} = \{18, 22; 7, 8, 10, 15\}$, $\{5, 5, 5; 5, 5, 5\}$, $\{8, 12, 14; 7, 11, 16\}$ and $\{10, 10, 10; 10, 10, 10\}$, for which the numbers of tables are respectively 698, 231, 1969 and 2211.

If the conjecture is true, it can be proved [**7**] that the maximum occurs at $x = \infty$ (if that can be called local) just if

$$\sum_i \sum_j \binom{a_{ij}}{2} \leqslant \sum_i \binom{r_i}{2} \sum_j \binom{c_j}{2} \Big/ \binom{N}{2}.$$

A similar conjecture, applying to the multinomial distribution, was made in 1965, nearly proved by Good [**5**] and completely proved by Levin & Reeds [**9**].

Phil M. Locke, Department of Mathematics, University of Maine at Orono, ME 04473, asks, in an athletic conference of n evenly matched teams,

What is the Probability of a Unique Champion?

First solve the problem under the assumption that no game can end in a tie. How is the situation altered if there is a probability $q > 0$ that any given game will end in a tie?

Locke remarks that the six-team Yankee Conference has a tie-breaker rule for overtimes so that it will be "easier to determine a conference champion." In the fall of 1982 the Conference had three co-champions. Ironically, had tied games been allowed to stand, Maine would have been the undisputed champion with a 3-0-2 record.

This was originally submitted to the regular Problems section. Several referees made interesting comments. We summarize four sets, in part to show how much hard work goes into securing the excellence of the MONTHLY.

Referee A noted that the problem as posed is open-ended and probably too hard. She formulated two specific problems which the reader can use as warm-ups:

PROBLEM 1 (Elementary?). Is the tie-breaker argument that Locke quotes a valid one? More precisely, let the points be 2 for a win, 1 for a tie, 0 for a loss. Suppose all six teams to be of equal strength, each having a probability p of winning, p of losing and $1 - 2p$ of tying. What value of p, $0 \leqslant p \leqslant \frac{1}{2}$, maximizes the probability of a unique conference champion, exactly one team with maximum total points?

PROBLEM 2 (Advanced?). Consider a round-robin tournament with n teams, evenly matched with probability $\frac{1}{2}$ of winning each game; no ties. How does p_n, the probability of there being a unique champion, behave as $n \to \infty$?

Referee B simulated several tournaments, estimating the probability $p_n(q)$ of a unique winner among n equally matched teams, each having a probability $q = 1 - 2p$ of a tie. He used 200 tournaments each time and assumed that the scoring is as in chess and the National Hockey League, where a tie counts as half a win. He noted that the estimations seem to indicate that the probability of a unique winner is highest when the probability of a tie is small but positive. In subsequent correspondence he (I may as well reveal that it is Paul Smith of the University of Victoria) observed that $p_n(q)$ is a polynomial of degree $\binom{n}{2}$ in q. Thus $p_2(q) = 1 - q$ and $p_3(q) = \frac{3}{4}(1 - q)(1 + q + 2q^2)$. The maximum of $p_3(q)$ is $7/9$, at $q = 1/3$. The constant terms $p_n(0)$ are p_n (see Referee D's values below). It is obvious that $p_n(0) \geqslant n/2^{n-1}$, with equality just if $n \leqslant 4$. It seems difficult to find an upper bound, but Paul Smith is sure that $p_n(0) \to 0$ as $n \to \infty$ and he asks what can be said about $\max p_n(q)$ as q varies.

The original proposal also suggested considering the situation with the teams *un*evenly matched, but admitted that this might be a nasty problem. Paul Smith notes that the case for three players and no ties is easy. If p_{ij} is the probability that team i beats team j, then the probability of a clear winner is $p_{12}p_{13} + p_{23}(1 - p_{12}) + (1 - p_{13})(1 - p_{23})$. For n players and no ties the answer is "an ugly polynomial of degree $\binom{n}{2} - 1$ in $\binom{n}{2}$ variables." He proposes a related problem:

A chess championship match is played. The champion's probability of winning any game is p and the probability of a draw is q. Find the probability that (a) the match is tied after $2n$ games, (b) the challenger leads after $2n$ games. [Of course, for $q = 0$ the answer to (a) is $\binom{2n}{n}p^n(1 - p)^n$.]

Referee C notes that Epstein [4] considers the problem with no ties, gives numerical results for $4 \leqslant n \leqslant 8$, and asserts that the probability of a unique winner tends to 1 as $n \to \infty$. Referee C invites readers to prove or disprove this, noting that B's results make it appear to be false. He notes that there are many ways of looking at the problem: via directed graphs, via skew symmetric matrices with off diagonal entries ± 1, via restricted partitions of $n(n - 1)/2$. He also notes that the problem isn't well-posed when there are ties. In professional hockey, wins count 2, ties 1, losses 0 (equivalently 1, 0, -1). In some leagues and sports, ties are ignored; in others they count as a win and a loss. The first method preserves the total number of games and is probably the appropriate mathematical problem to attack. With no ties and n teams, there are $2^{n(n-1)/2}$

possible outcomes; but referee C suggests that enumeration isn't helpful.

However, Referee D assumes that there are n evenly matched teams, enumerates the tournaments and uses David's tables [3] to obtain the values

n	2	3	4	5	6	7	8
p_n	1	$\frac{3}{4}$	$\frac{1}{2}$	$\frac{75}{128}$	$\frac{321}{512}$	$\frac{76139}{131072}$	$\frac{312971}{524288}$
p_n'	0	0	0	$\frac{35}{68}$	$\frac{159}{279}$	$\frac{55503}{104843}$	$\frac{26469}{46195}$

The values p_n' in the last line are the probabilities of a unique champion when the tournaments are irreducible. He also gives the polynomial (cf. Referee B above)

$$p_4(q) = \frac{1}{8}(1-q)(4+19q-14q^2-12q^3+42q^4+9q^5)$$

and its maximum, 0.7418596767..., attained at $q = 0.3044071868...$.

Walter W. Piegorsch, Biometrics Unit, New York State College of Agriculture and Life Sciences, 337 Warren Hall, Cornell University, Ithaca, NY 14853, wants to know

Can We Generate a Bivariate Poisson Distribution with a Negative Correlation?

Suppose that X_i $(i = 0,1,2)$ are independent Poisson random variables with parameters $\tau_i > 0$. That is, the probability density function of X_i is

$$f(x_i;\tau_i) = e^{-\tau_i}\tau_i^{x_i}/x_i!$$

where x_i ranges over the nonnegative integers. Then the vector $(X_0 + X_1, X_0 + X_2)$ has a bivariate Poisson distribution with marginal means $\tau_0 + \tau_1$, $\tau_0 + \tau_2$ and (obviously positive) correlation

$$\rho = \tau_0(\tau_0+\tau_1)^{-\frac{1}{2}}(\tau_0+\tau_2)^{-\frac{1}{2}}.$$

I. Olkin [**10**] has shown that such a bivariate distribution can also be derived from a properly-taken limit of a bivariate binomial distribution with any given correlation (possibly negative-valued).

Can a bivariate Poisson distribution with a negative correlation be generated by some simple transform, or limiting process, such as the above?

J. G. F. Thiemann, Department of Mathematics and Computing Science, Technische Hogeschool, P.O. Box 513, 5600 MB Eindhoven, The Netherlands, asks

When Is There a Probability Measure on a σ-Algebra?

A is a set and $\mathscr{A} = \{A_n | n \geqslant 0\}$ is a **semi-compact** sequence of subsets of A, i.e., for each subset $\mathscr{A}'$ of $\mathscr{A}$ with $\cap\mathscr{A}' = \varnothing$, there is a finite subset $\mathscr{A}''$ of $\mathscr{A}'$ such that $\cap\mathscr{A}'' = \varnothing$. $\{a_n\}$ is a sequence of real numbers. Let $\sigma(\mathscr{A})$ be the σ-algebra on A generated by $\mathscr{A}$, and, for each $m \geqslant 0$, suppose there is a probability measure μ on $\sigma(\mathscr{A})$ such that $\mu(A_n) \geqslant a_n$ for $0 \leqslant n \leqslant m$.

Is there a probability measure μ on $\sigma(\mathscr{A})$ such that $\mu(A_n) \geqslant a_n$ for all n?

When the answer is "no," what can be said when A is the set of reals and $\{A_n\}$ is a sequence of Borel sets?

Several useful comments on some of these problems were made by E. G. Enns and by Lajos Takács.

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NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

AN ELEMENTARY PROOF THAT THE HILBERT CUBE IS COMPACT

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The Hilbert cube is defined to be the product of a countably infinite family $\{I_n : n = 1, 2, \dots\}$ of homeomorphic copies of the closed unit interval $[0,1]$. That it is compact follows, of course, from Tychonoff's Theorem which says that any product of compact spaces is compact. Our proof of the compactness of the Hilbert Cube is, however, of a very different flavor from the usual proofs of Tychonoff's Theorem. We call the proof elementary because it can be easily understood by the average student of topology and also because (when written out with some care) it avoids the Axiom of Choice. The approach is as follows.

Define the Cantor space, G, in the usual way so that it is seen to be a closed subspace of $[0,1]$ and so is compact. Next, observe that each point in G has a unique ternary representation $\sum_{n=1}^{\infty} a_n/3^n$ with $a_n \in \{0,2\}$, for each n. For each positive integer n, define A_n to be the discrete space $\{0,2\}$. Then it is easily verified that the mapping ϕ from the product space $\prod_{n=1}^{\infty} A_n$ onto G given by

$$\phi((a_1, a_2, \dots, a_n, \dots)) = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

is a homeomorphism. (See [**1**, p. 104].)

Now we state two lemmas, the proofs of which are quite straightforward.

LEMMA 1. *For each positive integer n, let G_n be homeomorphic to the Cantor space G. Then the product space $\prod_{n=1}^{\infty} G_n$ is homeomorphic to G.*

Lemma 1 follows from the fact that a countable product of a countable product of copies of $\{0,2\}$ is homeomorphic to a countable product of copies of $\{0,2\}$. (There is nothing special about $\{0,2\}$ in this statement, it is equally true for any topological space: [**1**, p. 102].)

LEMMA 2. *There exists a continuous mapping ψ of the Cantor space onto* $[0,1]$.

Lemma 2 is obtained by observing that the mapping $\theta : \prod_{n=1}^{\infty} A_n \to [0,1]$ given by

$$\theta((a_1, a_2, \ldots, a_n, \ldots)) = \sum_{n=1}^{\infty} \frac{a_n}{2^{n+1}}$$

is continuous [**1**, p. 104] and surjective. The required map $\psi = \theta \circ \phi^{-1}$, where ϕ is as above.

We can now prove the main result.

THEOREM. *The Hilbert cube is compact.*

Proof. With G_n and I_n as above, Lemma 2 implies that there is a continuous mapping ψ_n of G_n onto I_n, for each positive integer n. Thus there is a mapping Ψ of $\prod_{n=1}^{\infty} G_n$ into $\prod_{n=1}^{\infty} I_n$ given by

$$\Psi((g_1, g_2, \ldots, g_n, \ldots)) = (\psi_1(g_1), \psi_2(g_2), \ldots, \psi_n(g_n), \ldots)$$

where $g_n \in G_n$, for each n. It is easily verified that Ψ is continuous and surjective. Caution is required when proving surjectivity, so as to avoid the Axiom of Choice. Let $(x_1, x_2, \ldots, x_n, \ldots) \in \prod_{n=1}^{\infty} I_n$. Observing that (i) each G_n has an ordering inherited from $[0,1]$, (ii) each G_n is a closed subset of $[0,1]$, and (iii) each $\psi_n : G_n \to I_n$ is surjective, we can let g_n be the *smallest* element of G_n such that $\psi_n(g_n) = x_n$. Then

$$\Psi((g_1, g_2, \ldots, g_n, \ldots)) = (x_1, x_2, \ldots, x_n, \ldots).$$

Then Lemma 1 says that $\prod_{n=1}^{\infty} G_n$ is homeomorphic to the Cantor space, and so it is compact. Thus the Hilbert cube $\prod_{n=1}^{\infty} I_n$ is a continuous image of a compact space and hence is compact.

The above approach has several advantages. The Cantor space has been introduced not as an oddity, but rather as a tool. Also, it is now but a small step (see [**1**, p. 104]) to show that the n-cube $[0,1] \times \cdots \times [0,1]$ is a continuous image of $[0,1]$—so space filling curves appear quite naturally. Another advantage, and this is quite subjective, is that it is a good idea to spend some time on countable products before moving on to uncountable products. Finally, one can proceed to prove that every compact metric space is homeomorphic to a subspace of the Hilbert cube, from which one can then deduce that any countable product of compact metric spaces is compact and also that every compact metric space is a continuous image of (a closed subspace of) the Cantor space. (See [**2**].)

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A PARTITION OF THE UNIT INTERVAL

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By a first category subset of the real line R we mean the union of countably many nowhere dense subsets of R. A second category set is a set that is not a first category set. The subset E of R is said to be "large" in an interval J if $E \cap J$ is a second category Lebesgue measurable set with positive measure. In [**2**] Walter Rudin provides a pleasantly uncomplicated method of constructing a measurable subset A of $[0,1]$ such that $0 < m(A \cap J) < m(J)$ for any subinterval J of $[0,1]$ where m denotes Lebesgue (outer) measure. His set A, however, is small in the sense that A is a first category set. We modify his technique to prove

THEOREM 1. *The unit interval $I = (0,1)$ can be partitioned into a countably infinite collection of mutually disjoint measurable sets T_k such that each T_k is large in every subinterval of I.*

It is not so difficult to obtain either the desired category or the desired measure, but it is harder

to obtain both. We will use nonmeasurable sets and other things to define the measurable sets T_k. We begin with

LEMMA 1. *There exists a sequence (S_n) of mutually disjoint first category measurable subsets of I such that $m(S_n \cap J) > 0$ for all n and all subintervals J of I, and such that $\Sigma_n m(S_n) = 1$.*

Proof. For any open interval J, by a "special set in J" we mean a perfect nowhere dense subset A of J satisfying $m(A) > \frac{1}{2}m(J)$. Now for $n \geqslant 1$, let $\{I_n\}_n$ be the set of all open intervals I_n with rational endpoints whose closures $\overline{I_n}$ lie in I. For $n \geqslant 1$ let C_n denote a nonvoid perfect set of measure 0 with $C_n \subset I_n$. Let C_0 denote the Cantor set.

We define two infinite sequences of perfect sets $(A_{1j})_j$, $(B_{1j})_j$, as follows. For each j let A_{1j} be a special set in a component interval of the open set $I_1 \setminus (C_1 \cup C_0)$, there being exactly one in each such component. Likewise for each j let B_{1j} be a special set in a component interval of the open set $(I \setminus C_0) \cap (I \setminus \overline{I_1})$, there being exactly one in each such component. Then

$$D_1 = \cup_j (A_{1j} \cup B_{1j}) \cup C_0 \cup C_1$$

is a perfect set, and $m(D_1) > \frac{1}{2}$. We proceed by induction to define sequences of perfect sets $(A_{ij})_j$ and $(B_{ij})_j$ for each i such that

$$A_{ij} \subset I_i, \quad B_{ij} \subset I \setminus \overline{I_i}, \quad D_n = \cup_{i=1}^n \cup_j (A_{ij} \cup B_{ij}) \cup C_i \cup C_0$$

is a perfect set, and $m(D_n) > 2^{-1} + 2^{-2} + \cdots + 2^{-n}$.

Assume that appropriate sequences $(A_{ij})_j$ and $(B_{ij})_j$ have been chosen for $i = 1, \ldots, n-1$. Let A_{nj} be a special set in a component interval of the open set $(I \setminus D_{n-1}) \cap (I_n \setminus C_n)$, there being exactly one in each component. Let B_{nj} be a special set in a component interval of the open set $(I \setminus D_{n-1}) \cap (I \setminus \overline{I_n})$, there being exactly one in each component. It follows that D_n is a perfect set. Note that any accumulation point of $\cup_{i=1}^n \cup_j (A_{ij} \cup B_{ij})$ either lies in this set or in $\cup_{i=0}^n C_i$. Moreover

$$m(D_n) > m(D_{n-1}) + \frac{1}{2}(1 - m(D_{n-1})) = \frac{1}{2} + \frac{1}{2}m(D_{n-1})$$

$$> \frac{1}{2} + \frac{1}{2}\sum_{i=1}^{n-1} 2^{-i} = \sum_{i=1}^{n} 2^{-i}.$$

This completes the induction. It follows that $D = \cup_{k=1}^\infty D_k$ is a first category set and $m(D) = 1$.

Now put $S_k = \cup_{i=1}^\infty (A_{ik} \cup B_{ik})$. Then the S_k are mutually disjoint first category sets, and $1 = m(D) = \Sigma_k m(S_k)$. It remains only to show that if J is any subinterval of I, $m(S_k \cap J) > 0$ for each k. There is an $I_n \subset J$, and hence

$$S_k \cap J \supset S_k \cap I_n \supset A_{nk}.$$

So $m(S_k \cap J) \geqslant m(A_{nk}) > 0$.

Next we tackle category.

LEMMA 2. *The unit interval I can be partitioned into countably infinitely many mutually disjoint second category sets W_n such that for any subinterval J, $W_n \cap J$ is a second category set for all n.*

Proof. Regard R as an additive vector space over the field of rational numbers Q. Let $\{x_c\}$ be a basis of this vector space where $x_{c_0} = 1$. Let V be the subspace of R generated by the basis vectors $\{x_c\}_{c \neq c_0}$. Now let (a, b) be any open interval.

I claim that $(a, b) \cap (r + V)$ is a second category set for each r. Suppose not. Put $E = (a - r, b - r)$. Then $E \cap V$ is a first category set, and since $qV = V$ for $q \in Q$, $q \neq 0$, so is

$$q \cdot (E \cap V) = (qE) \cap (qV) = V \cap (qE).$$

Hence the union of the sets $q \cdot (E \cap V)$, over all $q \in Q$, is a first category set, but this union is V. Likewise all the translates $s + V$ of V are first category sets. But $R = \cup_{s \in Q}(s + V)$ is a

second category set, which is a contradiction. The sets $W_r = I \cap (r + V)$ $(r \in Q)$ suffice.

Unfortunately W_n is not measurable.

Proof of Theorem 1. Let S_k be as in Lemma 1 and W_n be as in Lemma 2. Put

$$T_k = (W_k \cup S_k) \setminus (\cup_{j \neq k} S_j) \quad (k = 1,2,3,\ldots).$$

Then the T_k are mutually disjoint because the S_k are. For any k and any subinterval J of I, $T_k \cap J$ is a second category set because $W_k \cap J$ is a second category set and $\cup_{j \neq k} S_j$ is a first category set. Now

$$T_k \setminus S_k \subset I \setminus \cup_j S_j$$

and

$$m(I \setminus \cup_j S_j) = 1 - \sum_j m(S_j) = 0.$$

Thus $T_k \setminus S_k$ is a subset of a set of measure 0. Hence $T_k \setminus S_k$ is measurable and $m(T_k \setminus S_k) = 0$. By construction $T_k \cap S_k = S_k$, so

$$T_k = (T_k \cap S_k) \cup (T_k \setminus S_k)$$

is a measurable set. Moreover $S_k \cap J \subset T_k \cap J$ and

$$m(T_k \cap J) \geqslant m(S_k \cap J) > 0.$$

Finally, if the set of measure 0, $I \setminus \cup_k T_k$, is nonvoid, just adjoin it to T_1. Thus $\{T_k\}_k$ is the desired partition of I.

It follows that if U is an open set meeting I, then $T_k \cap U$ is a second category set with positive measure for each k. We mention that it is impossible to partition I into uncountably many mutually disjoint measurable sets, each with positive measure. So we have as many T_k as possible.

In conclusion we show that none of the sets T_k in Theorem 1 can be a Borel set. Suppose T_1 were a Borel set. By [1], there exist an open set U and first category sets E_1 and E_2 such that

$$T_1 = (U \setminus E_1) \cup E_2.$$

Since T_1 is a second category set, U is nonvoid. Let J be an interval with $J \subset U$. Then $J \setminus T_1 \subset E_1$, and for $k \neq 1$, $T_k \cap J \subset E_1$ and $T_k \cap J$ is a first category set. But this is impossible because it conflicts with Theorem 1.

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GEOMETRY OF BINOMIAL COEFFICIENTS

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This note describes the geometrical pattern of zeroes and ones obtained by reducing modulo two each element of Pascal's triangle formed from binomial coefficients. When an infinite number of rows of Pascal's triangle are included, the limiting pattern is found to be "self-similar," and is characterized by a "fractal dimension" $\log_2 3$. Analysis of the pattern provides a simple derivation of the result that the number of even binomial coefficients in the nth row of Pascal's triangle is $2^{\#_1(n)}$, where $\#_1(n)$ is a function which gives the number of occurrences of the digit 1 in the binary representation of the integer n.

Pascal's triangle modulo two appears in the analysis of the structures generated by the

evolution of a class of systems known as "cellular automata." (See [1], [2], [3] for further details and references.) These systems have been investigated as simple mathematical models for natural processes (such as snowflake growth) which exhibit the phenomenon of "self organization." The self-similarity of the patterns discussed below leads to self-similarity in the natural structures generated.

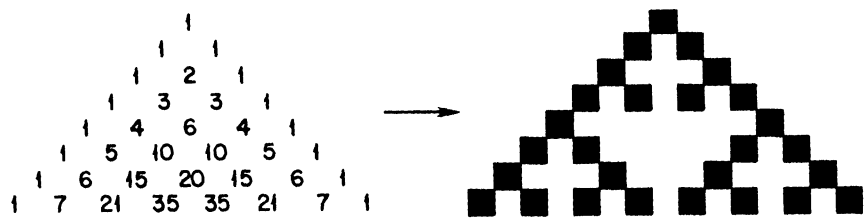


FIG. 1. The first few lines of Pascal's triangle modulo two.

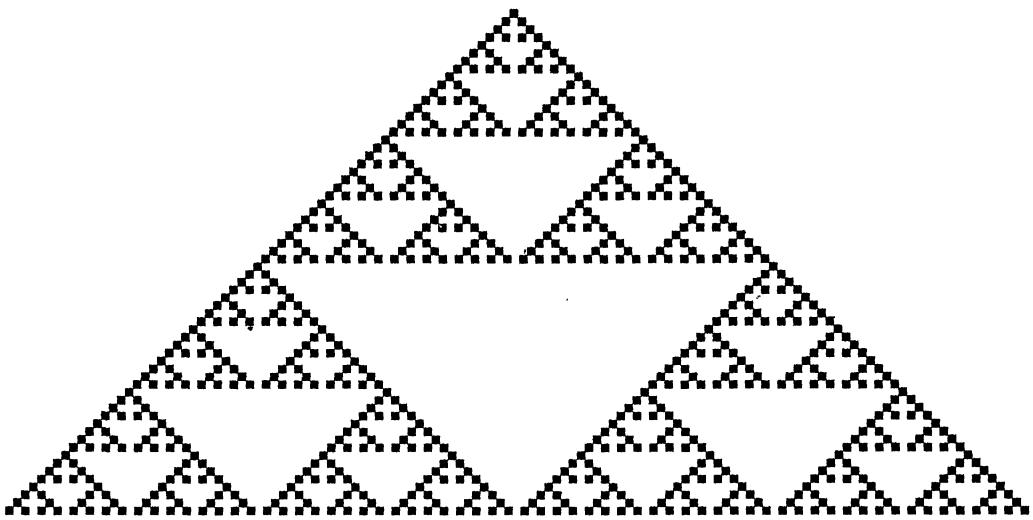


FIG. 2. The first sixty-four lines of Pascal's triangle modulo two (black squares indicate ones, white squares indicate zeroes).

Fig. 1 shows the first few rows of Pascal's triangle, together with the figure obtained by reducing each element modulo two, and indicating ones by black squares and zeroes by white (blank) squares. Fig. 2 gives sixty-four rows of Pascal's triangle reduced modulo two. A regular pattern of inverted triangles with various sizes differing by powers of two is clear. Large inverted triangles spanning the whole of Pascal's triangle begin at rows $n = 2^j$. Consider the pattern down to the beginning of one such large inverted triangle (say down to the sixty-third row). A striking feature of the pattern is that the largest upright triangle contains three smaller triangles whose contents are similar (except at the scale of very small triangles) to those of the largest triangle, but reduced in size by a factor of two. Inspection of each of these three smaller triangles reveals that each is built from three still smaller similar triangles. This "self similarity" continues down to the smallest triangles. At each stage, one upright triangle from the pattern could be magnified by one or more factors of two to obtain essentially the complete pattern. The pattern obtained differs from the original complete pattern at the scale of very small triangles. If, however, Pascal's triangle were extended to an infinite number of rows, then for all finite triangles this effect would disappear, and the original and magnified patterns would be identical. In fact, triangles of any size could be reproduced by taking smaller triangles and then magnifying them. The limiting pattern

obtained from Pascal's triangle modulo two is thus "self similar" or "scale invariant," and may be considered to exhibit the same structure at all length scales. Many examples of other "self similar" figures are given in [4], [5].

If the number of inverted triangles with base length i is denoted T_i, then Fig. 2 indicates that $T_{i/2} = 3T_i$. For large i, therefore

$$T_i \sim i^{-\log_2 3}. \tag{1}$$

The exponent $\log_2 3 \simeq 1.59$ appearing here gives the "fractal dimensionality" [4], [5] of the self-similar pattern.

Consider a ("filled in") square. Reduce the square by a factor of two in each of its linear dimensions. Four copies of the resulting reduced square are then required to cover the original square. Alternatively, one may write that the number of squares S_i with side length i contained in the original square satisfies $S_{i/2} = 4S_i$, so that $S_i \sim i^{-2}$. The exponent two here gives the usual dimensionality of the square. One may then by analogy identify the exponent $\simeq 1.59$ in Equation (1) as the generalized or "fractal" dimension of the figure formed from Pascal's triangle modulo two.

Fig. 2 suggests that the number $N(n)$ of ones in the nth row of Pascal's triangle modulo two (or, equivalently, the number of odd binomial coefficients of the form $\binom{n}{i}$ is a highly irregular function of n. However, when n is of the form 2^i, the simple result $N(2^i) = 2$ is obtained. This can be considered a consequence of the algebraic relation $\binom{p^j}{i} = 0 \bmod p$ for $0 < i < p^j$ and all primes p, which may be proved by considering the base p representations of factorials. Algebraic methods [6]–[12] have been used to obtained the general result

$$N(n) = 2^{\#_1(n)}. \tag{2}$$

The function $\#_1(n)$ gives the number of occurrences of the digit 1 in the binary representation of the integer n. Hence, for example, $\#_1(1) = 1$, $\#_1(2) = \#_1(10_2) = 1$, $\#_1(3) = \#_1(11_2) = 2$, $\#_1(4) = 1$, and so on. A graph of $\#_1(n)$ for n up to 128 is given in Fig. 3. Note that although the function is defined only for integer n, values at successive integers have been joined by straight lines on the graph. For $n > 0$, $1 \leqslant \#_1(n) \leqslant \lceil \log_2 n \rceil$. The lower bound is reached when n is of the form 2^j; the upper one when $n = 2^j - 1$. Clearly $\#_1(2^j n) = \#_1(n)$ (since multiplication by 2^j simply appends zeroes, not affecting the number of 1 digits), and for $n < 2^j$, $\#_1(n + 2^j) = \#_1(n) + 1$ (since the addition of 2^j in this case prepends a single 1, without affecting the remaining digits).

The result (2) for $N(n)$ may be obtained by consideration of the geometrical pattern of Fig. 2, continued for $2^{\lceil \log_2 n \rceil}$ rows, so as to include the complete upright triangle containing the nth row. By construction, the nth row corresponds to a line which crosses the lower half of the largest upright triangle. Each successive digit in the binary decomposition of n determines whether the line crosses the upper (0) or lower (1) halves of successively smaller upright triangles. The upper halves always contain one upright triangle smaller by a factor two; the lower halves contain two such smaller triangles. The total number of triangles crossed by the line corresponding to the nth row is thus multiplied by a factor of two each time the lower half is chosen. The total number of ones in the nth row is therefore a product of the factors of two associated with each 1 digit in the binary representation of n, as given by Equation (2).

There are several possible extensions and generalizations of the results discussed above.

One may consider Pascal's triangle reduced modulo some arbitrary integer k. Fig. 4 shows the resulting patterns for a few values of k. In all cases, a self-similar pattern is obtained when sufficiently many rows are included. For k prime, a very regular pattern is found, with fractal dimension

$$D_k = \log_k \sum_{i=1}^{k} i = 1 + \log_k\left(\frac{k+1}{2}\right),$$

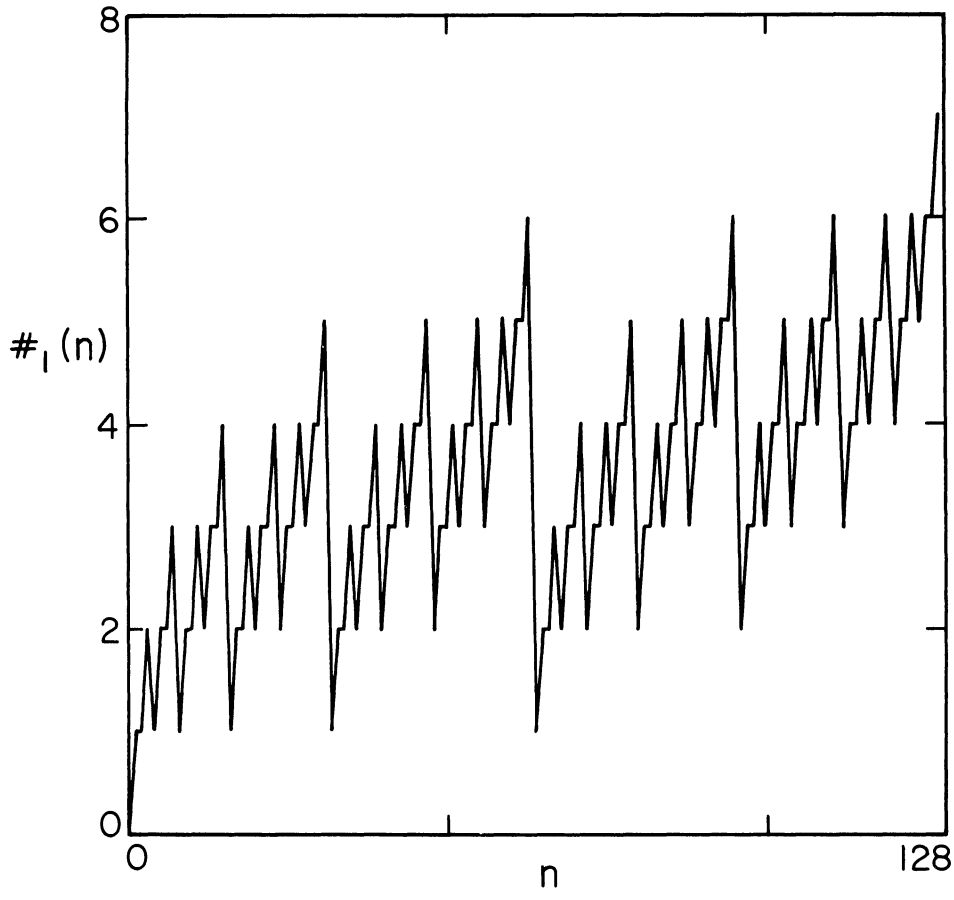


FIG. 3. The number of ones in the binary representation of the integer n.

so that $D_3 = 1 + \log_3 2 \simeq 1.631$, $D_5 \simeq 1.683$, and so on. In general, for large k, one finds that $D_k \sim 2 - 1/\log_2 k$; when $k \to \infty$, the elements of Pascal's triangle modulo k become ordinary integers, which are all nonzero by virtue of the nonzero values of binomial coefficients. By a simple generalization of Equation (2), the number of entries with value r in the nth row of Pascal's triangle modulo k is found to be $N^{(r)}(n) = 2^{\#_r^{[k]}(n)}$, where now $\#_r^{[k]}(n)$ gives the number of occurrences of the digit r in the base-k representation of the integer n.

One may also consider the generalization of Pascal's triangle to a three-dimensional pyramid of trinomial coefficients. Successive rows in the triangle are generalized to planes in the pyramid, with each plane carrying a square grid of integers. The apex of the pyramid is formed from a single 1. In each successive plane, the integer at each grid point is the sum of the integers at the four neighbouring grid points in the preceding plane. When the integers in the resulting three-dimensional array are reduced modulo k, a self-similar pattern is again obtained. With $k = 2$, the fractal dimension of the pattern is $\log_2 5 \simeq 2.32$. In general, the pattern obtained from the d-dimensional generalization of Pascal's triangle, reduced modulo two, has fractal dimension $\log_2(2d + 1)$.

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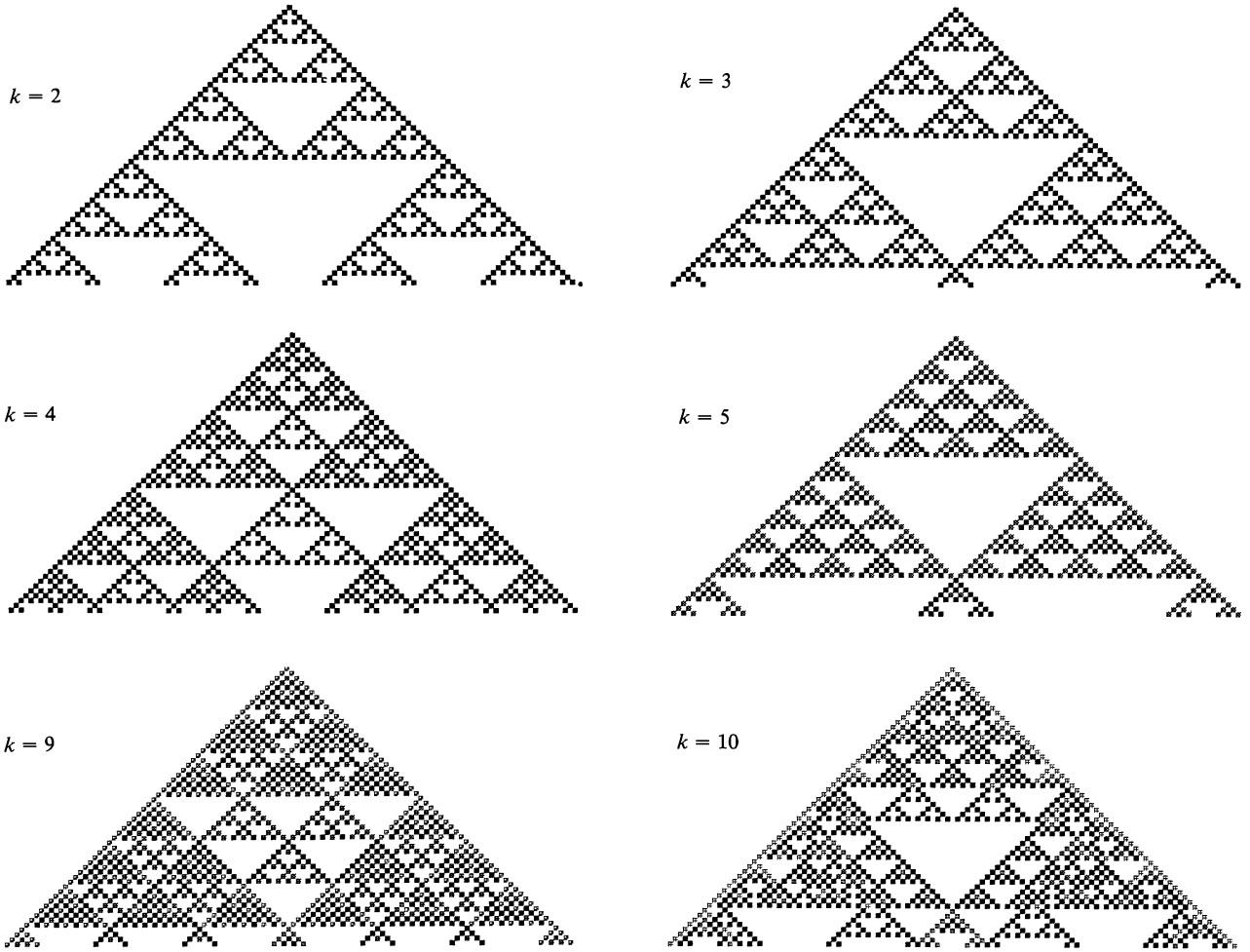


FIG. 4. Patterns obtained by reducing Pascal's triangle modulo k for several values of k. White squares indicate zeroes; progressively blacker squares indicate increasing values, up to $k - 1$.

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A REMARK ON THE NUMBER OF CYCLIC SUBGROUPS OF A FINITE GROUP

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It is common enough in books on elementary group theory to prove that if G is a finite cyclic group, then, for each divisor d of the order of G, there is exactly one subgroup of order d, and that this subgroup is cyclic. Yet it seems that it has not been noted that the number of cyclic subgroups, and therefore the number of subgroups, is minimized precisely in the case of the cyclic group. We prove therefore this theorem.

THEOREM. *Let G be a group of order n. The number of cyclic subgroups of G is greater than or equal to $d(n)$, the number of divisors of n. Furthermore, the number of cyclic subgroups of G is equal to $d(n)$ if and only if G is cyclic.*

Proof. Let G be a group of order n. We consider an action of U_n, the group of invertible residues modulo n, on G. With each $s \in U_n$ we associate the permutation of G, ψ_s, defined by $\psi_s(g) = g^s$ for all $g \in G$. Under this action two elements belong to the same orbit if and only if they generate the same cyclic subgroup. Thus the number of orbits, which we shall denote by $c(G)$, is equal to the number of cyclic subgroups of G. By Burnside's Lemma [4], we have the relation:

$$c(G) = \frac{1}{\phi(n)} \sum_{s \in U_n} |F(s)|. \tag{1}$$

Here $\phi(n)$ is the Euler totient function, while $F(s)$ is the fixed set of ψ_s, that is, the set of elements in G which satisfy the equation $x^{s-1} = 1$. By a theorem of Frobenius [2], or see [1], $|F(s)| = k_s \cdot (s-1, n)$, where k_s is a positive integer. Formula 1 therefore becomes

$$c(G) = \frac{1}{\phi(n)} \sum_{s \in U_n} k_s \cdot (s-1, n). \tag{2}$$

We first specialize Formula 2 to the case where G is the cyclic group of order n. Then $c(G)$ is equal to $d(n)$ and, for each $s \in U_n$, we have $k_s = 1$. We conclude that

$$d(n) = \frac{1}{\phi(n)} \sum_{s \in U_n} (s-1, n). \tag{3}$$

Returning to the general case of Formula 2, we recall that each constant k_s is greater than or equal to 1, so that

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Telegraphic Reviews

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General, S, P, L*. Kahn On Codes: Secrets of the New Cryptology. David Kahn. Macmillan Pub, 1983, viii + 343 pp, $19.95. [ISBN: 0-02-560640-9] A collection of essays on codes (including four book reviews) on topics ranging from the historical (origins of codes, World War II codebreaking) to the contemporary (public key cryptography, privacy vs. government intrusion). Although the essays are reprinted from a wide variety of sources (e.g., Foreign Affairs, Playboy, The New York Times) and span several decades of writing, they form a remarkably coherent portrait of codes and cryptography. LAS

General, S*, P*, L.** Mathematics: People, Problems, Results. Ed: Douglas M. Campbell, John C. Higgins. Wadsworth, 1984, $27.95 (P); $39.95 set [ISBN: 0-534-22884-5; 0-534-02879-9]. Volume I, xvi + 304 pp; Volume II, iv + 275 pp; Volume III, iv + 292 pp. A splendid collection of 90 expository articles, mostly by distinguished twentieth-century mathematicians, dealing with all aspects of mathematics: history, biography, famous problems, philosophy, computers, education, statistics, and more. All selections are reprints, and none contain more than incidental impediments (mathematical formulas) to the serious lay reader. A valuable sequel to Newman and Newman's classic The World of Mathematics, marred only (but significantly) by lack of an index. LAS

General, S*, P*, L*. Turing's Man: Western Culture in the Computer Age. J. David Bolter. U of North Carolina Pr, 1984, xii + 264 pp, $19.95; $8.95 (P). [ISBN: 0-8078-1564-0; 0-8078-4108-0] An innovative, refreshing interpretation of the meaning of computers in our age, as giving us a new definition of nature as information and of man as "information processor." Bolter suggests that in many respects, Turing's man is akin to Athenian man--limited by a finite mathematics, creating private universes like the craftsman potter in Plato's Timaeus. An imaginative, stimulating analysis of the philosophical impact of "embodied mathematics," as Bolter calls computers. LAS

General, T(13), S. Introduction to Mathematics, Fifth Edition. Bruce E. Meserve, Max A. Sobel. Prentice-Hall, 1984, xiii + 509 pp, $25.95. [ISBN: 0-13-487348-3] This Fifth Edition (Second Edition, TR, November 1969; Third Edition, TR, December 1973; Fourth Edition, TR, August-September 1978) includes new chapters on problem solving and mathematics for consumers. JG

General, P.** Renewing U.S. Mathematics: Critical Resource for the Future. National Research Council (Board on Mathematical Sciences, 2101 Constitution Ave., Washington, DC 20418), 1984, xiv + 207 pp, (P). The report of the "David Committee," which has been widely cited in the mathematical community, documenting a significant deterioration in federal support for mathematics research. Especially notable is Arthur Jaffe's 50-page appendix "Ordering the Universe: The Role of Mathematics," setting forth in sophisticated yet non-mathematical terms the amazing scope of contemporary mathematical applications. LAS

General, S, L. Frontiers in Science and Technology: A Selected Outlook. COSEPUP: National Academy of Sciences. WH Freeman, 1983, xii + 227 pp, $16.95 (P). [ISBN: 0-7167-1517-1] Third in a series of NSF-commissioned reports providing five year outlooks in seminal areas of science and technology. This volume includes reports on genetics, lasers, robotics, turbulence, et al. (Earlier reports were published in 1979 and 1982.) LAS

Elementary, T(13: 1). Trigonometry for College Students, Third Edition. Karl J. Smith. Brooks/Cole, 1984, xiii + 316 pp, $24.95. [ISBN: 0-534-02688-5] New edition of a text first published in 1977 (First Edition, TR, June-July 1977; Second Edition, TR, October 1980). JS

Elementary, T(13: 1). College Algebra, Third Edition. Raymond A. Barnett. McGraw-Hill, 1984, xv + 557 pp, $23.95. [ISBN: 0-07-003861-9] Changes from the Second Edition (First Edition, TR, December 1974; Second Edition, TR, December 1979) include: rewritten chapter on graphs and functions, new chapter on review of algebraic operations, chapter review sections. JRG

Elementary, T(13: 1). Trigonometry with Applications. Patrick J. Boyle. Harper & Row, 1983, xii +

372 pp, $22.95. [ISBN: 0-06-040898-7] Triangle trigonometry, concluding with complex numbers and an appendix on logarithms and exponential functions. "At least as much attention is given to using a calculator as was formerly given to calculation with logarithms." LAS

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Education, P. Teacher Shortage in Science and Mathematics: Myths, Realities, and Research. Ed: John L. Taylor. NIE (Teaching and Instruction Division, Mail Stop 5, 1200 19th St., NW, Washington, DC 20208), 1984, iv + 281 pp, (P); Summary, 1983, vi + 53 pp, (P). Report with separately-published Summary of a February 1983 conference sponsored by the National Institute of Education. Papers (followed by transcribed discussions) cover supply and demand data, current conditions in science and mathematics education, and various responses. An excellent current source of facts and proposals. LAS

History, P, L*. The Analytic Art. François Viete. Transl: T. Richard Witmer. Kent St U Pr, 1983, 450 pp, $45. [ISBN: 0-87338-282-X] Viete lived in the 16th century, between Cardano and Descartes; his seminal work In Artem Analyticem Isagox introduced letters of the alphabet as symbolic variables--he called them "species"--in equations, thereby freeing algebra from cumbersome reliance on geometric proofs in the Euclidean tradition. This volume contains the first complete English translation of all Viete's works in algebra, plus a few others. LAS

History, S*, L**. Fire in the Valley: The Making of the Personal Computer. Paul Freiberger, Michael Swaine. Osborne/McGraw-Hill, 1984, xv + 288 pp, $9.95 (P). [ISBN: 0-88134-121-5] A substantive, sprightly history of personal computers, from the first Altair kit (1975; 256 bytes) to the Macintosh (1984; 128K bytes), based on interviews with the founders (e.g., Jobs and Wozniak of Apple, Kildall of Digital Research, and Gates of Microsoft). Full of fascinating details (why is it called "apple?"), almost unbelievable anecdotes, and archival photographs. LAS

History, L. A Convergence of Lives, Sofia Kovalevskaia: Scientist, Writer, Revolutionary. Ann Hibner Koblitz. Birkhauser Boston, 1983, xx + 305 pp, $19.95. [ISBN: 0-8176-3162-3] Highly readable, popular account of Kovalevskaia's life, with emphasis on scientific and political interests. Extensive bibliography. JRG

History, S*(16-18), P, L*. From Error-Correcting Codes Through Sphere Packings to Simple Groups. Thomas M. Thompson. Carus Math. Mono., No. 21. MAA, 1983, xiv + 228 pp, $21. [ISBN: 0-88385-023-0] Filled with fascinating mathematical and historical detail, this monograph traces the serendipitous events leading from Hamming and Golay's discovery of error-correcting codes in 1948 through Conway's explication of the related group ".0" two decades later. Thompson supplements his detailed exposition with frequent quotations from interviews with the principals, thereby blending personal and logical motivation in a compelling tale of mathematical discovery. LAS

Foundations, T*(15-16: 1), S, L*. Numbers, Sets and Axioms: The Apparatus of Mathematics. A.G. Hamilton. Cambridge U Pr, 1982, ix + 255 pp, $42.50; $17.95 (P). [ISBN: 0-521-24509-5; 0-521-28761-8] Excellent text for course in foundations of mathematics. Covers number systems, axiomatic set theory (including chapter on axiom of choice), infinite sets, ordinal and cardinal numbers. Very readable; comments on history of subject and philosophical issues. Good exercises with hints for many at end of book. KS

Foundations, S(15-16), L*. Elements of Logic and Foundations of Mathematics in Problems. Wiktor Marek, Janusz Onyszkiewicz. D Reidel Pub, 1982, viii + 276 pp, $39.50. [ISBN: 90-277-1084-8] Problem book covering topics in undergraduate courses in set theory and mathematical logic. Each chapter includes summary of basic definitions and theorems followed by problems arranged according to difficulty. Complete solutions to representative problems and hints or short answers for all others. Translation of 1972 Polish edition. KS

Foundations, T(17-18: 1,, 2), L. Computability. George J. Tourlakis. Reston Pub, 1984, x + 566 pp, $27.95. [ISBN: 0-8359-0876-3] Thorough, rigorous introductory text on recursion theory. Emphasizes recursive functions but covers machines, Markov algorithms, Post systems. Includes nondeterminism and complexity. Extensive examples and exercises. No mathematical prerequisites but high level of mathematical maturity essential. KS

Graph Theory, T(13-15: 1), S, L. Map Coloring, Polyhedra, and the Four-Color Problem. David Barnette. Dolciani Math. Expos., No. 8. MAA, 1983, x + 168 pp, $26. [ISBN: 0-88385-309-4] An informative yet relaxed exploration of those parts of graph theory that blossomed during work on the Four Color Conjecture: Euler's equation, Hamiltonian circuits, duality, reductions, etc. Emphasizes his-

tory and examples rather than theorems and proofs. Each brief chapter concludes with exercises (mostly requests for proofs or counterexamples), followed immediately by a section giving solutions. LAS

Graph Theory, T(17-18: 1). Graphes, Third Edition. Claude Berge. Gauthier-Villars, 1983, ix + 400 pp, 180 FF (P). [ISBN: 2-04-15555-4] Third edition of first part of Graphs and Hypergraphs (TR, January 1974). Includes recent results and new proofs of old theorems developed since the original edition appeared in 1970. Contains French-English-German lexicon of graph theory terms. KS

Combinatorics, T(17-18: 1), P. Symmetric Designs: An Algebraic Approach. Eric S. Lander. London Math. Soc. Lect. Note Ser., No. 74. Cambridge U Pr, 1983, xii + 306 pp, $29.95 (P). [ISBN: 0-521-28693-X] Monograph demonstrates use of algebraic coding theory and group representation theory in study of designs. Emphasizes difference sets and subsets of finite groups that give rise to symmetric designs. Extensive problem sets. Six appendices present algebraic results used in book. KS

Number Theory, T(15). Number Theory in the Quadratic Field with Golden Section Unit. Fred Wayne Dodd. Polygonal Pub, 1983, 155 pp, $16.95. [ISBN: 0-936428-08-2] Elementary number theory done in the ring $R = Z[(1+\sqrt{5})/2]$, although algebra à la Herstein and basic arithmetic properties of the integers are assumed. Topics such as primitive roots and quadratic residues in R are covered. Applications to Z are given. No exercises. SG

Number Theory, S(18), P. Upper Bounds for the Numbers of Solutions of Diophantine Equations. J.H. Evertse. Math. Centre Tracts No. 168. Math Centrum, 1983, v + 125 pp, Dfl. 16,50 (P). [ISBN: 90-6196-265-X] Upper bounds for the numbers of solutions of diophantine equations from several classes are derived. A version of the author's Ph.D. thesis. CEC

Linear Algebra, S(14), L. Linear Algebra, Problems Book. H.D. Ikramov. Transl: Oleg Efimov. MIR Pub, 1983, 327 pp. A wealth of problems organized to accompany the linear algebra text by Voyevodin, a text with emphasis on computational aspects of linear algebra written for a course in Moscow University's Computational Mathematics and Cybernetics Department. Great source of problems with answers. AWR

Linear Algebra, T(16-17: 1). Estructuras Algebraicas Tensoriales. F. Gonzalez de Posada. Editorial Alhambra, 1983, xii + 170 pp, (P). [ISBN: 84-205-0941-8] Third volume of planned eight volume work on vector and tensor algebra and analysis. Covers tensor products in general vector spaces and Euclidean spaces. Exercises at ends of chapters; complete solutions in fourth volume of series. KS

Algebra, P. Combinatorics and Commutative Algebra. Richard P. Stanley. Progress in Math., V. 41. Birkhauser Boston, 1983, viii + 88 pp, $12.95. [ISBN: 0-8176-3112-7] Lecture notes surveying solutions to linear equations in non-negative integers, and the face ring of a simplicial complex. Introductory chapter provides necessary background in algebra, combinatorics, and topology. JRG

Calculus, T**(13-14: 3, 4). Calculus with Analytic Geometry, Second Edition. Howard Anton. Wiley, 1984, xxiv + 1238 pp, $37.95. [ISBN: 0-471-08271-6] Parts rewritten and reorganized. Some new material. Changes made to improve exposition. Exercise sets modified to provide more flexibility. Still long. Still readable. Stunning computer graphics remain. (First Edition, TR, April 1981.) JK

Calculus, T(13-14: 2, 3). Calculus with Analytic Geometry, Third Edition. Earl W. Swokowski. Prindle, Weber & Schmidt, 1984, xii + 996 pp. [ISBN: 0-87150-443-X] Major changes from the Second Edition (First Edition, TR, December 1975; Second Edition, TR, June-July 1979): a delay in the treatment of analytic geometry, to make possible earlier coverage of exponential functions and logarithms (with associated separable differential equations); extensive rewriting of the chapter on multivariable calculus to emphasize role of level curves; and many new examples and exercises. LAS

Real Analysis, P. Lecture Notes in Mathematics-1033: Measure Theory and its Applications. Ed: J.M. Belley, J. Dubois, P. Morales. Springer-Verlag, 1983, xv + 317 pp, $14.60 (P). [ISBN: 0-387-12703-8] Consists of papers presented at a workshop held at the Université de Sherbrooke in June 1982. Topics include ergodic theory, Choquet representation theory, vector measures, measure theory, and topology. BH

Complex Analysis, S(18), P. Quasikonforme Abbildungen--neue Methoden und Anwendungen. Samuil L. Kruschkal, Reiner Kühnau. Teubner-Texte zur Math., B. 54. BG Teubner, 1983, 172 pp, 17,50M (P). On some recent results concerning quasiconformal mappings and their applications. An extensive bibliography. JD-B

Differential Equations, T*(14-15: 1). Differential Equations, A First Course. Martin M. Guterman, Zbigniew H. Nitecki. Saunders Coll Pub, 1984, xv + 636 pp. [ISBN: 0-03-062502-5] Standard first-course fare plus a chapter on partial differential equations and Fourier series. Student-oriented. Well-written. Motivation provided throughout. Especially nice self-contained treatment of systems. JK

Differential Equations, P*. Introduction to the Theory of Nonlinear Elliptic Equations. Jindřich Nečas. Teubner-Texte zur Math., B. 52. BG Teubner, 1983, 204 pp, 19M (P). Lecture notes. Continuation of the author's book Les méthodes directes en théorie des équations elliptiques, 1967. Boundary value problems for nonlinear, second order, elliptic partial differential equations. Short introduction to Sobolev and Morrey-Campanato spaces. Some non-standard applications to elasticity.

Examples. Scattered exercises. References, many to recent work. JK

Differential Equations, T(15: 1). Introduction to Ordinary Differential Equations. Roger C. McCann. HarBrace J, 1982, xi + 451 pp, $24.95. [ISBN: 0-15-543485-3] Standard topics in standard order, richly supported by dozens of applications, many rather innovative. Integrates numerical techniques throughout. An appealing, straightforward presentation. LAS

Differential Equations, T(17-18: 1), S, P. Partielle Differentialgleichungen. Wolfgang Tutschke. Teubner-Texte zur Math., B. 27. BG Teubner, 1983, 192 pp, 19M (P). A brief and difficult introduction to the theory of partial differential equations, treating classical, complex and function-theoretic methods. JD-B

Numerical Analysis, P. Multiple Grid Methods for Equations of the Second Kind with Applications in Fluid Mechanics. H. Schippers. Math Centre Tracts, No. 163. Math Centrum, 1983, iii + 133 pp, Dfl. 17,60 (P). [ISBN: 90-6196-260-9] This monograph, based on the author's doctoral thesis, presents multiple grid methods for solving the algebraic systems that arise in numerical methods for the solution of Fredholm equations of the second kind. AO

Functional Analysis, P. Banach Bundles, Banach Modules and Automorphisms of C*-algebras. M.J. Dupré, R.M. Gillette. Research Notes in Math., No. 92. Pitman Pub, 1983, 111 pp, $15.95 (P). [ISBN: 0-273-08626-X] First half introduces Banach bundles and reviews representation theory of C*-algebras. Second half applies this theory to classify certain automorphisms of C*-algebras. BH

Analysis, S(16). Differential Equations, Multiple Integrals, Series, Theory of Functions of a Complex Variable. Ya. S. Bugrov, S.M. Nikolsky. Transl: Leonid Levant. Higher Math. MIR Pub, 1983, 475 pp. Translated from Russian and written for engineers. Oriented towards physical applications. Rather dry with few problems. Awkward grammar in places. BH

Analysis, P. Modules Differentiels et Equations Differentielles p-Adiques. Christol Gilles. Papers in Pure & Appl. Math., No. 66. Queen's U, 1983, vi + 218 pp, (P). An introduction to non-Archimedean analysis (including analytic functions) together with a study of p-adic differential equations. SG

Analysis, P. Lecture Notes in Mathematics-1034: Orlicz Spaces and Modular Spaces. Julian Musielak. Springer-Verlag, 1983, 222 pp, $11 (P). [ISBN: 0-387-12706-2] An enlarged version of Modular Spaces published in 1978 by Mickiewicz University. Contains an additional chapter on generalized Orlicz spaces. BH

Algebraic Geometry, P. Théorèmes de Bertini et Applications. Jean-Pierre Jouanolou. Progress in Math., V. 42. Birkhauser Boston, 1983, 127 pp, $12.95. [ISBN: 0-8176-3164-X] Lengthy first part devoted to proofs of the theorems of Bertini with application to the connectedness results of Fulton and Hansen; second part discusses various general theorems on the structure of projective modules, including Serre's theorem, and theorems of Bass and Suslin. JRG

Algebraic Geometry, P. Complex Algebraic Surfaces. Arnaud Beauville. London Math. Soc. Lect. Note Ser., No. 68. Cambridge U Pr, 1983, ix + 132 pp, $19.95 (P). [ISBN: 0-521-28815-0] Based on a 1976-77 course at Orsay. Goal is proof of Enriques' classification theorem for complex algebraic surfaces. Assumes knowledge of sheaf theory, basic constructs and some hard theorems of algebraic geometry. First two chapters are elementary. Appendix treats classification of surfaces in characteristic p case. PZ

Differential Geometry, S(18), P. Riemannian Geometry. Wilhelm Klingenberg. Stud. in Math., No. 1. Walter de Gruyter, 1982, x + 396 pp, $44.50. [ISBN: 3-11-008673-5] Selected topics, including classical local Riemannian geometry, curvature and topology, and structure of the geodesic flow. Manifolds are modelled on separable Hilbert space. JG

Geometry, T(17-18: 2), S, P. Metamathematische Methoden in der Geometrie. W. Schwabhäuser, W. Szmielew, A. Tarski. Hochschultext. Springer-Verlag, 1983, viii + 482 pp, $26.80 (P). [ISBN: 0-387-12958-8] A careful development of Euclidean geometry from a set of axioms due to Tarski, followed by an extensive treatment of metamathematical questions such as consistency and completeness for Euclidean and other geometries. JD-B

Topology, S(18), P, L. The Geometric Topology of 3-Manifolds. R.H. Bing. AMS Colloq. Pub., V. 40. AMS, 1983, x + 238 pp, $54. [ISBN: 0-8218-1040-5] An elaboration of Bing's 1970 Colloquium Lectures, reflecting his efforts "to bridge the gap between the study of topological objects in 3-space and the corresponding polyhedral objects." Builds from the specific to the general, from the classical Schoenflies theorem to the important side approximation theorem. LAS

Probability, P. Spectral Theory of Random Fields. M.I. Yadrenko. Transl. Ser. in Math. & Eng. Springer-Verlag, 1983, iii + 259 pp, $24 (P). [ISBN: 0-387-90823-4] Presents a unified treatment of the current state of knowledge in the area described by the title of the monograph. AO

Statistics, P. Studies in Econometrics, Time Series, and Multivariate Statistics. Ed: Samuel Karlin, Takeshi Amemiya, Leo A. Goodman. Academic Pr, 1983, xx + 570 pp, $40. [ISBN: 0-12-398750-4] This book is published in commemoration of the 65th birthday of Theodore W. Anderson. It is a collection of papers in the areas of econometrics, time series processes, and multivariate statistical analysis, all areas where Anderson has made significant contributions. MT

Statistics, S*(14-18), P, L*. Encyclopedia of Statistical Sciences. Samuel Kotz, Norman L. Johnson. Wiley, 1983, $79.50 each. Volume 3: Faàdi Bruno´s Formula--Hypothesis Testing, ix + 722 pp [ISBN: 0-471-05549-2]; Volume 4: Icing the Tails to Limit Theorems, ix + 657 pp. [ISBN: 0-471-05551-4] Third and fourth volumes of a planned eight-volume set. (See TR, October 1982, of Series and Volume 1.) RSK

Statistics, S(15). Understanding Randomness: Exercises for Statisticians. David Salsburg. Lecture Notes in Stat., V. 6. Dekker, 1983, vii + 105 pp, $17.75 (P). [ISBN: 0-8247-7057-9] Essentially a set of exercises designed to illustrate the effects of various residual structures on data and statistical inference. Salsburg hopes that his book will help students and practicing statisticians develop a feel for the data they work with. MT

Computer Literacy, S, L*. Electronic Life: How to Think About Computers. Michael Crichton. Alfred A Knopf, 1983, ix + 209 pp, $12.95. [ISBN: 0-394-53406-9] A delightfully opinionated series of brief demythologizing comments, from A (Afraid of Computers) to Z (Zenith: The Final Days of Man) conveying an attitude towards computers that preserves the dignity of man. Software produces "images of rubber table settings;" programming is like "talking to a cranky aunt who doesn´t hear well. You say, ´I´ve just been outside,´ and she replies ´I never told you to hide.´ That´s what programming is like." LAS

Computer Programming, S(14-15), L. Numerical BASIC. Bruce Douglass. Howard Sams, 1983, 222 pp, $12.95 (P). [ISBN: 0-672-22048-2] A cookbook presentation of numerical methods in TRS-80 BASIC, intended for those many microcomputer users who have no more mathematical training than one year of calculus. A clear, elementary presentation, with good references to more scholarly sources of the algorithms. LAS

Data Structures, T*(14-15: 1, 2), L. Data Structures and Program Design. Robert L. Kruse. Prentice-Hall, 1984, xxi + 486 pp, $28.95. [ISBN: 0-13-196253-1] An example-oriented Pascal-based presentation of the "precision tools" of computer science (e.g., linked lists, hashing, sorting, binary trees, recursion), intended for courses CS2 and CS7 from ACM Curriculum ´78. Includes several substantial case studies, some developed in the text (e.g., Conway´s game of Life, an indexing program), others suggested for group work by teams of students. LAS

Software Systems, S(16-17), L.** Computer Graphics Programming: GKS--The Graphics Standard. G. Enderle, K. Kansy, G. Pfaff. Symbolic Computation. Springer-Verlag, 1984, xvi + 542 pp, $39. [ISBN: 0-387-11525-0] A thorough introduction to the GKS (Graphical Kernel System) standard for computer graphics. A valuable source of information on the standard. Includes a section on the process that led to the standard. AO

Computer Science, T*(16-17: 1), L*. Compiler Construction. William M. Waite, Gerhard Goos. Springer-Verlag, 1984, xiv + 446 pp, $24.50. [ISBN: 0-387-90821-8] A textbook for a first course on compiler design. Emphasizes concepts and techniques of practical importance (e.g., LL and LR parsing, attribute grammars). A good textbook for an undergraduate course. AO

Computer Science, P. Data Base File Organization. Ed: Sakti P. Ghosy, Yahiko Kambayashi, Witold Lipski, Jr. Notes & Reports in Comp. Sci. & Appl. Math., No. 6. Academic Pr, 1983, x + 352 pp, $21.50. [ISBN: 0-12-281860-1] Proceedings of a conference on consecutive retrieval property (CRP) held in Warsaw, Poland, August 1981. A query set has CRP with respect to record set if there is linear arrangement of records such that for every query all records pertinent to query are stored in consecutive locations. Papers cover: decomposition of query sets, algorithms for file organization, physical constraints, applications to data base design. KS

Control Theory, T???, L?? Contemporary Principles to Control Complex Objects. L.A. Rastrigin. Transl: Michael Burov. MIR Pub, 1983, 248 pp. The author´s own philosophical control theory, shunning systems analysis and other standard theories, with interest in cybernetics. "Complex object" is left undefined: an ant is a complex object, but a ballistic missile is not. Popular science. RB

Control Theory, P. Lecture Notes in Control and Information Sciences-53: General Decoupling Theory of Multivariable Process Control Systems. Liu Chen Hui. Springer-Verlag, 1983, xi + 474 pp, $24.50 (P). [ISBN: 0-387-12496-9] Written for advanced graduate students in process control engineering. Presents a systematic description of decoupling design theory for multivariable process control systems. AO

Applications (Artificial Intelligence), P. Self-Organization and Associative Memory. Teuvo Kohonen. Ser. in Infor. Sci., V. 8. Springer-Verlag, 1984, xii + 255 pp, $29. [ISBN: 0-387-12165-X] Substantial revision and expansion of Associative Memory--A System-Theoretical Approach, by the same author. Focus on principles and mechanisms of memory and learning in adaptive physical systems. Bibliography. JRG

Applications (Communication Theory), P. Statistical Communication Theory and Its Applications. Ed: B.R. Levin. Transl: M. Edelev, A. Repyev. MIR Pub, 1982, 319 pp, $9.95. Translated from Russian, this book covers much of modern statistical communication theory and its application to telecommunication. Topics addressed include stochastic models of signals, noise and communications channels, coding and decoding, pseudonoise signals, communication networks, and computer network design. MT

Applications (Economics), P. Lecture Notes in Economics and Mathematical Systems-220: Notes on Economic Time Series Analysis: System Theoretic Perspectives. Masanao Aoki. Springer-Verlag, 1983,

ix + 249 pp, $17 (P). [ISBN: 0-387-12696-1] Monograph on analysis and modeling of vector-valued time series using results from system theory. Emphasis on Hankel matrices. Appendices cover mathematical results used in text. KS

Applications (Economics), P. Pricing Policies of Financial Intermediaries. Jean Dermine. Stud. in Contemp. Econ. Springer-Verlag, 1984, vii + 174 pp, $12.50 (P). [ISBN: 0-387-13080-2] Technical tract which mixes concepts and models from finance and economics to explain the pricing policies of financial intermediaries such as savings and loan institutions. GHM

Applications (Engineering), P. Pictorial Data Analysis. Ed: Robert M. Haralick. Computer & Systems Sci., No. 4. Springer-Verlag, 1983, viii + 468 pp, $49.70. [ISBN: 0-387-12288-5] Proceedings of NATO Advanced Study Institute held at de Bonas, France, August 1-12, 1982. First part covers theory including quadtree data structure, image feature extraction and pattern recognition methods, computer vision systems. Second part presents applications including medical, satellite and radar images. KS

Applications (Engineering), S(16-17), P, L*. Lecture Notes in Engineering-1: The Boundary Element Method Applied to Inelastic Problems. J.C.F. Telles. Springer-Verlag, 1983, ix + 243 pp, $18 (P). [ISBN: 0-387-12387 3] First in a new series of Springer Lecture Notes. Boundary element methods for solving differential equations require first a transformation to an integral equation, followed by numerical integration of polynomial functions (akin to those used in the finite element method) over certain boundary elements. This monograph provides a thorough introduction to the method and extends its applicability to certain inelastic problems. LAS

Applications (Linguistics), T(17-18: 1, 2), S, P, L. Language as a Cognitive Process, Volume I: Syntax. Terry Winograd. Addison-Wesley, 1983, xiv + 640 pp, $29.95. [ISBN: 0-201-08571-2] Introductory text for course on models of natural language for graduate students in linguistics or computer science. Covers context free, transformational, automatic transition network, and systemic grammars. Concludes with overview of computer systems for natural language parsing. Exercises and extensive references at end of each chapter. KS

Applications (Mechanics), P. Trends in Applications of Pure Mathematics to Mechanics, Volume IV. Ed: J. Brilla. Pitman Pub, 1983, 266 pp, $39.95. [ISBN: 0-273-08606-5] A collection of papers from the Fourth Symposium of Trends in Application of Pure Mathematics to Mechanics held at the Institute of Applied Mathematics and Computing Technique of Comenius University, Bratislava, August 18-21, 1981. The papers focus on nonlinear problems of continuum mechanics. AO

Applications (Physics), T(18: 1), P. Scattering Theory for Diffraction Gratings. Calvin H. Wilcox. Appl. Math. Sci., No. 46. Springer-Verlag, 1984, ix + 163 pp, $16 (P). [ISBN: 0-387-90924-9] Develops a theory of the scattering of two-dimensional acoustic and electromagnetic waves by periodic surfaces. Should be of interest to mathematicians interested in spectral analysis and scattering theory for partial differential operators. AO

Applications (Physics), P. Workshop on Non-Perturbative Quantum Chromodynamics. Ed: Kimball A. Milton, Mark A. Samuel. Progress in Physics, V. 8. Birkhauser Boston, 1983, xii + 265 pp, $24.95. [ISBN: 0-8176-3127-5] Contains the texts of twenty-three papers presented at a workshop on non-perturbative quantum chromodynamics held March 7-9, 1983 at Oklahoma State University, Stillwater. AO

Applications (Physics), P. Scaling and Self-Similarity in Physics: Renormalization in Statistical Mechanics and Dynamics. Ed: Jürg Fröhlich. Progress in Physics, V. 7. Birkhauser Boston, 1983, 426 pp, $27.50. [ISBN: 0-8176-3168-2] Twelve articles based on seminar talks given at the I.H.E.S., Bures-sur-Yvette, between spring 1981 and summer 1982. Provides a brief status report on results in rigorous statistical mechanics and dynamical systems theory that are related to the ideas of scaling and self-similarity. AO

Reviewers

RJA: Richard J. Allen, St. Olaf; MA: Melissa Anderson, St. Olaf; FA: Fahrad Anklesaria, Macalester; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JD-B: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; PH: Paul Humke, St. Olaf; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; LN: Linda Ness, Carleton; YN: Yves Nievergelt, St. Olaf; AO: Arnold Ostebee, St. Olaf; TR: Teresa Reardon, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MT: Michael Tveite, St. Olaf; ML: Milton Ulmer, Carleton; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

Section Reports

An asterisk (*) by the title of a paper indicates that copies of the paper are available from the author. Papers presented under special sponsorship as part of joint meetings are so noted in parentheses.

New Jersey Section

The New Jersey section of the Mathematics Association of America met jointly with the Mathematics Association of Two-Year Colleges of New Jersey on March 10, 1984 at Montclair State College in Upper Montclair, New Jersey. There were approximately 70 in attendance.

Invited Addresses:

"An Application of Geometry to Medicine: The Surgical Correction of Astigmatism," by Helen Strassberg, St. John's University.
"Graph Theoretical Aspects of VLSI Design," by Anni Bruss, IBM Watson Research Center.
"Mathematics in the Computer Science Curriculum," by Donald Yee, Essex County College.
"Report of the Mathematicians' Trip to China, June, 1983," by Evan B. Alderfer, Ocean County College.

Michigan Section

The annual spring meeting of the Michigan Section was held May 4-5, 1984 at the University of Michigan in Ann Arbor, Michigan. Meeting attendance was approximately 115.

Invited Addresses:

"Statistical Education for an Industrial Career," by Harvey Arnold, Oakland University.
"Using Students in Teaching Calculus," by Kenneth Cummings, Kent State University.
"Folding Regular Convex and Regular Star Polygons," by Jean Pedersen, University of Santa Clara.
"Number-theoretical Properties of Folded Polygons," by Peter Hilton, SUNY at Binghamton.
"Observations of Mathematics and Mathematicians," by Ivan Niven, President, MAA, University of Oregon.
"Problem Solving in the Classroom," by Kenneth Cummins, Kent State University.
"Some Peculiar Properties of Rationals and Irrationals," by Ivan Niven, University of Oregon.
"The Half Binomial Sum and Two-party Elections," by J. Sutherland Frame, Michigan State University.

Short Presentations:

"Computer Movies," by Tim Sipka, Alma College.
"Enumerators: Permutations, Combinations, and Markings," by Michael Gilpin, Michigan Tech University, Houghton.
"Algebraic Coding Theory in the Undergraduate Curriculum," by L.W. Brinn, University of Michigan, Dearborn.
"Mathematical Constructions are Computer Programs," by Karil Cubik, Wayne State University.
"Educating America for the 21st Century," by Karen Sharp, Mott Community College.
"The State of K-12 Mathematics and Science Today," by Wayne Scott, State Department of Education, Michigan.

Student Papers:

"Some Problems Associated with the Euler ϕ Function," by Paul Odea, Central Michigan University.
"Progress and Problems in Attempts to Solve an Open Question in Robbins Algebra with an Automated Theorem Proving Program," by Rick Stevens, Western Michigan University.
"On Related Sums and Products of Integers," by Amy Aldridge and Peter B. Young, Northern Michigan University.

Panel Discussion:

Section involvement in state legislation regarding educational issues.

Wisconsin Section

The Wisconsin Section met at St. Norbert College in De Pere, Wisconsin on April 13-14, 1984. There were 180 registrants.

Invited Addresses:

"Intuitive and Counterintuitive Mathematics," by Warren Page, Editor, The College Mathematics Journal.
"Solving Polynomial Equations," by Sylvan Burgstahler, University of Minnesota, Duluth.
"Nearness Relations Among Measures of Central Tendency and Dispersions," by Warren Page, Editor,

The College Mathematics Journal.

Short Presentations:

"A Solution to the Bandwidths Problem for the Cartesian Product on N Paths," by Hosien Moghadam, University of Wisconsin, Oshkosh.
"Reflections, Computer Graphics and Calculus," by N.L. Petrakopoulos and R. Davies, University of Wisconsin, Green Bay.
"A Mathematical Model for Predicting Baseball Standings," by Steve Krevisky, University of Wisconsin, West Bend Campus.
"Finite and Modular Ramanujan Pairs," by Richard Blecksmith, University of Wisconsin, Green Bay.
"A Few Computer Inspired Math Problems for Students," by Ro Cahill, University of Wisconsin, West Bend Campus.
"The Arithmetic of Differentiation," by Louis Rall, University of Wisconsin, Madison.
"Math Careers in Chemistry," by A.J. Hopfinger, Searle Research and Development.
"Computer Assisted Instruction in Trigonometry," by Lyle Espenscheid, University of Wisconsin, Marinette Campus.
"A Note on Partial Fractions in Calculus," by Roger Peterson, University of Wisconsin, West Bend Campus.
"Some Differences Among Difference Equations," by David Beran, University of Wisconsin, Superior.
"Three Person Games with McCarthy's Revenge Rule," by Philip Straffin, Beloit College.
"Sharp Bounds for Derivatives of Univalent Functions," by Bruce O'Neill, University of Wisconsin, Green Bay.
"Acceleration of One-Step Stationary Root-Finding Algorithms," by Don Leake, University of Wisconsin, River Falls.
"Gauss' Theorem and Harmonic Functions," by Andrew Matchett, University of Wisconsin, LaCrosse.

Panel Discussions:

"Apple Users Group."
"Undergraduate Training of Mathematicians for Careers in Industry," by Paul Campbell (Moderator), Beloit College; A.J. Hopfinger, Searle Research and Development; John Hunt, University of Wisconsin, Stout; Dan Klos, Distribution Management Services; Gil Walter, University of Wisconsin, Milwaukee.
"Resolutions Pertaining to Mathematics Teacher Shortage," by Rick Poss (Moderator), Wisconsin Section Chairman; Gail Burrill, Whitnall High School; Donald Chambers, Wisconsin Department of Public Instruction; Bruce Staal, University of Wisconsin, Stevens Point; Marshall Wick, University of Wisconsin, Eau Claire.

Student Papers:

"Alternate Ways of Evaluating $\int \csc u \, du$," by Judy Carlson, University of Wisconsin, Janesville Campus.
"Integration by Parts, A Second Route," by Hoang Luong, University of Wisconsin, Janesville Campus.
"Design and Implementation of an Electronic Mail System," by Gerald Bull and Mark Wonsil, Beloit College.
"Computer Graphics for Black and White Symmetry Patterns," by Michael Case, Beloit College.
"Summation Formulas and Bernoulli Numbers," by Randy Cox, University of Wisconsin, Green Bay.
"Twenty-five Point Geometry," by Sue Anderson, St. Norbert College.

Southwestern Section

The annual spring meeting of the Southwestern Section was held April 12-13, 1984 at Arizona State University in Tempe, Arizona. There were 20 registrants.

Invited Addresses:

"Geometrical Gems," by D. Albers, Menlo College.
"The Method of Equivalent Integrals in the Calculus of Variations," by H. Rund, University of Capetown.

Short Presentations:

"SAS-FUNCAT-BEWARY," by G. Rogers, New Mexico State University.
"Golden Arithmetization: Useful or Useless," by G. Miel, Arizona State University.
"An Application of the Baire Category Theorem," by J. McDonald, Arizona State University.
"The Converse of the Poincare Lemma--An Elementary Proof," by W. Schulz, Northern Arizona University.
"Robots, Resultants, and Representations," A. Swimmer, Arizona State University.
"Problem Solving in Community College Classes," by E. Fitzgerald, Phoenix College.
"Notes on an Undergraduate Course in Problem Solving," by C.G. Moore, Northern Arizona University.
"J.F. Herbert and the Metaphysics Which Underlies the Hypotheses Which Underlie Geometry," by J.G. MacCarthy, Deming, New Mexico.
"The Relative Difference Sets of Butson (and Bose)," by P. Leonard, Arizona State University.

Special Sessions:

"Mathematicians Observed," by D. Albers, Menlo College.
"The Knowledge Explosion in Mathematics," by H. Rund, University of Capetown.

Missouri Section

The annual spring meeting of the Missouri Section was held April 27-28, 1984 at Southeast Missouri State University, Cape Girardeau, Missouri. There were 70 registrants.

Invited Addresses:

"Some Problems I Couldn't Solve," by Paul R. Halmos, Editor, American Mathematical Monthly.

Short Presentations:

"A Report on Results of Lipman and Anderson on Wreath Products of Graphs," by Gerald Schrag, Central Missouri State University.
"Fixed Point Theory: A Revisitation," by Troy L. Hicks, University of Missouri, Rolla.
"Fixed Point Theory: Computational Aspects," by Alberta Bollenbacher and Troy L. Hicks, University of Missouri, Rolla.
"Regular Polygons and Lattice Points," by John R. Wolffer, Defense Mapping Agency, and Mangho Ahuja, Southeast Missouri State University.
"On the Natural Density of the Niven Numbers," by Robert E. Kennedy and Curtis N. Cooper, Central Missouri State University.
"On an Asymptotic Formula for the Niven Numbers," by Curtis N. Cooper and Robert E. Kennedy, Central Missouri State University.

Iowa Section

The seventy-first meeting of the Iowa Section was held on April 13-14, 1984 at Wartburg College, Waverly. The meeting was attended by 49 persons.

Invited Addresses:

"Another Approach to Riemann-Stieltjes Integrals," by Kenneth Ross, University of Oregon.
"A Nice Property of Armadillos and How to Take Advantage of It," by Daniel Mihalko, University of Nebraska, Lincoln.

Short Presentations:

"Unified Field Based Mathematics at Maharishi University," by Catherine Gorini Wadsworth, Maharishi International University.
"Computer Graphics at UNI," by Walter Beck, University of Northern Iowa.
"Interpretation of Discarded Roots of Algebraic Equations," by Bernadette Baker, Drake University.
"Mathematical Models for the Dendritic Tree of a Nerve Cell," by Jack Heidel, Iowa State University.
"Graph Generators," by Milan Randic, Drake University.
"Quotient Structures in Binary Systems With No Identity Element," by Don Pilgrim, Luther College.
"Extending Minty's Algorithm for Network Minimization Problems," by Robert Arnold, Iowa State University.
"Sweep Operator in Statistics: Multivariate Analysis on a Micro," by George Woodworth, University of Iowa.
"Least Squares Projections Onto Intersections of Convex Cones," by Richard Dykstra, University of Iowa.
"Bayes, Buehler and Buehler-Bayes Confidence Bounds," by Margarita Guerrero, Iowa State University.

Student Papers:

"On Hermite Quintic Interpolation," by Frank A. Odoom, Iowa State University.
"Counting Self-Avoiding Walks on a Regular 3-Dimensional Network," by Jennifer L. Meyer, Drake University.
"Construction of Complex Transitive Graphs," by Grant Izmirlian, Drake University.

Rocky Mountain Section

The sixty-seventh annual meeting of the Rocky Mountain Section was held on April 27-28, 1984 at the United States Air Force Academy in Colorado Springs, Colorado. The meeting was attended by 100 members.

Invited Addresses:

"A New Look at Sub-Rings," by Gene Abrams, University of Colorado, Colorado Springs.
"New Mathematical Techniques in Image Understanding," by Dean Lucas, Interactive Systems.
"Tackling a Ticklish Type of Tic-Tac-Tow (or The Case of the (Almost) Total Tactics): II," by Ira Rosenholtz, University of Wyoming.
"Math and Computer Science: A Case Study in Marriage and Divorce," by Lynn Arthur Steen, St. Olaf College.
"Geometry and the Universe," by Ron O. Wells, Rice University.

Banquet Address:

"1 + 1 = 0: Learning Our Lesson From the New 'New Math'," by Lynn Arthur Steen, St. Olaf College.

Contributed Papers:

"A Note on Approximation of n!" by Genji Yoshino, student, University of Colorado, Boulder.
"Applications of Harmonic Analysis to Number Theory," by Boris Lerner, student, University of Colorado, Boulder.
"A Summer of Applied Math with NASA," by T.J. Sakulich, student, United States Air Force Academy.
"Symmetric Colorings vs. Color Groups," by Richard L. Roth, University of Colorado, Boulder.
"On Unique Critical Points in the Plane," by Lowell Smylie, student, University of Wyoming.
"Recursion Relations in Summation Formulas," by Shane Hubler, student, Colorado College.
"Is Mathematical Thinking Artistic or Artistic Thinking Mathematical?" by Cecile Duray-Bito, student, University of Colorado, Boulder.
"Group Theory and Crystallography," by Kent D. Choquette, student, University of Colorado, Boulder.
"Relational Approximation, Continued Fractions and Fixed Points," by Christopher Baltus, University of Northern Colorado.
"The Bliss Principle," by Richard A. Gibbs, Fort Lewis College.
"Making Sense Out of Dollars: A Study of Faculty Salaries at SDSM&T," by Bonnie Henrikson, South Dakota School of Mines and Technology.
"Necessary and Sufficient Conditions for Metrizability of C(X) With the Compact-Open Topology," by Paul O'Meara, University of Colorado, Denver.
"Computer Generated Insights Into Number Theory Results," by Robert S. Fisk, Colorado School of Mines.
"Programs for the Gifted and Talented," by Terry L. Jenkins, University of Wyoming.
"How to Handle Large Classes," by David Ballew, South Dakota School of Mines and Technology.
"Something Old," by Thomas E. Kelley, Colorado School of Mines.
"Something New," by Lloyd Best, student, Colorado School of Mines.
"Real Division Algebras of Dimension >1 Contain C," by John Petro, United States Air Force Academy.
"Integral Solutions of the Gamma Equation $\Gamma(x) = \Gamma(y)\Gamma(z)$," by Hung C. Li, University of Southern Colorado.
"Algorithms for Sorting," by Bob Tolar, University of Northern Colorado.
"My Impressions of Chinese Education," by James C. Davis, Mesa College.

Panel Discussion:

"Discrete and Indiscrete Thoughts on the First Two Years," by David Ballew (Moderator), South Dakota School of Mines and Technology; Karen Whitehead, South Dakota School of Mines and Technology; John Maybee, University of Colorado, Boulder; Paul Myers, Denver University.

Intermountain Section

The annual spring meeting of the Intermountain Section was held April 27-28, 1984 at Ricks College in Rexburg, Idaho.

Banquet Address:

"Mathematical Modeling," by Harvey Fletcher, Brigham Young University.

Invited Addresses:

"A Probabilistic Approach to Studying Groups," by Kenneth A. Ross, University of Oregon.
"Classification of Surfaces," by C.E. Burgess, University of Utah.

Contributed Papers:

"Pythagorean Quadruples," by Steven Heath, Southern Utah State College.
"Teaching Partial Fractions, Some New Thoughts on an Old Ordeal," by Vencil Skarda, Brigham Young University.
"The Cone Property of Quasilinear Systems," by Paul Talaga, Weber State College.
"Tactical Maneuvers in Problem Solving," by Jay Huber, Ricks College.
"On Counting Subcubes of a Hypercube," by Lee Badger, Weber State College.
"Overview of the Next FORTRAN Standard," by N.H. Marshall, Member of X3J3, ANSI.

"Torsion Subgroup of Elliptic Curves," by Jasbir S. Chahal, Brigham Young University.
"Test Sets for Generalized Matrix Functions," by Leroy Beasley, Utah State University.
"When are Tautochrones Brachistochrones?" by Bryce Parry, Utah State University.
"A Correction for the Partial Sums of Positive-term Series," by Donald R. Snow, Brigham Young University.
"Iterations of Polynomials," by Charles Cunkle, Emeritus, Slippery Rock.

Student Papers:

"Symmetric Tri-diagonal Matrices and Sturm Sequences," by Ross Blankley, Montana State University.
"NUSAT I Satellite Project," by LaMont Sheridan, Weber State College.
"Sun Sensor Data Simulation Model for NUSAT I," by Jeff Salt, Weber State College.
"The Ulam Sequence," by Steven Cannon, Logan High School and Utah State University.
"An Alternate Proof of the Jordan Canonical Form," by Gary Lawlor, Brigham Young University.

Panel Discussion:

"Maintaining and Stimulating Student Interest in Mathematics," by Dewey Furness (Moderator), Ricks College; Larry Kratz, Idaho State University; David Tolman, Weber State College; E.E. Underwood, Utah State University.

Texas Section

The annual meeting of the Texas Section was held at the University of Texas at Tyler in Tyler, Texas on April 6-7, 1984. There were 277 registrants for the meeting.

Invited Addresses:

"Mathematical Time Exposures: Some Recurrent Geometry," by Mary Ellen Rudin, Univ. of Wisconsin.
"Mathematics and Computer Science: A Case Study in Marriage and Divorce," by Lynn Arthur Steen, St. Olaf College.

Contributed Papers:

"The James Space Revisited," by Tommy Leavelle, North Texas State University.
"On a Pair of Non-linear Functional Equations," by Elias Deeba, Univ. of Houston, Downtown.
"Some Results on Boxer's Hyperspace," by Stuart Anderson, East Texas State University.
"A Class of Mappings Containing All Continuous Mappings," by J.K. Kohli, Lamar University.
"Computer Software Packages in the Applied Mathematical Sciences," by Sudir Goel and Ron Barnes, University of Houston Downtown College.
"A Computer Investigates the Bernoolli Roses," by E. Dennis Huthnance, Midwestern State Univ.
"The Influence of Numerical Linear Algebra on the Development of Computers," by George Poole, Lamar University.
"Collapsible Series," by Donald F. Bailey, Trinity University.
"The American Invitational Mathematics Examination," by George Berzsenyi, Lamar University.
"How Not to Teach Math," by Joe F. Allison, Eastfield College.
"Pascal's Tetrahedron: A Geometric Interpretation of the Trinomial Theorem and Its Applications," by Richard A. Cooper, Trinity University.
"Geometric Formulations of Concavity," by Victor A. Belfi, Texas Christian University.
"Cooperative Education--What Can It Do for Me?" by Robert E. Goad, Sam Houston State University.
"The Polar Compensating Planimeter," by Chris Boldt, Eastfield College.
"P-Commutative Banach *-Algebras," by Robert S. Doran, Texas Christian University.
"How Many Squares in a Grid?" by David Hughes, Abilene Christian University.
"An Example of a Kaplansky Structure Without the Singleton Property," by Don E. Edmonson, University of Texas at Austin.
"Solving Polynomial Equations by Radicals," by Joseph Kung, North Texas State University.
"On 2x2 Matrix Square Roots," by Timothy P. Donovan, East Texas State University.
"Uniqueness Results for Homeomorphism Groups," by Robert R. Kallman, North Texas State University.
"Numbers with Missing Digits," by Michael Laidacker, Lamar University.
"Existence in Uniqueness of Solutions of the Second Order Boundary Value Problem," by Reza Aftabazadeh, Pan American University.
"The TEX Typesetting System and Its Status at TAMU," by Norman Naugle, Texas A&M University.

Student Paper:

"The Mathematical Foundations of Sentence Structure," by Sandra Wheeler, East Texas State Univ.

Special Sessions:

"Forum on Mathematics Resource Centers and Other Aids in Managing Large Classes," by Eddie Warren, University of Texas at Arlington; John Hardy, University of Houston at University Park; John Durbin, University of Texas at Austin; Bill Perry, Texas A&M University.

Special Reports:

"A Mathematician's Tour of China," by Linda Kyle, Tarrant County Junior College.
"New School Mathematics Curriculum and New Standards for Mathematics Teacher Certification," by Barbara Montalto, Texas Education Agency.
"Report on MAA TEAM Project (Teaching Experimental Applied Mathematics)," by Bo Green, Abilene Christian University.
"Computers in Schools in Texas (Pre-college and College): Impact on Curriculum and Pedagogy," by John Huber, Pan American University.

Ohio Section

The annual spring meeting of the Ohio Section was held on April 13-14, 1984 at Bowling Green State University. There were 174 registrants.

Invited Addresses:

"Recent Progress in Numerical Conformal Mapping," by Peter Henrici, University of North Carolina and Eidgen Tech. Hochschule, Switzerland.
"Programming Languages for Simulation Modeling," by Zaven Karian, Denison University.
"Curves of Calculus," by V. Frederick Rickey, Bowling Green State University.
"Some Conjectures and Open Problems in Function Theory and Approximation," by Richard Varga, Kent State University.
"Some Things My Mathematics Professors Never Taught Me--Mathematical Things That Is," by Richard Little, Baldwin-Wallace College.

Short Presentations:

"Advice to Young Math Majors," by David Kullman, Miami University.
"Solving Messy Math Problems on a Micro," by Ann Albert, Findlay College.
"Exploratory Data Analysis on a Timex Microcomputer," by James Albert, Bowling Green State University.
"The Computerized Biomechanical Man-model (Combiman)," by Medhat Korna, University of Dayton--Research Institute.
"A Probability Distribution Generator," by William M. Wagner, CALYPSO.
"Graph of a Polynomial Function of Degree-Three," by Byung G. Hahn, Sinclair Community College.
"Collineations of Finite Projective Planes," by John Hayden, Bowling Green State University.
"Automated Reasoning," by Sister Teresemarie McCloskey, Notre Dame College.
"Mathematical Content of 1984 AHSME," by Leo Schneider, John Carroll University.
"A Note on Functions with Interlacing Roots," by Theodore Kilgore, Ohio State University, Marion.

Special Session:

"Approximation and Numerical Methods," by Charles Groetsch (Moderator), University of Cincinnati.
"Ada Tasking and Numerical Analysis," by J. Buoni, Youngstown State University.
"Finding a Bracketed Zero by Larkin's Method(s)," by V. Norton, Bowling Green State University.
"Some Numerical Techniques for Inverse Problems," by L.D. Sabbagh, Indiana University and Bowling Green State University.
"On the Acceleration of Convergence of Vector Sequences," by S.D. Silliman, Cleveland State University.
"Some New Results on the Limit Points of Zeros and Poles," by T. Norfolk, University of Akron.
"Polynomial Interpolation at the n-th Roots of Unity," by T. Price, Jr., University of Akron.

Student Papers:

"The Circular Sections of a Torus, or How to Dunk Your Donut Four Different Ways," by Lee Ann Shollenberger, Miami University.
"An Introductory Look at Coding Theory," by Mark Goddard, Youngstown State University.
"Algorithm Development for Prime Number Generators," by Andrew Moore, College of Wooster.
"A Branch and Bound Method for the Traveling Salesman Problem," by Mark Toskin, Youngstown State University.
"Finding a Point Estimator on the Cartesian Plane or Calculating the Center of Complaints," by Michael H. Cox, Marshall University.
"An Infinite Series Representation of $\int_0^1 x^x dx$," by Jeff Kubina, Youngstown State University.
"Taking It With You," by Mark S. Russell, Miami University.
"The Kuhn-Munkres Algorithm for Maximal Weight Bipartile Matching," by Kirk Markopolis, Youngstown State University.
"The Simulation of the 6502+ Microprocessor on the VAX-11/780," by Stephen Kennedy, Denison University.
"The Simulation of an Assembler and Operating System for the 6502+ on the VAX-11/780," by Daryl Steen, Denison University.
"A Twenty-Five Point Geometry," by David Cameron, Miami University.
"Representation of Trees by Prufer Sequences: A Computer Implementation," by Albert Vargo, Youngstown State University.
"Bezier Polynomials in Robotic Path Design," by Douglas Meade, Bowling Green State University.
"The Greedy Algorithm," by Marilyn Shasho, Youngstown State University.

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A REMARK ON THE NUMBER OF CYCLIC SUBGROUPS OF A FINITE GROUP

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It is common enough in books on elementary group theory to prove that if G is a finite cyclic group, then, for each divisor d of the order of G, there is exactly one subgroup of order d, and that this subgroup is cyclic. Yet it seems that it has not been noted that the number of cyclic subgroups, and therefore the number of subgroups, is minimized precisely in the case of the cyclic group. We prove therefore this theorem.

THEOREM. *Let G be a group of order n. The number of cyclic subgroups of G is greater than or equal to $d(n)$, the number of divisors of n. Furthermore, the number of cyclic subgroups of G is equal to $d(n)$ if and only if G is cyclic.*

Proof. Let G be a group of order n. We consider an action of U_n, the group of invertible residues modulo n, on G. With each $s \in U_n$ we associate the permutation of G, ψ_s, defined by $\psi_s(g) = g^s$ for all $g \in G$. Under this action two elements belong to the same orbit if and only if they generate the same cyclic subgroup. Thus the number of orbits, which we shall denote by $c(G)$, is equal to the number of cyclic subgroups of G. By Burnside's Lemma [4], we have the relation:

$$c(G) = \frac{1}{\phi(n)} \sum_{s \in U_n} |F(s)|. \tag{1}$$

Here $\phi(n)$ is the Euler totient function, while $F(s)$ is the fixed set of ψ_s, that is, the set of elements in G which satisfy the equation $x^{s-1} = 1$. By a theorem of Frobenius [2], or see [1], $|F(s)| = k_s \cdot (s-1, n)$, where k_s is a positive integer. Formula 1 therefore becomes

$$c(G) = \frac{1}{\phi(n)} \sum_{s \in U_n} k_s \cdot (s-1, n). \tag{2}$$

We first specialize Formula 2 to the case where G is the cyclic group of order n. Then $c(G)$ is equal to $d(n)$ and, for each $s \in U_n$, we have $k_s = 1$. We conclude that

$$d(n) = \frac{1}{\phi(n)} \sum_{s \in U_n} (s-1, n). \tag{3}$$

Returning to the general case of Formula 2, we recall that each constant k_s is greater than or equal to 1, so that

$$c(G) \geqslant \frac{1}{\phi(n)} \sum_{s \in U_n} (s-1, n). \tag{4}$$

The first part of the theorem is now immediate from Formulae 3 and 4.

A consideration of Formulae 3 and 4 indicates that the number of cyclic subgroups of a finite group, of order n, can only equal $d(n)$ if, for each s such that $(s, n) = 1$, the number of solutions of the equation $x^{(s-1,n)} = 1$ in G is exactly $(s-1, n)$. To prove the second part of the theorem, it suffices to show that if a group G has this property, then G is cyclic. It is necessary to ask therefore for a description of the values of $(s-1, n)$ as s runs over all invertible residues modulo n. The answer is simply stated.

LEMMA 1. *Let* $D_n = \{(s-1, n); s \in U_n\}$. *If* n *is even, then* $D_n = \{d; d|n, d \text{ is even}\}$. *If* n *is odd, then* $D_n = \{d; d|n\}$.

The proof of this result is an elementary exercise in number theory, using the Chinese Remainder Theorem. We next quote a well-known result of group theory; for proof see [3].

LEMMA 2. *Let* G *be a group of order* n. *Suppose that for each* d *dividing* n *the number of solutions of the equation* $x^d = 1$ *in* G *is less than or equal to* d. *Then* G *is cyclic.*

From Lemmas 1 and 2 and the remarks preceding them, it follows that if G is a group of odd order n such that the number of cyclic subgroups of G is $d(n)$, then G is cyclic. Therefore let G be a group of even order n. Suppose that the number of cyclic subgroups of G is $d(n)$. By Lemma 1 this amounts to the assumption that for each even divisor, d, of n, the number of solutions of the equation $x^d = 1$, in G, is exactly d. In particular, the number of solutions of the equation $x^2 = 1$ is 2. Therefore G has a unique involution. Let this involution be called z. Since it is fixed under conjugation, z is necessarily central. We consider m, an odd divisor of n. Let $y^m = 1$. Then both $y^{2m} = 1$ and $(zy)^{2m} = 1$. Thus to each solution of the equation $x^m = 1$ there correspond two distinct solutions of the equation $x^{2m} = 1$. Since the number of solutions of the equation $x^{2m} = 1$ is only $2m$, then the number of solutions of the equation $x^m = 1$ cannot exceed m. It is therefore true of any divisor, d, of n, that the number of solutions of $x^d = 1$, in G, does not exceed d. Lemma 2 now suffices to prove that G is cyclic and so the second part of the theorem is proved.

Note that Formula 3 gives an unusual relation between the divisor function and the Euler totient function. It may be generalized as follows. Let $f(x)$ be any polynomial in $Z[x]$. Let U_n once again denote the invertible residues modulo n. Then

$$\sum_{s \in U_n} (f(s), n) = \phi(n) \sum_{d|n} |\{r \in U_d; f(r) \equiv 0 \bmod d\}|. \tag{5}$$

In conclusion the author wishes to thank his supervisor at Exeter University, Dr. R. Odoni, for his assistance in the preparation of this note. While this note was written, the author was in receipt of a grant from the Science and Engineering Research Council.

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ANSWER TO PHOTO ON PAGE 550

From left to right, Gábor Szegő and George Pólya, authors of the path-breaking and still outstanding problem book of the century. The picture was taken in 1925, when the subjects were in Berlin for the signing of the contract for their book.

MATRIX INVARIANTS

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Let M_n be the set of all $n \times n$ complex matrices. Consider the functions $c_i : M_n \to \mathbb{C}$ defined by

$$\det(\lambda I - A) = \lambda^n - c_1(A)\lambda^{n-1} + c_2(A)\lambda^{n-2} - \cdots + (-1)^n c_n(A).$$

Recall that $c_1(A) = \operatorname{tr} A$ and $c_n(A) = \det A$.

If B is invertible, then

$$\det(\lambda I - A) = \det\big(B(\lambda I - A)B^{-1}\big) = \det(\lambda I - BAB^{-1}),$$

so $c_i(A) = c_i(BAB^{-1})$. Thus whenever $L : V \to V$ is a linear transformation and A is the matrix for L obtained by choosing a basis for V, the numbers $c_i(A)$ depend only on L and not on a particular basis.

It is natural to ask if the c_i are the only similarity invariants. Obviously, combinations like $c_3 + c_2^5 c_7$ are invariant; we might conjecture that every similarity invariant has the form $F(c_1(A), \ldots, c_n(A))$ for some $F : \mathbb{C}^n \to \mathbb{C}$. This conjecture is false; the c_i do not distinguish $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, but

$$f(A) = \begin{cases} 1, & \text{if } A = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \\ 0, & \text{otherwise,} \end{cases}$$

is invariant. However, if continuity is required we get the following satisfying characterization:

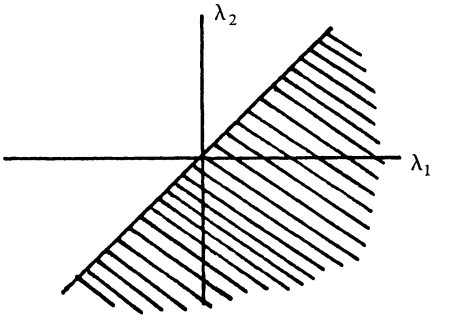
MAIN THEOREM: *Let $f : M_n \to \mathbb{C}$ be continuous and suppose $f(BAB^{-1}) = f(A)$ for every A in M_n and every invertible B in M_n. Then there is a unique $F : \mathbb{C}^n \to \mathbb{C}$ such that*

$$f(A) = F(c_1(A), \ldots, c_n(A)).$$

This result is seldom included in linear algebra books because it has a topological component, but it is well known and easily proved. I like the following unnecessarily long proof because it seems to get to the heart of the matter.

Let $\mathscr{S}$ denote the set of equivalence classes of $n \times n$ matrices under the equivalence relation $A \sim BAB^{-1}$; we call $\mathscr{S}$ the *classifying space*. Using the canonical map $\pi : M_N \to \mathscr{S}$, define a topology on $\mathscr{S}$ by calling $\mathscr{U} \subseteq \mathscr{S}$ open just in case $\pi^{-1}(\mathscr{U}) \subseteq M_n$ is open.

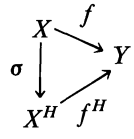
Let us stop to draw a picture of $\mathscr{S}$ when $n = 2$ (the picture is not needed in the proof). Every 2×2 matrix is similar to $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$; $\left(\begin{smallmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \lambda_2 & 0 \\ 0 & \lambda_1 \end{smallmatrix}\right)$ are similar. Drawing only the real portion of the picture, we obtain the shaded region below from points with diagonal representatives:



We must add a line corresponding to the matrices $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$. I claim this line lies "infinitesimally close" to the line $\lambda_1 = \lambda_2$ in the picture above. Indeed, for each positive integer m, $\begin{pmatrix}\lambda & \frac{1}{m}\\ 0 & \lambda\end{pmatrix}$ is similar to $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$. Since $\begin{pmatrix}\lambda & \frac{1}{m}\\ 0 & \lambda\end{pmatrix} \to \begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$, the sequence which is identically $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ in $\mathscr{S}$ converges to $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$. (In particular, $\mathscr{S}$ is not Hausdorff.)

The basic idea of the proof is to construct a simpler space $\mathscr{S}^H$ by gluing the line $\begin{pmatrix}\lambda & 1\\ 0 & \lambda\end{pmatrix}$ directly to the line $\begin{pmatrix}\lambda & 0\\ 0 & \lambda\end{pmatrix}$. We need the following easy topological result:

THEOREM: *Let X be a topological space. There is a Hausdorff space X^H and continuous map $\sigma: X \to X^H$ such that whenever Y is Hausdorff and $f: X \to Y$ is continuous, the following diagram can be uniquely completed*:

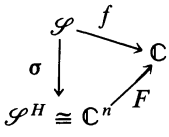


Proof. Call x_1 and x_2 in X equivalent if $g(x_1) = g(x_2)$ whenever $g: X \to Z$ is a continuous map from X to a Hausdorff space Z. Let X^H be the set of equivalence classes, and continue in the obvious way.

We can easily determine the structure of $\mathscr{S}^H$. The continuous maps $c_i: M_n \to \mathbb{C}$ induce maps $c_i: \mathscr{S} \to \mathbb{C}$. Since $\mathbb{C}$ is Hausdorff, there are associated maps $c_i^H: \mathscr{S}^H \to \mathbb{C}$. These can be combined to give a map $c^H = (c_1^H, \ldots, c_n^H): \mathscr{S}^H \to \mathbb{C}^n$. The main theorem now follows from:

THEOREM A: *$c^H: \mathscr{S}^H \to \mathbb{C}^n$ is a homeomorphism.*

Proof of Main Theorem: If $f: M_n \to \mathbb{C}$ is a continuous invariant, let F be defined by the following diagram:



Proof of Theorem A: The map c^H is onto; if $(d_1, \ldots, d_n) \in \mathbb{C}^n$, factor $\lambda^n - d_1\lambda^{n-1} + \cdots + (-1)^n d_n = \Pi(\lambda - \lambda_i)$ and notice that $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ maps to $(d_1, \ldots, d_n)$.

The map c^H is one-to-one on elements with diagonal representatives; if $A = \begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ and $B = \begin{pmatrix}\mu_1 & & 0\\ & \ddots & \\ 0 & & \mu_n\end{pmatrix}$ and $c_i(A) = c_i(B)$, then $\Pi(\lambda - \lambda_j) = \Pi(\lambda - \mu_j)$, so A and B are similar.

Finally, every element of $\mathscr{S}^H$ has a diagonal representative. Indeed, every matrix is similar to a triangular matrix. Let $A = \begin{pmatrix}\lambda_1 & & *\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$ and define D_m to be the diagonal matrix with diagonal entries $1, m, m^2, \ldots, m^{n-1}$. Let $A_m = D_m A D_m^{-1}$ and notice that A_m converges to $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}$. Since A_m and A are similar, they represent the same point in $\mathscr{S}^H$. Therefore the sequence A_m

converges both to A and to $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ in $\mathscr{S}^H$. But $\mathscr{S}^H$ is Hausdorff, so A and $\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ represent the same element.

THE TEACHING OF MATHEMATICS

EDITED BY MARY R. WARDROP AND ROBERT F. WARDROP

Material for this department should be sent to Professor Robert F. Wardrop, Department of Mathematics, Central Michigan University, Mount Pleasant, MI 48859.

THE ROLE OF MATHEMATICS EDUCATION SPECIALISTS IN A MATHEMATICS DEPARTMENT

WILLIAM F. BURGER
Department of Mathematics, Oregon State University, Corvallis, OR 97331

JAMES E. SCHULTZ
Department of Mathematics, Ohio State University, Columbus, OH 43210

Faculty in university mathematics departments frequently can be separated into various groups which specialize in such areas as algebra, analysis, topology, and applied mathematics. Many departments also have faculty who specialize in mathematics education. The purpose of this article is to point out some consequences of having an active mathematics education group within a department of mathematics. We will include in such a group anyone in a department of mathematics whose primary responsibility is carrying out research in mathematics education, developing and teaching mathematics courses for teachers, developing precalculus courses, or providing service to the department such as overseeing learning resource centers and supervising teaching assistants. This group of faculty is distinguished from those in a school of education who develop mathematics teaching methods courses and who supervise student teachers. It should be emphasized that we speak of faculty in a mathematics department whose primary function is in mathematics education as opposed to research mathematicians who have limited involvement with such activities. The group may include people with or without formal training in mathematics education.

Research in Mathematics Education. Research in mathematics education is fundamentally different from that in pure mathematics in that it tends to deal with teaching and learning problems, and not with abstract mathematical problems. Investigating acquisition and development of mathematical concepts in students, effective teaching practices, and improvement of problem solving abilities are examples of ongoing research efforts in mathematics education. For example, see [2], [7], and [10]. Such research efforts can be valuable to mathematics departments for their applications in program development. In addition, research findings aid mathematics education specialists in developing text and course materials. This activity frequently has direct benefit to the department's students, and it has the indirect benefit of bringing professional recognition and, in some cases, outside funding to the department. While time spent by research mathematicians developing course materials may be at the expense of doing research, such efforts are consonant with the research interests of mathematics educators.

Courses for Teachers. Teacher education is usually shared by several units on campus.

5. Franklin D. Demana and Joan R. Leitzel, Transition to College Mathematics, Addison-Wesley, Reading, MA, 1984.

6. Ramesh Gangolli, Some observations on issues of educational policy for mathematics departments. Talk given June 2, 1983, at Oregon State University.

7. Alan R. Hoffer, Geometry is more than proof, The Mathematics Teacher, 74 (1981) 11–18.

8. Joan R. Leitzel, James E. Schultz, and Arthur L. White, Alternatives in the mathematics preparation of elementary teachers, this MONTHLY, 87 (1980) 572–574.

9. Gary Musser and Linda Thompson, A learning center based system of instruction, this MONTHLY, 84 (1977) 290–293.

10. Alan H. Schoenfeld, Teaching problem-solving skills, this MONTHLY, 87 (1980) 794–805.

STILL MORE REBUSES

According to the American Heritage Dictionary, a *rebus* is a riddle composed of words or syllables depicted by symbols or pictures that suggest the words or syllables they represent. Can you decipher the ones below?

A few additional mathematical rebuses will appear in this department from time to time. If you can think of others, please send them to the editor.

1.



—Bruce Hampton

2. SOL $\dot{+}$ DARITY

—Jim Griffith

3.

N
O
I
T
C
N
U
F

4.

$$f(0) = n$$
$$f(n) = e$$
$$f(e) = 0$$

5. nomial nomial

6.

S
E
I
R
E
S
E
R
I
E
S

—Carl Morris

Answers on p. 590.

135. **MISCELLANEA**

Some Clerihews

John Napier
Might have used a rapier
In defending his schisms,
If not his logarithms.

Galileo Galilei
Refuted Ptolemy
And made the Greek look stupider
By observing the moons of Jupiter.

Pierre de Fermat
Had a great deal of éclat;
But he did not have a margin
To enlarge in.

The family Bernoulli
Could be quite unruly;
They had many altercations
Over the calculus of variations.

Samuel Bates
Department of Mathematics
Northwestern University
Evanston, IL 60201

unless $f = 0$ a.e. in (x_0, ∞).

Suppose that (*) holds with an f that is not zero a.e. in (x_0, ∞). Let $G(x) = \int_0^x |f|^p$. Then $G(x_1) > 0$ for some $x_1 \geqslant x_0$ and $G'(x) = |f(x)|^p$ a.e. in (x_0, ∞), and hence, by (*), $G'(x) \geqslant G^p(x)$ a.e. in (x_0, ∞). Integrating G'/G^p on $[x_1, y]$ we get the contradiction that, for all $y > x_1$,

$$\frac{G^{1-p}(x_1) - G^{1-p}(y)}{p-1} \geqslant y - x_1.$$

Also solved by Preben Alsholm, Miroslav D. Ašić (Yugoslavia), F. S. Cater, Roy O. Davies (England), W. R. Emerson, Michael Golomb, O. P. Lossers (The Netherlands), J. R. Partington (England), Andrew Vogt, and the proposers.

Non-Invertible Operators

6423 [1983, 289]. *Proposed by P. R. Halmos, Indiana University.*

If A is a bounded linear operator on a Hilbert space, and if

$$\|1 - A\| < 1,$$

then A is invertible. That assertion, not only for operators on Hilbert spaces, but for arbitrary elements of Banach algebras, is usually proved by consideration of the infinite series $\sum_{n=0}^{\infty}(1-A)^n$, and, therefore, the proof uses completeness. Is the assertion true without completeness? Is there, in other words, a bounded linear operator A on an inner product space, such that $\|1 - A\| < 1$ but A is not invertible?

Solution. Twenty-two solvers gave simple examples to show that the assertion is false without completeness. A typical example is afforded by taking the inner product space to be the space $\mathcal{P}$ of all polynomials with inner product $(p, q) = \int_0^1 p(t)\overline{q(t)}\,dt$ and defining A by

$$(Ap)(x) = \left(1 - \frac{x}{2}\right)p(x).$$

Then $((1-A)p)(x) = \frac{x}{2}p(x)$, so that

$$\|(1-A)f\|^2 = \frac{1}{4}\int_0^1 t^2|p(t)|^2\,dt \leqslant \frac{1}{4}\int_0^1 |p(t)|^2\,dt = \frac{1}{4}\|f\|^2.$$

Hence, $\|1 - A\| \leqslant \frac{1}{2}$, but A is not invertible since it transforms no polynomial into the constant polynomial 1.

A paper in this MONTHLY, 91 (1984) 49–51, "Geometric series in incomplete normed algebras" by Fuster and Marquina, deals with the problem in a slightly different setting.

ANSWERS TO REBUSES ON PAGES 578–579

1. Ring with identity
2. Disjoint union
3. Steep function
4. One-to-one function
5. Binomial
6. Divergent series.

5. nomial nomial

6.

S
E
I
R
E
S
E
R
I
E
S

—Carl Morris

Answers on p. 590.

135. **MISCELLANEA**

Some Clerihews

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Could be quite unruly;
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PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS),
H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*Send all **proposed** problems, typed and in duplicate if possible, to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara CA 95053. Please include solutions, relevant references, etc.*

*An asterisk (*) indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the addresses given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by March 31, 1985. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgment).

E 3059. *Proposed by Meigu Guan, Weixuan Li, University of Waterloo, and Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

Let H denote the regular n-gon with side length equal to one, $n \geqslant 4$. Show that if K is any n-gon inscribed in H with side-length x_i $(i = 1, 2, \ldots, n)$, then

$$n(1 - \cos\theta)/2 \leqslant \sum_{i=1}^{n} x_i^2 \leqslant n(1 - \cos\theta),$$

where θ denotes the internal angle of H. Discuss the cases when either equality holds.

E 3060. *Proposed by Ira Gessel, Massachusetts Institute of Technology.*

A popular computer game works as follows: The computer picks an integer from 1 to n at random. The player is given k opportunities to guess the number. After each guess the computer responds "correct," "too high," or "too low." For fixed k and n, find the probability of success with optimal play, and find all optimal strategies.

E 3061. *Proposed by T. Ferguson and C. Melolidakis, University of California, Los Angeles.*

For each of n independent trials, a ball is placed in cell i with probability $2^{-(i+1)}$ for $i = 0, 1, 2, \ldots$. Let K denote the number of empty cells less than the largest occupied cell. Show that for all $n \geqslant 1$, $P_n(K = k) = 2^{-(k+1)}$ for $k = 0, 1, 2, \ldots$.

E 3062. *Proposed by Roger B. Nelsen, Lewis and Clark College.*

Show that for all integers $n \geqslant 0$ and all real x,

$$\sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} (x + j)^n = \frac{1}{2} \sum_{j=0}^{\infty} (x + j)^n 2^{-j}.$$

E 3063. *Proposed by Ion Cucurezeanu, Constanta, Rumania.*

Find all natural numbers $n > 1$ and $m > 1$ such that

$$1!3!5!\cdots(2n-1)! = m!.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Positive Integer Solutions j of $\{mj/n\} \geq 2j/n$

E 2935* [1982, 212]. *Proposed by Clark Kimberling, University of Evansville.*

Let n be a prime greater than 10, and let $\{\ \}$ denote fractional part. Prove or disprove that the number $S(m)$ of positive integer solutions j of the inequality $\{mj/n\} \geq 2j/n$ satisfies

$$S\left(\frac{n+1}{2}\right) \leq S(m) \leq S\left(\frac{n-1}{2}\right)$$

for $m = 5, 6, 7, \ldots, n-5$.

Solution by William J. Gilbert, University of Waterloo, Canada. We prove that if n is an odd prime, then:

(i) The values of $S(m)$ are symmetrical about $\frac{n}{2} + 1$, i.e.,

$$S(m) = S(n+2-m), \qquad 3 \leq m \leq n-1.$$

(ii) $S(0) = 0$, $S(1) = 0$, $S(2) = \left\lfloor \frac{n}{2} \right\rfloor$, $S(3) = \left\lfloor \frac{n}{3} \right\rfloor$.

(iii) $\left\lfloor \frac{n+1}{6} \right\rfloor = S\left(\frac{n+1}{2}\right) \leq S(m) \leq S\left(\frac{n-1}{2}\right) = \left\lfloor \frac{3n}{10} \right\rfloor$ for $4 \leq m \leq n-2$ when $n \geq 7$.

Here $\lfloor n \rfloor$ denotes the largest integer $\leq n$.

$S(m)$ is the number of points of the lattice $\left\{\left(j, \frac{m}{n}j - k\right) : j, k \in Z\right\}$ lying inside the triangle with vertices $A = (0,1)$, $B = (0,0)$ and $C = \left(\frac{n}{2}, 1\right)$ plus the number of such points lying inside the segment BC. The hypothesis that n is prime implies that there are no lattice points inside the segment BC if $m \neq 2$, since the Diophantine equation

$$(2-m)x = ny \tag{1}$$

has no solutions with $1 \leq x \leq \frac{n}{2}$ if $m \neq 2$.

To prove (i) we make the transformation

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ (n-2m)/n & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

which takes the original lattice points into the integral points satisfying $x'' \equiv y'' \pmod 2$ and, for $m \neq 2$, $S(m)$ is the number of these lattice points inside the triangle $A'' = (0,2)$, $B'' = (0,0)$, $C'' = (n/2, (n-2m+4)/2)$. By symmetry in the line $y'' = 1$, $S(m)$ is the number of such lattice points in the triangle $A'' = (0,2)$, $B'' = (0,0)$, $D'' = (n/2, (2m-n)/2)$; hence $S(m) = S(n+2-m)$ for $3 \leq m \leq n-1$.

For the rest of the proofs we make the transformation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ m/n & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix},$$

which shows that $S(m)$ is the number of integer lattice points inside the triangle $A' = (0,0)$, $B' = (0,1)$, $C' = (n/2, m/2)$ plus the number of lattice points on $B'C'$, which is zero if $m \neq 2$.

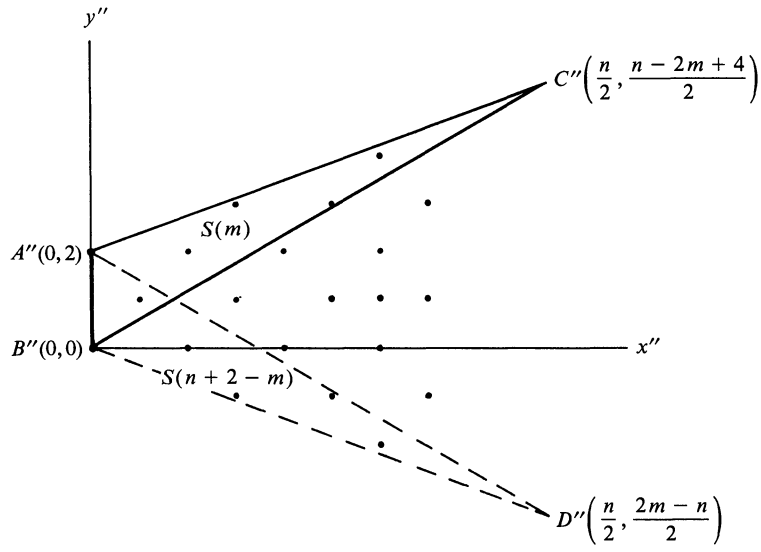


FIG. 1. The symmetry about $y'' = 1$ showing $S(m) = S(n + 2 - m)$ for $3 \leqslant m \leqslant n - 1$.

To prove (ii), in case $m = 2$ equation (1) has $\left\lfloor \frac{n}{2} \right\rfloor$ solutions, and there are no points inside the triangle, so $S(2) = \left\lfloor \frac{n}{2} \right\rfloor$. It is easy to check $S(0) = S(1) = 0$, $S(3) = \left\lfloor \frac{n}{3} \right\rfloor$.

To prove (iii), we shall estimate the general values of $S(m)$ by counting the integer lattice points inside the triangle $A'B'C'$ that lie on parallel lines whose slopes are rational, close to m/n, in a Farey series of small order f. The uncertainty in this estimate will be the number of parallel lines crossing the triangle. If n is large, the triangle is long and thin and the relative uncertainty can be reduced by an appropriate choice of f.

The Farey series of order f consists of the fractions l/k with $0 \leqslant l \leqslant k \leqslant f$ and $\gcd(l, k) = 1$. If l/k is such a fraction (close to m/n), then either (i) $km - ln = 2s$ or (ii) $km - ln = 2s + 1$ for some (small) integer s.

LEMMA 1. *If* $km - ln = 2s$ *and* $4 \leqslant m \leqslant n - 2$, *then* $S(m)$ *lies in the range*

$$\frac{n}{4} \pm \max(k - 1, s - 1, k - s - 1).$$

Proof. Let λ_i be the line $ky' - lx' = i$. Consider the three cases (ia) $0 \leqslant s \leqslant k$, (ib) $k < s$ and (ic) $s < 0$. In case (ia), if $0 < i \leqslant s$, then λ_i intersects the triangle $A'B'C'$ on $A'B'$ where $x' = 0$ and on $A'C'$ where $x' = in/(km - ln) = in/2s$. Since $\gcd(l, k) = 1$ it follows that the integer solutions, for x', to the Diophantine equation λ_i differ by multiples of k and the line λ_i contains either $\lfloor in/2ks \rfloor$ or $\lfloor in/2ks \rfloor + 1$ integer lattice points inside the triangle $A'B'C'$. That is, the number of lattice points lies in the range $(in/2ks) \pm 1$.

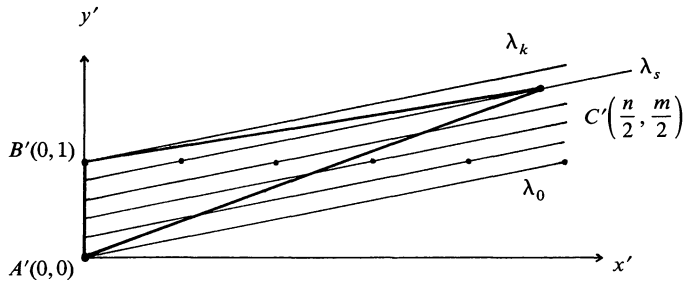
Similarly, if $s < i < k$, the line λ_i meets the sides $A'B'$ and $B'C'$ and the number of lattice points on λ_i inside $A'B'C'$ lies in the range $((k - i)n/2k(k - s)) \pm 1$. However,

$$\sum_{i=1}^{s} \frac{in}{2ks} + \sum_{i=s+1}^{k-1} \frac{(k - i)n}{2k(k - s)} = \frac{n}{4}$$

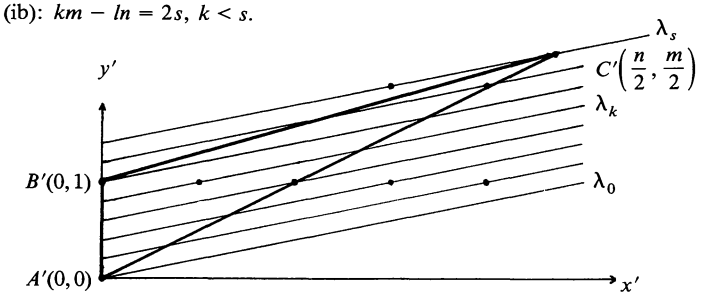
so $S(m)$, the total number of lattice points inside $A'B'C'$, lies in the range $(n/4) \pm (k - 1)$.

Similarly, in case (ib), $S(m)$ lies in the range $(n/4) \pm (s - 1)$ and, in case (ic), $S(m)$ lies in the range $(n/4) \pm (k - s - 1)$. □

Case (ia): $km - ln = 2s$, $0 \leqslant s \leqslant k$.



Case (ib): $km - ln = 2s$, $k < s$.



Case (ic): $km - ln = 2s$, $s < 0$.

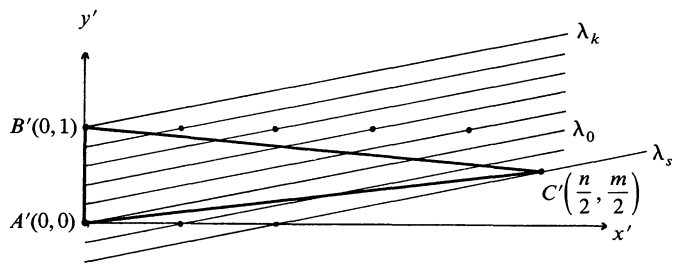
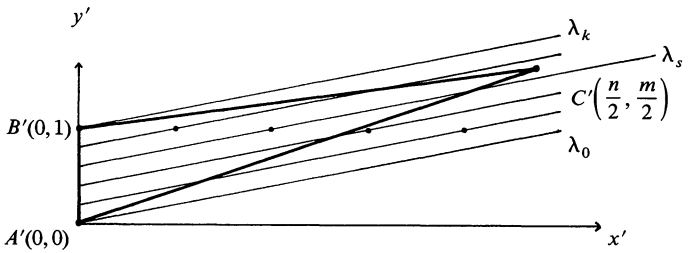
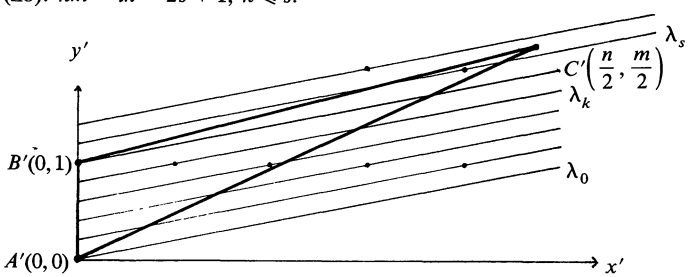


FIG. 2(i). The three cases when $km - ln$ is even.

Case (iia): $km - ln = 2s + 1$, $0 \leqslant s < k$.



Case (iib): $km - ln = 2s + 1$, $k \leqslant s$.



Case (iic): $km - ln = 2s + 1$, $s < 0$.

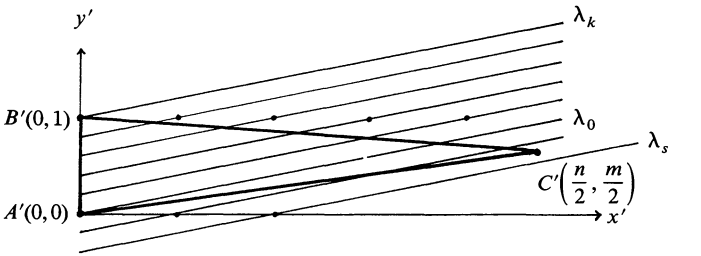
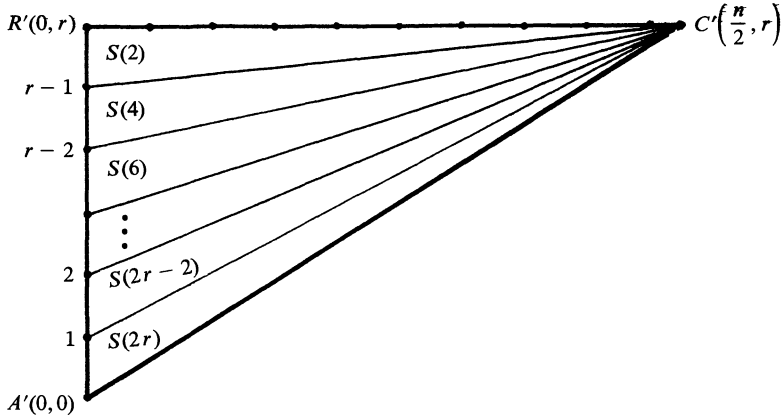


FIG. 2(ii). The three cases when $km - ln$ is odd.



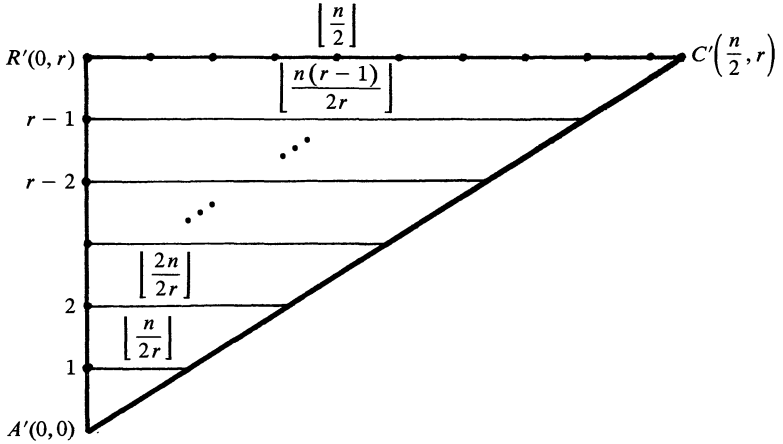
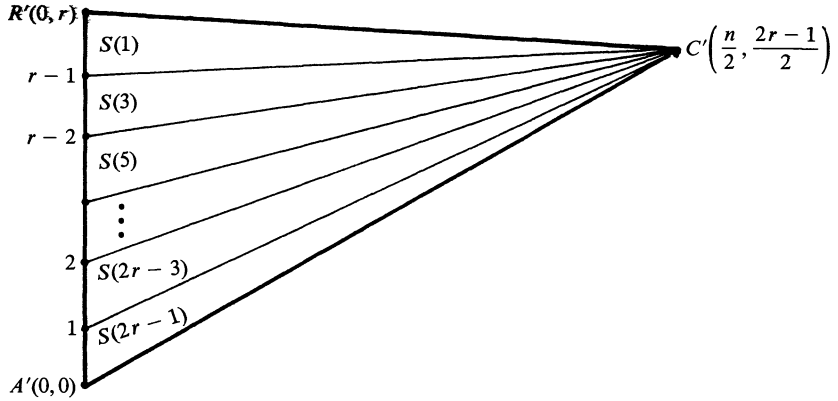


FIG. 3(i). Illustration of the equality $\sum_{i=1}^{r} S(2i) = \sum_{j=1}^{r} \left\lfloor \frac{n_j}{2r} \right\rfloor$.



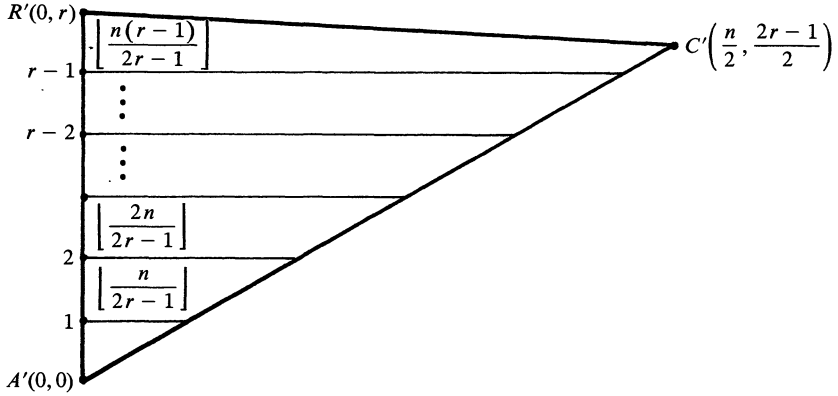


FIG. 3(ii). Illustration of the equality $\sum_{i=1}^{r} S(2i-1) = \sum_{j=1}^{r-1} \left\lfloor \frac{n_j}{2r-1} \right\rfloor$.

LEMMA 2. *If* $km - ln = 2s + 1$ *and* $4 \leqslant m \leqslant n - 2$, *then* $S(m)$ *lies in the range*

$$\frac{n}{4}\left(1 + \frac{1}{(2s+1)(2s-2k+1)}\right) \pm \max(k-1, s, k-s-1).$$

This lemma is proved in a similar way to Lemma 1 by considering the three cases (iia) $0 \leqslant s < k$, (iib) $k \leqslant s$ and (iic) $s < 0$.

In particular, if $m = (n+1)/2$, take $l/k = 1/2$ so that $km - ln = 1$ and $S((n+1)/2)$ lies in the range $(n/6) \pm 1$. Looking more closely at the line λ_1, we see that its integer points occur when x' is odd and $S((n+1)/2) = (n-1)/6$ if $n \equiv 1 \pmod 6$ and $S((n+1)/2) = (n+1)/6$ if $n \equiv 5 \pmod 6$. Hence $S((n+1)/2) = \lfloor (n+1)/6 \rfloor$. In a similar way, if $m = (n-1)/2$, take $l/k = 1/2$ so that $km - ln = -1$ and $S((n-1)/2)$ lies in the range $(3n/10) \pm 2$. Examining λ_0 and λ_1 in more detail for the separate cases $n \equiv 1, 3, 7, 9 \pmod{10}$, we see that $S((n-1)/2) = \lfloor 3n/10 \rfloor$.

LEMMA 3. *If there exists a Farey fraction* l/k *of order* $f = \lfloor (n-150)/20 \rfloor$ *such that* $2(k-f) \leqslant km - ln \leqslant 2f$, *then* $\lfloor (n+1)/6 \rfloor \leqslant S(m) \leqslant \lfloor 3n/10 \rfloor$ *for* $4 \leqslant m \leqslant n-2$.

Proof. In case (i) when $km - ln = 2s$, it follows from Lemma 1 that $S(m)$ lies in the range $(n/4) \pm (f-1)$. Now $(n/4) + f - 1 < 3n/10$ since $f < n/20$ and $(n/4) - f + 1 > (n+1)/6$ since $f < (n/12) - 1$. Hence $\lfloor (n+1)/6 \rfloor \leqslant S(m) \leqslant \lfloor 3n/10 \rfloor$.

In case (iia) when $km - ln = 2s + 1$ and $0 \leqslant s < k$, it follows from Lemma 2 that $S(m)$ lies in the range

$$\frac{n}{4}\left(1 - \frac{1}{(2s+1)(2k-2s-1)}\right) \pm (k-1).$$

Hence $S(m) < (n/4) + (f-1) < 3n/10$ since $f < n/20$. The minimum value of $(2s+1) \times (2k-2s-1)$ is $2k-1$ and occurs when $s = 0$ or $k-1$. Therefore, since $k \leqslant f < (n-29)/12$, it follows that

$$\frac{n}{4}\left(1 - \frac{1}{2k-1}\right) - k + 1 - \frac{n+1}{6} = \frac{n(k-2)}{6(2k-1)} - \frac{6k-5}{6}$$

$$= \frac{(k-2)}{6(2k-1)}\left(n - 12k - 8 - \frac{21}{k-2}\right) > 0$$

if $k > 2$. Hence, if $k > 2$,

$$S(m) \geqslant \frac{n}{4}\left(1 - \frac{1}{2k-1}\right) - k + 1 > \frac{n+1}{6} \geqslant \left\lfloor \frac{n+1}{6} \right\rfloor.$$

Now k cannot be 1 since $m \neq 1$. If $k = 2$, then $s = 0$ or 1 and so $m = (n+1)/2$ or $(n+3)/2$. By the symmetry,

$$S((n+3)/2) = S((n+1)/2) = \lfloor (n+1)/6 \rfloor.$$

In case (iib) when $km - ln = 2s + 1$ and $k \leqslant s \leqslant f - 1$, it follows from Lemma 2 that $S(m)$ lies in the range

$$\frac{n}{4}\left(1 + \frac{1}{(2s+1)(2s-2k+1)}\right) \pm s.$$

Hence $S(m) > (n/4) - s > (n+1)/6 \geqslant \lfloor (n+1)/6 \rfloor$, since $s < f < (n/12) - 1$. The minimum value of $(2s+1)(2s-2k+1)$ is $2s+1$ and occurs when $k = s$. Therefore, since $s < f < (n-150)/20$, it follows that

$$\frac{n}{4}\left(1 + \frac{1}{2s+1}\right) + s - \frac{3n}{10} = \frac{-n(s-2)}{10(2s+1)} + s$$

$$= -\frac{(s-2)}{10(2s+1)}\left(n - 20s - 50 - \frac{100}{s-2}\right) < 0$$

if $s > 2$. Hence, if $s > 2$,

$$S(m) \leqslant \frac{n}{4}\left(1 + \frac{1}{2s+1}\right) + s < \frac{3n}{10};$$

therefore $S(m) \leqslant \lfloor 3n/10 \rfloor$. Now s cannot be 1 since $m \neq 3$. If $s = 2$, then $k = 1$ or 2 and so $m = 5$ or $(n+5)/2$. Now, by Lemma 2, $S(5) \leqslant (4n/15) + 2 < 3n/10$, if $n > 60$. However, the conditions of Lemma 3 implicitly imply that $n > 170$, since $f \geqslant 1$. By the symmetry, $S((n+5)/2) = S((n-1)/2) = \lfloor 3n/10 \rfloor$.

The proof of the case (iic), when $km - ln = 2s + 1$ and $s < 0$, is similar to the case (iib). □

LEMMA 4. *If* $n > 530$ *and* $f = \lfloor (n-150)/20 \rfloor$, *then*

$$\frac{ln + 2f}{2k} > \frac{l'n + 2(k' - f)}{2k'},$$

where l/k *and* l'/k' *are successive terms in the Farey series of order* f.

Proof. Successive terms in a Farey series satisfy $l'k - lk' = 1$ and $k + k' \geqslant f + 1$. (See G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Ch. III.) In the range $1 \leqslant k \leqslant f$, $1 \leqslant k' \leqslant f$ and $k + k' \geqslant f + 1$, the minimum value of $2f(k + k') - 2kk'$ is $(3f^2 + 2f - 1)/2$. If $f = \lfloor (n-150)/20 \rfloor$ and $n > 530$, then $(3f^2 + 2f - 1)/2 > n$. Hence, if $n > 530$,

$$2f(k + k') - 2kk' > n = (l'k - lk')n.$$

Dividing by $2kk'$ we obtain the desired result. □

Lemma 4 implies that the conditions of Lemma 3 are satisfied when $n > 530$ and so the result is true in these cases. (This value of 530 could be reduced to about 300 by considering each of the low values of $(2s+1)(2s-2k+1)$ separately in Lemma 3 and raising the value of f appropriately.)

For $n < 530$, the result was checked using a programmable calculator.

A Generalized Vandermonde Determinant

E 2942 [1982, 274]. *Proposed by Vic Norton, Bowling Green State University.*

Show that

$$\det[a_{ij}]_1^n = \prod_{1 \leqslant p < q \leqslant r} (x_q - x_p)^{n_p n_q},$$

where $n_1, n_2, \ldots, n_r$ are positive integers with sum n and

$$a_{ij} = \binom{i-1}{k-1} x_q^{i-k} \; (= 0 \text{ if } i < k)$$

with

$$q = \min\{p: j \leqslant n_1 + n_2 + \cdots + n_p\}, \qquad k = j - (n_1 + n_2 + \cdots + n_{q-1}.)$$

(Note: if each $n_q = 1$, this reduces to the standard Vandermonde formula.)

Solution by the proposer. We view the generalized Vandermonde determinant as an appropriate limit involving the standard Vandermonde determinant, $V(h)$, on the nodes $x_j + kh$ $(j = 0, 1, \ldots, m;\ 0 \leqslant k < n_j)$ with $h \neq 0$.

Let $R(x)$ denote the column vector $R(x) = (1, x, x^2, \ldots, x^{n-1})^T$. Set $R_{jk} = R(x_j + kh)$ and let Δ be the forward difference operator on the second index; i.e., $\Delta R_{jk} = R_{j,k+1} - R_{jk}$. Then

$$V(h) = \det\left[R_{00}, R_{01}, \ldots, R_{0,n_0-1}, R_{10}, \ldots, R_{m0}, R_{m1}, \ldots, R_{m,n_m-1}\right]$$
$$= \det\left[R_{00}, \Delta R_{00}, \ldots, \Delta^{n_0-1}R_{00}, R_{10}, \ldots, R_{m0}, \Delta R_{m0}, \ldots, \Delta^{n_m-1}R_{m0}\right].$$

If we denote $D^kR_j = D^kR(x_j)$ (and $R_j = R(x_j)$), then the generalized Vandermonde determinant, V, can be expressed as

$$V = \det\left[R_0, \frac{DR_0}{1!}, \ldots, \frac{D^{n_0-1}R_0}{(n_0-1)!}, R_1, \ldots, R_m, \frac{DR_m}{1!}, \ldots, \frac{D^{n_m-1}R_m}{(n_m-1)!}\right].$$

But

$$D^kR_j = \lim_{h\to 0}\frac{\Delta^kR_{j0}}{h^k}.$$

Consequently (after factoring out the limits and scalars)

$$V = \lim_{h\to 0}\left(\frac{V(h)}{\prod_{j=1}^{m}\prod_{k=0}^{n_j-1}k!h^k}\right).$$

The final formula $V = \prod_{0\leqslant i<j\leqslant m}(x_j - x_i)^{n_in_j}$ results from directly evaluating the last limit, using the standard formula for the Vandermonde $V(h)$.

Editorial note. Th. Grundhöfer, Universität Tübingen, West Germany, pointed out that this problem is an exercise on page 119 in A. C. Aitken's book, *Determinants and Matrices*. He also mentioned that a solution of the problem, using Hilbert's Nullstellensatz, can be found in the paper "Über die Summation von Reihen aus linearen Schieberegisterfolgen" by Th. Grundhöfer and H. Lüneburg in *Archiv der Mathematik* (Basel), 34 (1980) 428–439.

In the *Mathematics Magazine*, vol. 57, no. 1, January 1984, readers may again see the result of the problem in an article by Dan Kalman entitled "The Generalized Vandermonde Matrix".

Relationships Among Eigenvalues of A, B and AB

E 2945 [1982, 333]. *Proposed by H. Kestelman (deceased), University College, London.*

A and B are $n \times n$ matrices and both have only positive eigenvalues; can AB have only negative eigenvalues? How is the result affected if A, B are both Hermitian?

Solution by B. Richter (student), University of Waterloo, Waterloo, Ontario, Canada. For general A and B, AB can have only negative eigenvalues. If A and B are both Hermitian, then AB has only positive eigenvalues.

For the general case, we present infinitely many counterexamples when $n = 2$. Let r and s be any positive real numbers and choose x to be any real number such that $x^2 \geqslant (r+s)^2$. Now let $A = \begin{bmatrix} r & 0 \\ x & s \end{bmatrix}$ and $B = \begin{bmatrix} r & -x \\ 0 & s \end{bmatrix}$. The eigenvalues of A and B are positive while the eigenvalues of AB are negative.

Let C be any Hermitian matrix. The following properties of C may be found on pages 212 and 213 of G. Strang, *Linear Algebra and its Applications*:

P1: if $x \in \mathbb{C}^n$, then x^HCx is real;
P2: if λ is an eigenvalue of C, then λ is real;
P3: there are n orthonormal eigenvectors for C.

We can deduce the following:

P4: If C has only positive eigenvalues, then for every x in $\mathbb{C}^n - \{0\}$ we have $x^HCx > 0$.

Proof. Let $x_1, x_2, \ldots, x_n$ be orthonormal eigenvectors of C, with associated eigenvalues

$\lambda_1, \lambda_2, \ldots, \lambda_n$, respectively. If x is in $\mathbb{C}^n - \{0\}$, then there are complex numbers $c_1, \ldots, c_n$, not all 0, such that $x = \sum_{i=1}^n c_i x_i$. Thus

$$x^H C x = \left(\sum_i \bar{c}_i x_i^H\right) C\left(\sum_j c_j x_j\right) = \sum_{i,j} \bar{c}_i c_j \lambda_j x_i^H x_j = \sum_i \|c_i\|^2 \lambda_i > 0.$$

Now suppose that λ is a complex eigenvalue of AB and x is an associated nonzero eigenvector. Since B is invertible, Bx is not zero. Hence $(Bx)^H A(Bx)$ is positive. Hence $\lambda x^H B^H x = \lambda x^H B x$ is positive. But $x^H B x$ is positive, so this implies that λ is positive, as required.

Also solved by M. D. Ašić (Yugoslavia), P. Bhimasankarem (India), J. M. Borwein (Canada), D. Broline, C. S. K. Chetty (India), A. J. De Santi, A. L. Duarte (Portugal), F. Gaines (Ireland), C. Georghion (Greece), M. Henriksen, O. P. Lossers (The Netherlands), W. A. Newcomb, E. T. Wong, P. Y. Wu (Taiwan) and the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by March 31, 1985. The solver's full post-office address should be on each sheet.

6474. *Proposed by Heinz-Jürgen Seiffert, Berlin, West Germany.*

Let $\Psi_n(z)$ be the $n \times n$ matrix $(\psi(z + j + k))_{j,k=0,1,\ldots,n}$, where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. Prove that

$$\det \Psi_n(z) = \left(\prod_{j=0}^{n-1} (j!)^2\right) \sum_{k=0}^{n} (-1)^k \binom{n}{k} \psi(z+k) \prod_{\substack{r=0 \\ r \neq k}}^{n} \prod_{j=0}^{n-1} (z + j + r)^{-1}.$$

6475. *Proposed by Ignacy Icchak Kotlarski, Oklahoma State University.*

Let $\{X(t): t \geqslant 0\}$ and $\{Y(t): t \geqslant 0\}$ be stochastic processes defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with $X(t)$ and $Y(t)$ both having $N(0, t)$ distributions for all $t \geqslant 0$. Let Z be a random variable defined on $(\Omega, \mathcal{F}, \mathcal{P})$ with a $\chi^2(1)$ distribution such that $\{X(\cdot)\}$, $\{Y(\cdot)\}$ and Z are independent. Let $U = X(Z)$ [i.e., $U(\omega) = X(Z(\omega))(\omega), \omega \in \Omega$] and $V = Y(Z)$. Find the joint distribution of (U, V).

6476. *Proposed by Louis Funar, University of Craiova, Rumania.*

For positive integers n and k, define $R(n, k)$ to be the remainder when n is divided by k. Let

$$S(n, k) = \sum_{i=1}^{k} R(n, i).$$

(a) Prove that $\lim_{n \to \infty} \dfrac{S(n,n)}{n^2} = 1 - \dfrac{\pi^2}{12}$. (b)* Is it true that $\lim_{n \to \infty} \dfrac{S(2^n, n)}{n^2} = 1 - \dfrac{\pi^2}{12}$?

6477. *Proposed by Louis Funar, University of Craiova, Rumania.*

Let r be the radius of the incircle of an arbitrary triangle lying in the closed unit square. Prove or disprove that $r \leqslant (\sqrt{5} - 1)/4$.

6478. *Proposed by Louis Funar, University of Craiova, Rumania.*

Let r be the radius of the incircle of an arbitrary triangle lying in a closed convex figure F of width w, and let R be the radius of the incircle of F. Are the following inequalities valid?

(i) $1/4 \leqslant (\sup r)/w \leqslant 1/2$, (ii)* $1/2 \leqslant (\sup r)/R \leqslant 1$.

SOLUTIONS OF ADVANCED PROBLEMS

Sets with Dense Projections

6421 [1983, 134]. *Proposed by János Pach, Hungarian Academy of Sciences.*

Let E be a set of points in the plane with the property that every closed disc of radius 1 includes at least one element of E. Prove that there exists a straight line L such that the orthogonal projection of E onto L is everywhere dense in L.

Solution by Curt McMullen, Harvard University. We shall establish the assertion under the weaker hypothesis that the distance $d(z, E) = o(\|z\|)$ as $z \to \infty$. We first prove the following:

THEOREM. *Let* $E \subset \mathbb{R}^2$ *be such that* $\{x/y : (x, y) \in E, y \neq 0\}$ *is dense in* $\mathbb{R}$. *Then* $\{x + \alpha y : (x, y) \in E\}$ *is dense in* $\mathbb{R}$ *for some* $\alpha \in \mathbb{R}$.

Proof. For $r \in \mathbb{Z}^+, q \in \mathbb{Q}$, let

$$G_{rq} = \{\alpha : |x + \alpha y - q| < 1/r \text{ for some } (x, y) \in E\}.$$

Because $(q - x)/y \in G_{rq}$ for all $(x, y) \in E, y \neq 0$, we easily see that G_{rq} is a dense open set. By the Baire category theorem,

$$G = \bigcap_{\substack{r \in \mathbb{Z}^+ \\ q \in Q}} G_{rq}$$

is dense in $\mathbb{R}$. Then, for $\alpha \in G$, the set $\{x + \alpha y : (x, y) \in E\}$ is dense in $\mathbb{R}$ because, given $r \in \mathbb{Z}^+, q \in \mathbb{Q}$, there is an $(x, y) \in E$ such that $|x + \alpha y - q| < 1/r$.

COROLLARY. *Suppose* $E \subset \mathbb{R}^2$ *satisfies* $d(z, E) = o(\|z\|)$ *as* $z \to \infty$. *Then there is a line* L *such that the projection of* E *onto* L *is dense in* L.

Proof. Given $a \in \mathbb{R}$, let $z_n = (na, n)$ for $n \in \mathbb{Z}^+$. Then there exists $w_n = (x_n, y_n) \in E$ such that $\|z_n - w_n\|/\|z_n\| \to 0$ as $n \to \infty$. Hence, as $n \to \infty$, $(x_n + iy_n)/(na + ni) \to 1$ and so $x_n/y_n \to a$. Therefore, E satisfies the hypothesis of the theorem. Next, given $\alpha \in \mathbb{R}$, let L_α be the line through the origin at angle $\theta = \tan^{-1}\alpha$ to the positive x-axis. Then the projection of $(x, y) \in \mathbb{R}^2$ onto L_α has signed distance

$$x \cos\theta + y \sin\theta = (x + \alpha y)/\sqrt{(1 + \alpha^2)}$$

from the origin. It follows from the theorem that there is an $\alpha \in \mathbb{R}$ such that the projection of E onto L_α is dense in L_α. In fact the set of values of α for which this holds is itself dense in $\mathbb{R}$.

Also solved by Miroslav D. Ašić (Yugoslavia), Adam Fieldsteel, Attila Mate, Walter R. Stromquist, and the proposer.

On $f(x) - \int_0^x |f|^p$

6422 [1983, 213]. *Proposed by Harvey Diamond, West Virginia University, and Bruce Montgomery, Carnegie-Mellon University.*

Let f: $[0, \infty) \to (-\infty, \infty)$ be a measurable function. If $p > 1$, is it possible that $f(x) - \int_0^x |f|^p \to \infty$ as $x \to \infty$?

Solution by A. Meir, University of Alberta, Canada. The answer is no. It is not even possible to have

$$(*) \qquad |f(x)| \geq \int_0^x |f|^p \text{ for } x \geq x_0 \geq 0,$$

unless $f = 0$ a.e. in (x_0, ∞).

Suppose that (*) holds with an f that is not zero a.e. in (x_0, ∞). Let $G(x) = \int_0^x |f|^p$. Then $G(x_1) > 0$ for some $x_1 \geqslant x_0$ and $G'(x) = |f(x)|^p$ a.e. in (x_0, ∞), and hence, by (*), $G'(x) \geqslant G^p(x)$ a.e. in (x_0, ∞). Integrating G'/G^p on $[x_1, y]$ we get the contradiction that, for all $y > x_1$,

$$\frac{G^{1-p}(x_1) - G^{1-p}(y)}{p-1} \geqslant y - x_1.$$

Also solved by Preben Alsholm, Miroslav D. Ăsić (Yugoslavia), F. S. Cater, Roy O. Davies (England), W. R. Emerson, Michael Golomb, O. P. Lossers (The Netherlands), J. R. Partington (England), Andrew Vogt, and the proposers.

Non-Invertible Operators

6423 [1983, 289]. *Proposed by P. R. Halmos, Indiana University.*

If A is a bounded linear operator on a Hilbert space, and if

$$\|1 - A\| < 1,$$

then A is invertible. That assertion, not only for operators on Hilbert spaces, but for arbitrary elements of Banach algebras, is usually proved by consideration of the infinite series $\sum_{n=0}^{\infty}(1-A)^n$, and, therefore, the proof uses completeness. Is the assertion true without completeness? Is there, in other words, a bounded linear operator A on an inner product space, such that $\|1 - A\| < 1$ but A is not invertible?

Solution. Twenty-two solvers gave simple examples to show that the assertion is false without completeness. A typical example is afforded by taking the inner product space to be the space $\mathcal{P}$ of all polynomials with inner product $(p, q) = \int_0^1 p(t)\overline{q(t)}\,dt$ and defining A by

$$(Ap)(x) = \left(1 - \frac{x}{2}\right)p(x).$$

Then $((1-A)p)(x) = \frac{x}{2}p(x)$, so that

$$\|(1-A)f\|^2 = \frac{1}{4}\int_0^1 t^2|p(t)|^2\,dt \leqslant \frac{1}{4}\int_0^1 |p(t)|^2\,dt = \frac{1}{4}\|f\|^2.$$

Hence, $\|1 - A\| \leqslant \frac{1}{2}$, but A is not invertible since it transforms no polynomial into the constant polynomial 1.

A paper in this MONTHLY, 91 (1984) 49–51, "Geometric series in incomplete normed algebras" by Fuster and Marquina, deals with the problem in a slightly different setting.

ANSWERS TO REBUSES ON PAGES 578–579

1. Ring with identity

2. Disjoint union

3. Steep function

4. One-to-one function

5. Binomial

6. Divergent series.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Gabor Szegö Collected Papers. Volume I 1915–1927, Volume II 1927–1943, Volume III 1945–1972. Edited by Richard Askey. Birkhauser, Boston, 1982. Vol. 1, xx + 857 pp.; Vol. II, x + 869 pp.; Vol. III, x + 880 pp. $180.00.

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The publication of the three volumes of *The Collected Papers of Gabor Szegö* is a fitting tribute to an illustrious career of one of our most distinguished contemporaries. Handsomely produced and carefully (one might even say lovingly) edited by Richard Askey, the volumes should bring joy and edification to all who have even the tiniest classical component in their mathematical souls.

For who could be indifferent to the theorem that a power series with only finitely many different coefficients either represents a rational function or is not continuable beyond its circle of convergence! Or that if $f(\theta), 0 \leqslant \theta \leqslant 2\pi$, is real, nonnegative, Lebesgue integrable, and nonzero on a set of positive measure with the Fourier series:

$$f(\theta) \sim \sum_{k=-\infty}^{\infty} c_k \exp(ik\theta) \tag{1}$$

then:

$$\lim_{n\to\infty} \sqrt[n]{D_n(f)} = G(f), \tag{2}$$

where $D_n(f)$ is the determinant of the $n \times n$ (Toeplitz) matrix:

$$C_n = \left((c_{i-j})\right), i, j = 1, 2, \ldots, n, \text{ and} \tag{3}$$

$$G(f) = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} \log f(\theta)\, d\theta \right\}. \tag{4}$$

This says that the nth root of $D_n(f)$ is asymptotic to a constant; in symbols $\sqrt[n]{D_n(f)} \sim G(f)$. (One might ask for a much stronger result: an asymptotic formula for $D_n(f)$ itself.)

I am picking these two examples, both from Szegö's early years, because the first is one of the many isolated jewels scattered throughout the books, and the second the beginning of an important development which is likely to continue for many years to come.

It is characteristic of most, if not all, of Szegö's work that it begins with a concrete problem. That much of it flowered into elegant general theories (e.g., orthogonal polynomials on the unit circle) is a tribute to Szegö's impeccable taste in choosing problems and to the depth of his insight. Even then no one, not even Szegö himself, could have dreamed of the extent to which some of his work would ultimately influence mathematics and science.

It is a nice twist of fate that Szegö's earliest paper where (2) is proved for the first time (the result was conjectured by Pólya) marked the beginning of a development that brought its author the greatest measure of fame, even greater than the Pólya-Szegö book of problems. The latter is, of course, an all-time classic, but it is familiar mainly to mathematicians and, even among them, it is not as well known today as it deserves to be. The complex of ideas and results generated by Toeplitz matrices and forms, all going back to Szegö's first published paper, has reached a much wider audience. The two notes in General Commentary, by Kailath and McCoy, both, unfortunately, written in a somewhat telegraphic style (as is Askey's "An overview of Szegö's mathematics") bear witness to this.

McCoy ventures a guess that, of all of Szegö's papers, "On Certain Hermitian Forms

Associated with the Fourier Series of a Positive Function" (item [52-2] of the Bibliography) has had the most applications outside of mathematics. This is certainly true and inside mathematics the beautiful and remarkable result of this paper inspired a virtual flood of improvements, extensions, generalizations, and what have you.

The result was a far-reaching refinement of (2) which gives an asymptotic formula for the Toeplitz determinant $D_n(f)$ itself. It says that under certain regularity conditions on f (which were later relaxed by several mathematicians) one has:

$$D_n(f) \sim A(f)[G(f)]^{n+1} \tag{5}$$

where $A(f)$ can be explicitly computed in terms of f.

The most remarkable thing about this result is that such an accurate evaluation of the Toeplitz determinant $D_n(f)$ is at all possible. When I first heard it from Szegö himself in the fall of 1952, when we met in Los Angeles to discuss the final details of our joint paper with W. L. Murdock (item [53-4] of the Bibliography), I just couldn't believe it. My skepticism was reinforced when, upon returning to Ithaca, I discovered that Szegö's result implied a remarkable probabilistic identity—so remarkable in fact, that it had to be either wrong or known. It was neither, for Szegö's mastery of the methodology, based in orthogonal polynomials on the unit circle, which he himself invented, did not allow for error. It then seemed natural to reverse the procedure and derive Szegö's formula as a consequence of the probabilistic identity. This was done mainly through the efforts of Chung and Hunt. At the end, Szegö's purely analytic formula (4) turned out to be essentially equivalent to a purely combinatorial identity discovered by Hunt. This is only one of many facets of Szegö's result. Another is its algebraic face (exploited in a masterly fashion by Glen Baxter) bringing it into contact with Banach Algebras. It must have amused Szegö, the super classical analyst, to find that he had been speaking this brand of prose.

Throughout his scientific life, Szegö worked mainly in four areas: function theory (of one complex variable!), classical orthogonal polynomials, isoperimetric inequalities (mainly in collaboration with Pólya), and Toeplitz forms. That I have devoted most of my allotted space to the last of the four is not because Szegö's work in the other three was any less elegant or profound. It is just that Toeplitz forms turned out to be so spectacularly fruitful and I would venture a guess that they were Szegö's favorites. That "on revient toujours à ses premières amours" is literally true in Szegö's case. As late as 1960, he returned once more to his first love in an important paper, written jointly with Henry Helson, on prediction theory.

It is, of course, impossible in a brief review to do justice to so rich and varied a "Nachlass" as Szegö's collected papers are. They are a monument to the vitality of classical analysis and to the virtuosity of their author. Charles Löwner, himself one, called Szegö a virtuoso. The *Collected Papers* more than confirm this judgment.

A Hilbert Space Problem Book. Second Edition. By Paul R. Halmos. Graduate Texts in Mathematics No. 19. Springer-Verlag, New York, 1982. xvii + 369 pp. $28.00.

PETER FILLMORE AND NIGEL HIGSON
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The theory of Hilbert spaces and operators has its origins in the theory of integral equations as developed by Fredholm, Hilbert, and others around the turn of the last century. Thus the equation

$$g(x) = f(x) - \int_a^b k(x,y)f(y)\,dy,$$

which had arisen in the study of the Dirichlet problem, was completely analysed by Fredholm.

For example, he established the so-called Fredholm alternative: a solution f exists for every function g if and only if the homogeneous version of the equation has just the trivial solution $f = 0$. These studies were followed by a fundamental series of six papers published by Hilbert in the period 1904–10, which not only advanced the theory of integral equations but also laid the foundations for the field that now bears his name. Whereas Fredholm had proceeded by approximating the integral equation by systems of linear equations in finitely many unknowns, Hilbert's method was to replace the integral equation by an equivalent system of infinitely many linear equations in infinitely many unknowns. This replacement is accomplished using complete orthonormal systems of functions. For example, a square-integrable function may be represented by a sequence of "Fourier coefficients" f_n, and hence by a point of the infinite coordinate space l^2 of square summable sequences of real numbers. Fredholm's equation then amounts to a system of linear equations

$$g_i = f_i - \sum_j k_{ij} f_j,$$

relating the coefficients of f and g. In other words, the integral equation becomes an infinite matrix (or operator) equation

$$g = (I - K)f.$$

Thus Hilbert was led to study linear transformations (actually under the guise of bilinear forms) of the infinite dimensional space l^2. He introduced the notions of continuity and compactness for linear operators. He extended the spectral theorem of matrix theory to the context of bounded symmetric operators on l^2, and as a corollary obtained the spectral theorem for compact symmetric operators. Subsequently he developed Fredholm's theory in the new abstract context, proving, for example, the following version of the Fredholm alternative: If K is a compact operator, then $I - K$ is one to one if and only if it is onto.

Further investigations along these lines were made by F. Riesz, E. Schmidt and others during the period of publication of Hilbert's papers, and by 1912 Hahn and Hellinger had published solutions to the problem of classifying self-adjoint operators up to unitary equivalence. In the years immediately following, there was little significant activity in the area and interest was rekindled only in the latter half of the 1920's by the mathematical development of quantum mechanics, principally at the hands of von Neumann. Thus in two papers in 1929 he axiomatized Hilbert space, obtained the most general forms of the spectral theorem, and initiated the study of operator algebras. The theory of these algebras was developed by von Neumann and F. J. Murray in their famous series of papers "On rings of operators" in 1936–43, and in fundamental papers of M. H. Stone in 1940 and of Gelfand and Naimark in 1943 which began the theory of C^*-algebras. In a parallel development, the year 1930 brought the publication of Stone's theorem on one parameter groups of unitary operators, a result motivated by the study of the dynamical group in quantum mechanics. Stone's theorem, together with earlier work on symmetry by the physicist E. Wigner, marked the beginning of the theory of infinite dimensional unitary representations of topological groups, a subject having numerous close connections with the theory of operator algebras.

In the period since 1950 operator theory has developed in a number of directions. To begin with, Hilbert space methods have been introduced not only in the theory of integral equations, but also, for example, in the study of ordinary and partial differential equations, in complex analysis, and even in algebraic topology. Secondly, the theory of operator algebras has developed with great vigor since its beginnings forty years ago. A third direction has been the study of the behavior of individual Hilbert space operators. This is Halmos's field and the subject of his book. As a class these operators appear to be too varied to permit many interesting general statements to be made. Consequently attention has been focused more upon suitable interesting families of operators, with a view to obtaining unitary invariants and structure theorems for these. However one should note the extensive theories of triangular models (Brodskii and Livsic) and canonical models

(Sz. -Nagy and Foias). A general question which arises here is the existence of invariant subspaces, and although the solution in general seems beyond reach, several important advances have been made. In the 1930's von Neumann proved the existence of proper invariant subspaces for compact operators (although his proof was never published). This result was subsequently re-examined and generalized by a number of workers, most notably by Lomonosov in 1973 who extended it to operators which merely commute with a nonzero compact. Another notable result was Scott Brown's proof in 1978 of the existence of invariant subspaces for subnormal operators, a class of operators first studied by Halmos (see Ch. 21 of the *Hilbert Space Problem Book*).

A recurring theme is the perturbation of an operator by a compact operator, beginning with the Fredholm alternative and a 1909 result of Weyl on the invariance of the limit points of the spectrum under such perturbations. This line of development leads to some surprising connections between Hilbert space operators and topology, beginning with the fact (Atiyah and Janich, 1964) that the Fredholm operators form a classifying space for complex K-theory. The study of operators that are normal modulo the compacts (Brown, Douglas, and Fillmore, 1973) has led to a good deal of interest in "noncommutative topology." An example of recent progress is the astonishingly simple proof, due to J. Cuntz, of the Bott periodicity theorem—made possible by extending the context to C^*-algebras.

Because of the diversity of topics that it includes, single operator theory is ideally suited to a book of the nature of Halmos's (we can imagine a Homotopy Theory Problem Book being less successful). The second edition contains 250 problems as against 199 in the first. It remains an excellent, compact survey of single operator theory, as well as a valuable source for Hilbert space techniques. The form, content, and style of the second edition deserve the same high praise as the first.

The Fractal Geometry of Nature. By Benoit B. Mandelbrot. Freeman, San Francisco, 1982. 460 pp.

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> He judges a... critical book less by the text than by the quotations.... W. H. Auden [**1**, p. 47].
>
> Again, any conscientious critic who has ever had to review a new volume of poetry in a limited space knows that the only fair thing to do would be to give a series of quotations without comment.... W. H. Auden [**1**. pp. 11–12].
>
> A fractal is a mathematical set or concrete object that is irregular or fragmented at all scales.... B. B. Mandelbrot [**3**].
>
> ...I claim that many patterns of Nature are so irregular and fragmented, that, compared with *Euclid*—a term used in this work to denote all of standard geometry—Nature exhibits not simply a higher degree but an altogether different level of complexity.... The existence of these patterns challenges us to study those forms that Euclid leaves aside as being "formless," to investigate the morphology of the "amorphous." B. B. Mandelbrot [**4**].
>
> All pulchritude is relative.... We ought not to believe that the banks of the ocean are really deformed, because they have not the form of a regular bulwark; nor that the mountains are out of shape, because they are not exact pyramids or cones; nor that the stars are unskilfully placed, because they are not all situated at uniform distance. These are not natural irregularities, but with respect to our fancies only; nor are they incommodious to the true uses of life and the designs of man's being on earth.... Quoted from Richard Bentley by B. B. Mandelbrot [**4**].

Benoit B. Mandelbrot, in his book entitled *The Fractal Geometry of Nature*, systematically sets about convincing the reader's eye and mind that many of Nature's apparent irregularities can be efficiently and beautifully modelled by mathematical objects, some deterministic and some with a

random component, but all highly irregular or fragmented, which Mandelbrot chooses to call *fractals*.

Each newly created computer-generated illustration makes the argument more compelling. On the rear jacket of the book, a Brownian earth that never was rises over a moonscape that never was, both created by the computer artistry of one of Mandelbrot's coworkers at IBM, Richard F. Voss. The programs are based on simple geometric invariance principles. More recent programs create mountain ranges, valleys, rivers, clouds, and mists, all visually compelling, totally fake, again based on a few simple invariance principles.



Computer-generated fractal landscape surface of Hausdorff dimension approximately 2.1 realized by Richard F. Voss. From *The Fractal Geometry of Nature*, W. H. Freeman and Company©, Benoit B. Mandelbrot 1977, 1982, 1983.

Mandelbrot suggests tentative first fractal models for coastlines, galaxy clusters, turbulence, island clusters, trees, the lungs, Brownian motion, drainage systems, mountain ranges, irregular textures, error bursts in data transmission, stock market variations, and many more. The book is a rich source of beautiful pictures, interesting new mathematical models, and imaginative new terminology.

Mandelbrot presents a wide range of experimental evidence by other scientists to support his views that standard models fall short of explaining the same phenomena with the same efficiency, clarity, and adherence to fact. Mandelbrot describes the book as a manifesto and a casebook. It is full of strongly held opinions and claims of priority, full of historical anecdotes, apt illustration, and the best in computer art.

The classical mathematical fractals are the nowhere-differentiable continuous functions of Weierstrass, the space-filling curves of Peano, and the famous Cantor sets.

The classical view was that these examples were forced upon us by the abstractions of mathematics and that they bore little relationship to reality.

Here are some of the opposing views expressed by Mandelbrot.

As regards nowhere-differentiable functions. Mandelbrot quotes Jean Perrin, whose work on Brownian motion won Perrin the Nobel Prize.

> ...A good teacher... draws a well-defined curve and says, holding a ruler, "You see that there is a tangent at every point."... Mathematicians, however, are well aware that it is childish to try to show by drawing curves that every continuous function has a derivative.... Curves that have no tangents are the rule.... Those who hear of curves without tangents, or of functions without derivatives, often think at first that Nature presents no such complications.... The contrary however is true.... Consider, for instance, one

of the white flakes that are obtained by salting a solution of soap. At a distance its contour may appear sharply defined, but as we draw nearer its sharpness disappears. The eye can no longer draw a tangent at any point.... The use of a magnifying glass or microscope leaves us just as uncertain, for fresh irregularities appear every time we increase the magnification.... An essential characteristic of our flake... is that we suspect... that any scale involves details that absolutely prohibit the fixing of a tangent.... [4, pp. 6–7]

According to Norbert Wiener, the paths of Brownian motion may best be modelled by nowhere-differentiable, continuous, locally plane-filling curves. [4, p. 9]

As regards Peano curves, Mandelbrot writes,

...if a tree made of rivers of vanishing width is to drain an area thoroughly, it *must* penetrate everywhere. One who follows the rivers' combined bank performs a plane-filling motion.... [4, p. 59]

Thus a curve of positive area has intuitive value as a model and the analogous higher dimensional object, a two-dimensional surface with three-dimensional volume is just as important:

...Harvey led to a view of the circulation of the blood which asserts that both an artery and a vein are found within a small distance of every point of the body—except of course that points *within* an artery (or a vein) are prevented from being very close to a vein (or an artery).

Stated differently...: every point in nonvascular tissue should lie on the boundary between the two blood networks.

A second design factor is that blood is expensive. Hence the volume of all the arteries and veins must be a small percentage of the body volume, leaving the bulk to tissue.... (Therefore) tissue... is a fractal surface:... (it) must be topologically two-dimensional, because it forms the common boundary of two shapes that are topologically three-dimensional, but it is required to have a volume that not only is nonnegligible compared to the shapes it bounds, but is much larger.

(Thus) *fractal monsters are the very substance of our flesh*.... [4, pp. 149–150]

Mandelbrot notes that the best model for the clustering of the 3-dimensional mass of the stars and galaxies in the universe seen from afar seems to be not 3-dimensional but rather the Cantor set (or "dust" as Mandelbrot prefers to call it) of topological dimension 0; Mandelbrot then recalls an argument of E. E. Fournier d'Albe which suggests that the corresponding nontopological, geometric or fractal dimension of the clustering mass should be approximately 1. We quote only the conclusion of the argument.

...To paraphrase Fournier, an important conclusion may be drawn from the observation that no stellar velocity exceeds 1/300 of the velocity of light. It is that the mass comprised within a world ball increases as its radius, and not as its volume... . [4, p. 89]

Experimental evidence seems to suggest a more precise fractal dimension of about 1.23.

In conclusion, Mandelbrot argues that all of the classical fractal monsters are good intuitive models of real and important objects.

Though Mandelbrot spends the first half of his book (200 pages) on the classical and deterministic fractals, it seems clear to me that his love is attached to the random fractal models. The principal phenomena modelled are landscapes, galaxies, and coastlines.

...The full meaning and practical relevance of these (basic fractal) themes are not apparent until one tackles random fractals.... Let us go back to the question "How long is the coast of Britain?"... One might improve the (deterministic fractal) model by invoking more complicated deterministic algorithms. However, this approach would be not only tedious, but doomed to failure, because each coastline is molded throughout the ages by multiple influences that are not recorded and cannot be reconstituted in any detail.... The key out of this difficulty lies in statistics.... [4, pp. 200–201]

...Can chance bring about the strong degree of irregularity encountered, say, in coastlines? Not only does it, but in many cases it goes *beyond* the desired goal.... [4, p. 201]

...The preceding section argues that the theory of change is not *really* difficult. Unfortunately, it is not really easy.... Indeed, rules that generate acceptable random curves are very hard to describe, because geometric sets are imbedded in a space. By merely varying at random the shapes, the sizes, and the order of a coastline's parts, one tends to be left with pieces that will not fit together.... [4, p. 201]

> ... The nonrandom fractals' essential failing is that they are not symmetric enough.... [4, p. 205]

Mandelbrot's goals in introducing random fractals are twofold. First, he wants models that are efficient yet visually correct. Secondly, he wants models that satisfy the rather surprising statistical invariance properties that have been suggested by experiment. Amazingly, he discovers that satisfying the second property also seems to satisfy the first.

The most important thing to note about the invariance properties Mandelbrot describes is that many are demonstrably incompatible with standard Gaussian distributions.

> ... The study of scaling laws led me to recognize that nonstandard central limit behavior is in fact part of nature. Unfortunately, as soon as the central limit theorem argument is recognized to have more than one possible outcome, it ceases to be persuasive.... [4, p. 423]

Mandelbrot recognized that a large number of real-world, apparently random processes seemed to satisfy the following general notion of random self-similarity or scaling.

> ... Any natural fluctuation can be processed to be heard—as implied by the term *noise*. Tape it, and listen to it through a speaker that reproduces faithfully between, say, 40 Hz to 14,000 Hz. Then play the same tape faster or slower than normal. In general, one expects the character of what is heard to change considerably. A violin, for example, no longer sounds like a violin.... There is a special class of sounds, however, that behave quite differently. After the tape speed is changed, it suffices to adjust the volume to make the speaker output "sound the same" as before. I propose that such sounds or noises be called *scaling*.
>
> White Gaussian noise... is scaling. But other scaling noises can be made available for model making.... [4, pp. 249–250]

And the members of one important family of such scaling noises are distinguished by a certain fractal dimension [4, p. 250]. It is the members of this family that Mandelbrot often uses to wonderful advantage.

Fractals have existed in fact if not in name in mathematics ever since mathematicians began to think seriously about the foundations of their subject. Fractals have dominated certain areas of mathematics for decades. But Mandelbrot's thought and the interest evoked by his book are symptomatic of a real change in the mathematical milieu. What is that change?

I believe that the change has been wrought by the evolution of the electronic computer and the development of marvelous computer graphics. As Mandelbrot argued in a quotation cited in the previous section, any stream of data can be processed as sound or noise. But also it can be processed as geometric shapes or pictures. One discovers that algorithms have intrinsic shapes, that those shapes are complex and beautiful, that they have underlying geometries, underlying self-similarity or invariance properties. Mathematicians are in the new position of being able to explore extensively an almost unchartered area, the geometry of algorithms.

> What's in a name? That which we call a rose By any other name would smell as sweet.... Shakespeare, *Romeo and Juliet*, II, ii, 43.

Or would it? Let us choose as a name for each individual the algorithm encoded in its genetic structure or in the governing physical or spiritual laws of its creation. If we change the name, then we change the code, change the individual, change the shape, smell, and feel of the individual.

If we think in abstract terms, then it amazes us that algorithms have shapes, but when we realize that much of life about us is algorithmic, then we see and smell and hear in every face, bud, flower, and tree, the shape, smell, and sound of an algorithm, a marvelous creation.

It seems to me that what Mandelbrot is studying in his *Fractal Geometry of Nature* is the asymptotic geometric shape of algorithms. Irregularity, fragmentation, fractal dimension, and other important properties which he emphasizes are then simply important diagnostic measures of this asymptotic shape. Therefore I would redefine fractal geometry for the purpose of mathematics as follows.

> FRACTAL GEOMETRY is the study of the asymptotic geometric shape of algorithms. A FRACTAL is the asymptotic geometric shape of a particular algorithm.

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different... Goethe.

It is clear that many beautiful things are crying to be discovered in fractal geometry. To what extent we shall succeed is not clear. As we attempt to understand complex, seemingly random processes, we often feel like Mark Studdock who was being trained in Objectivity by his diabolic mentor in C. S. Lewis's science fiction novel, *That Hideous Strength*:

> ... Then (Mark) noticed the spots on the ceiling.... They were deliberately painted on: little round black spots placed at irregular intervals on the pale mustard-coloured surface.... He determined that he would not fall into the trap of trying to count them. They would be hard to count, they were so irregularly placed. Or weren't they? Now that his eyes were growing used to them (and one couldn't help noticing that there were five in that little group to the right), their arrangement seemed to hover on the verge of regularity. They suggested some kind of pattern. Their peculiar ugliness consisted in the very fact that they kept on suggesting it and then frustrating the expectation thus aroused. Suddenly he realised that this was another trap. He fixed his eyes on the table.
>
> There were spots on the table too: white ones. Shiny white spots, not quite round. And arranged, apparently, to correspond to the spots on the ceiling. Or were they? No, of course not... ah, now he had it! The pattern (if you could call it a pattern) on the table was an exact reversal of that on the ceiling. But with certain exceptions. He found he was glancing rapidly from one to the other, trying to puzzle it out.... [2, pp. 297–298]

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1. W. H. Auden, The Dyer's Hand, Vintage Books, New York, 1968.
2. C. S. Lewis, That Hideous Strength, Macmillan, New York, 1969.
3. B. B. Mandelbrot, Fractals: form, chance, and dimension, Freeman, San Francisco, 1977.
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LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405 (after December 15, 1984: Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053).

What follows was NOT submitted as a letter to the editor. It is an article that appeared in somewhat shortened forms in several newspapers, and several readers have suggested that it be made more widely available to the members of the Association. There is no standard way to include non-standard comments such as this in the MONTHLY: the decision to have it appear in this department has the author's permission.

Editor:

In 1968, when I was an undergraduate, many students were advised to major in Library Science: there was a need in large libraries for a new generation of librarians conversant with the modern concepts of data storage and retrieval. Within a few years, the need for such librarians dried up; indeed many find it hard to believe that there ever was such a need.

Today the hot college major is Computer Science. At UCLA nearly one third of entering freshmen express an interest in majoring in Computer Science. The situation is similar on campuses across the nation. Computer Science is very sexy stuff these days. Glossy magazines are

filled with stories about computers and hackers. Kids as young as 14 are becoming wealthy by designing video games and software. What college freshman would not be seduced by this apparently glamorous profession? But there's the rub. *What profession*? Who has taken the time to identify this profession? What are the employment prospects for Computer Science majors? The glossy magazine articles deal primarily with startling advances in hardware and software, and with the general move toward universal computer literacy; but little is said about employment for *professionals trained in Computer Science*. When will *these people* become a glut on the market?

The answer is that the glut has already begun. Job recruiters for major corporations (including IBM!) have taken a second look at Computer Science majors and decided to look elsewhere. Where are they looking? They are looking at a group which for the last 5000 years has sat quietly in the background: mathematicians. Why now? Here is one answer from a job recruiter for a big corporation who wanted to hire people to do software verification: "I know that software verification sounds like Computer Science, but I don't want to hire a Computer Science major. I want someone who can *think*. I want a math major." In a less acerbic form, this assertion is being repeated by recruiters at Westinghouse, GE, Hughes Aircraft, TRW, and other high tech companies across the country. How can this widespread dissatisfaction with Computer Science be reconciled with the obvious fact that there *is indeed* a computer revolution going on around us?

The answer is very simple. Of course we should all be computer literate. Even the smallest modern business office keeps its accounts on a microcomputer. Soon the preferred way to pay bills and to conduct correspondence will be on a terminal. But one doesn't need to major in Computer Science in order to do these things any more than one needs to study at the Cordon Bleu in order to prepare one's supper. I occasionally need to do computing in my mathematical research, and (with no prior experience) I learned all the Fortran that I needed in just one afternoon. Is this because I am some kind of Computer genius? No—it's because I am a mathematician. The ability to write a program requires that one be familiar with certain patterns of logical thought, and that he memorize a few details of a particular programming language. The latter are trivial, while the former are the stock in trade of the mathematician.

Students have been concentrating on learning the mechanics of using the computer and not considering what is done with a computer in the real world. Of course there are extensive real world applications to information storage and retrieval, but employment in this field consists primarily of glorified stenographic jobs. There are also reasonable job opportunities in designing software (although I, for one, would not want my future riding on the software market). But what of the Computer Science major who wants to become part of the forefront modern scientific activity? What will he be doing with a computer? The answer is that he will be doing mathematical modelling, solving differential equations, doing advanced probability theory, and manipulating matrices: in short, doing mathematics.

I have helped many of my students, all mathematics majors, to obtain permanent employment. When prospective employers telephone me for recommendations, the litany is always the same: "We have lots of engineers and computer scientists on our staff but we keep coming up against all sorts of mathematics problems. We need a math whiz." I always ask what computer background is required and the answer is generally "It would be nice if he could program in FORTRAN and APL, but if not we will train him. Math majors catch on quickly."

The last remark is the most telling. Math majors are not only trained in a number of advanced mathematical techniques, but they are also trained in abstract reasoning, modelling, and problem solving. These methods lend themselves well to many professions, from engineering to law to physics to medicine. And the *professions* are finally becoming aware of the value of the mathematical method.

Computer scientists, physicists, and engineers frequently do not realize that the technical problems with which they struggle on a daily basis are *mathematics*, pure and simple. I have often received phone calls from engineers which begin "We have a problem which we've been kicking around for a long time and we thought that perhaps you could help us." Invariably the problem

involves techniques which are well known to the working mathematician and I can answer the question instantly. People, *even technical professionals*, do not appreciate the pervasive nature of mathematics. One is reminded of the gentleman who discovered at an advanced age that all his life he had been speaking prose. For a long time now, the technical world has been speaking mathematics, but it is only beginning to become aware of it.

It is a sobering fact that the invention of the carburetion system on the Volvo, the creation of the fuel efficient wing on the Boeing 767, the use of prime numbers to create the new generation of "unbreakable" codes, the use of optical fibres in communications, the use of the fast Fourier transform in the resolution of photographs from space vehicles, and the use of satellites for communication purposes are all the products of work done by so-called pure, theoretical mathematicians. Why is the public unaware of the mathematician's important role in modern technology? Mathematicians are rarely encountered in the popular press. While Jobs, Wozniak, and other computer celebrities frequent the covers of popular magazines, only one mathematician has *ever* been on the cover of TIME magazine, and that was because *he invented the stored program computer*!

Mathematicians do not make good PR copy; we do not give splashy interviews. The average research mathematician cannot explain his work in a few sentences. A biologist can toss off a few generalities about genetic engineering and an astronomer can refer glibly to cosmology and black holes, but if I tell you that I work on "biholomorphic mappings of pseudoconvex domains in multidimensional complex space," you are unenlightened and perhaps even annoyed.

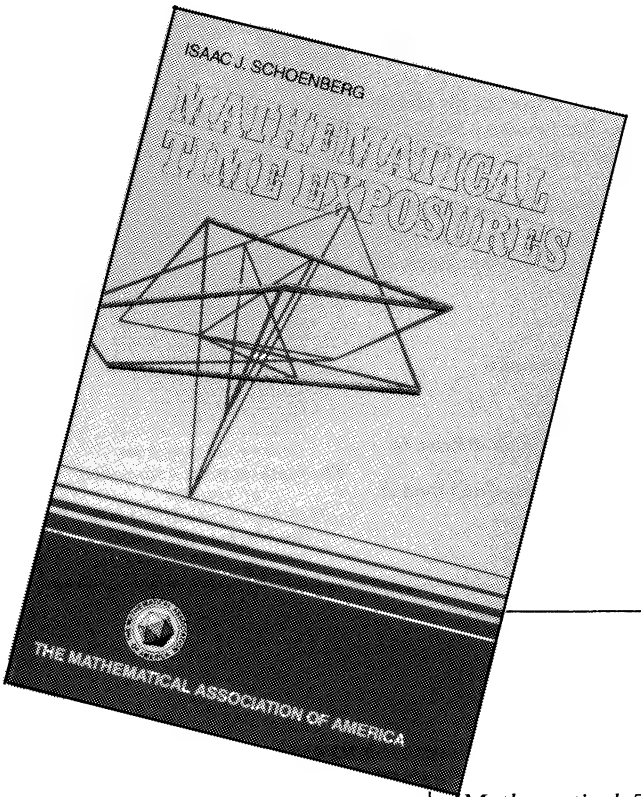
Abstruse we may appear, but industries and government are waking up to the fact that mathematicians are among the most gifted, creative, and able scientists around. Many students tell me "I want to major in math but my parents won't let me because they think that I won't get a job. They told me to major in Computer Science." Pity the parents *and* the students. The Department of Labor predicts that between now and the year 2000 there will be a need for about 150,000 Computer Scientists. Countrywide, there are over half that many students enrolled in university Computer Science programs *right now*.

Computer Science is in most instances not an end in itself: it is a *tool*. The world needs only a limited number of Computer Scientists. It is perfectly clear that we should all be able to use language effectively but that there should be few linguists; likewise, we should all be computer literate but there should be few Computer Scientists. I sincerely believe that many high school counselors and teachers are ignorantly misguiding their students into taking up Computer Science. Computer Science is fun and it is glamorous. Mathematics is hard work and it is not glamorous. But mathematics is important and it is rewarding. Computer Science did not exist twenty-five years ago; will it exist twenty-five years from now? Mathematics has been around for 5000 years and will be here for another 5000. While new mathematical ideas are always developing, the old ones still remain valid and useful. In Computer Science, however, last week's news is this week's trivia. The subject is all hardware and few ideas. If corporate job recruiters prefer math majors to Computer Science majors, isn't it time for a reevaluation?

If you are thinking of becoming a Computer Science major, or advising someone to do so, then ask yourself this question: "What is Computer Science?" The answer is that no one knows. The successful professional of the future will have a couple of computer science courses under his belt; he will know how to use a machine. But he will be an expert in a subject with content, with a past, and with a future. How about mathematics?

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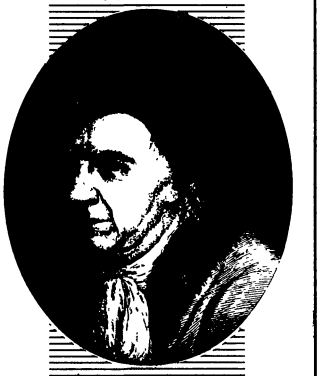
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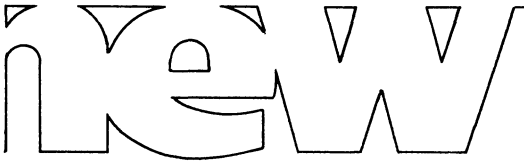
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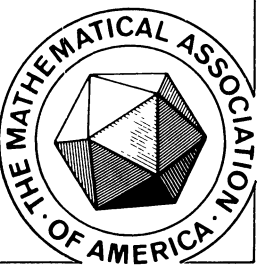
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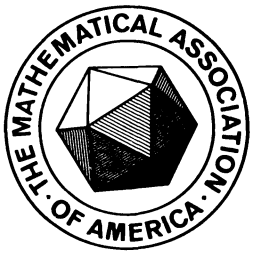
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Professor David E. Dobbs, University of Tennessee, has informed us that his paper, titled "Discs and shells," and published in the *MATYC Journal*, 11 (1977) 35–37, contains the results appearing in "The disk and shell method", this MONTHLY, 91 (1984) 139, by Professor Charles A. Cable.

Dr. Ruma Falk wishes to correct her affiliation in the paper "Multiplicative analogues of some statistics", this MONTHLY, 91 (1984) 198–202. She is in the Department of Psychology, The Hebrew University, Jerusalem 91–905, Israel.

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NOTICE TO AUTHORS

See statement of editorial policy (volume 89, p. 3). Follow the format in current issues. Put your full mailing address in a line at the head of the paper, following the title and the author's name. Send three copies of the manuscript, legibly typewritten on only one side of the paper, double-spaced with wide margins, and keep one as protection against loss. Prepare illustrations carefully, on separate sheets of paper in black ink, the original without lettering and two copies with lettering added. Rough copies of the illustrations should be included in the manuscript at the appropriate places. Supply the full five-symbol Mathematics Subject Classification number, as described in Mathematical Reviews, 1980 and later. (This is necessary for indexing purposes.) Send all main articles, letters to the editor, and editorial correspondence to the editor. Contributions to the other four special departments (Notes, Teaching of Mathematics, Unsolved Problems, and Problems and Solutions) should be sent to the appropriate one of the associate editors, whose addresses are given below.

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A LIFE IN MATHEMATICS REMEMBERED

ANGUS E. TAYLOR
82 Norwood Avenue, Kensington, CA 94707

When it was suggested to me that an article of reminiscences might be welcomed by the MONTHLY, I found that I could not resist the opportunity to review my love affair with mathematics and go public with some of my memories and reflections about student days, mathematicians who have influenced me, some of the mathematicians I have known, and about my life as a scholar and teacher.

I began life in the West (first Colorado and then California), went east to be an undergraduate at Harvard, returned to California to earn a Ph.D. at the California Institute of Technology, went east again to spend a year at Princeton as a Fellow of the National Research Council, and then came back to California for a career at UCLA (1938–66). During that career there were interludes: about a year in England during World War II as an operations analyst with the U.S. Army Eighth Air Force, a sabbatical year at Cambridge University soon after the war, another sabbatical as a Fulbright research scholar at the Johannes Gutenberg University in Mainz, West Germany, and still another sabbatical in Switzerland in 1961–62. During the academic year 1965–66 I made the transition from the life of a professor to the life of a full-time university administrator and essentially ended my career as a teacher and research scholar in pure mathematics. During my years in administration (1965–77), first as Vice President for Academic Affairs of the 9-campus University of California, then University Provost, and finally Chancellor of the UC's Santa Cruz campus, I could not keep up with the requirements of on-going research in my speciality—functional analysis. I did not give up scholarship, however. Since 1966 I have been able to write expository and historical papers about analysis, functional analysis, and the life and work of Maurice Fréchet; and I have been able, with the skilled assistance of my collaborators W. Robert Mann and David C. Lay (the latter one of the last of my Ph.D. students at UCLA), to keep alive my books on advanced calculus and functional analysis.

As a boy in school I did well in all subjects, but especially well in mathematics. However, I showed no really remarkable propensities as a budding mathematician. I learned only the usual things in my school mathematics before going to Harvard. In high school, in Pomona, California, I had an unusually good teacher, William A. Bartlett, gray-haired, wrinkled of face, precise in speech and possessed of dry humor, who impressed and inspired me, especially in plane geometry and trigonometry, as he made the subjects stimulating, satisfying, and powerful. In Euclidean geometry we were called on to go to the blackboard and demonstrate theorems. He scuttled the efforts of students to get by on memory without understanding by putting up a diagram in which a few of the important points were labelled in a way that mixed up the lettering used on the corresponding diagram in the text. He smiled gently and kindly when a student faltered in confusion as a result of this tactic; he then impressed on us all that we were supposed to grasp the reasoning in the proof of the theorem. Once we understood the reasoning, he said, we should be able to carry on and not be shackled by a need to adhere to the notation used in the book.

Mr. Bartlett challenged his class with a problem not in the text and promised a grade of A to any student who was able to provide a correct solution. Here is the problem: A triangle ABC is inscribed in a circle. A point P on the circle is given, distinct from A, B, C. Through P three lines are drawn, perpendicular respectively to the lines BC, CA, and AB (produced if necessary). Let D, E, F be the points where these perpendiculars intersect the lines forming the sides of the

Angus E. Taylor: I was born in 1911 on a homestead in northwestern Colorado. Information about my mathematical education is contained in the present memoir. Aside from mathematics and my wonderful wife and family, my great passions have been for mountains, music (especially opera), history, and literature. I have climbed a great many high mountains (by not inordinately difficult routes) in the California High Sierra and in the Swiss Alps, including many of the major summits accessible from Zermatt, Saas Fee, and Pontresina.

triangle. Prove that D, E, F are collinear. No member of the class was able to devise a proof. I tried long and hard without success, but I earned an A anyway. Only later did I find a proof, using trigonometry.

My parents were poor, but they wanted their children to have good educations. My older brother, after graduating from the University of Colorado, supported himself entirely while earning a Ph.D. in Economics at Harvard, after which he obtained a place on the Harvard faculty. Inspired and encouraged by my parents and my brother, I was able to make it through Harvard (1929–33) mainly on scholarships, supplemented by odd-job earnings and loans from my brother. Tuition at Harvard then was $400 a year, while room and board cost me about $475.

My first mathematics at Harvard was the freshman course in analytic geometry and calculus using a geometry text by W. F. Osgood and W. C. Graustein, and Osgood's *Introduction to the Calculus*. Both authors were on the Harvard faculty. My instructor was Marston Morse. He was a good teacher, but now and then he would fail to appear in class. We waited ten minutes and then left—that seemed to be an established custom in such cases. In the sophomore year we finished Osgood's calculus and had some analytic geometry in three dimensions; there was nothing on double or triple integrals and very little on partial differentiation. In that second year, as I recall, my teachers were Marshall H. Stone and George D. Birkhoff, in successive semesters. At that time I did not know anything about the research work of the various members of the Harvard faculty, except for a report that Professor Birkhoff had done something outstanding. I don't think I knew anything about contemporary research in mathematics except what could be inferred from looking at the advanced topics listed in the catalogue of courses. Not until a later year did I become acquainted with some mathematics journals and the library of mathematics books then in Widener Q, a room in the huge Widener library. In my sophomore year I got acquainted with Garrett Birkhoff, one year ahead of me as an undergraduate, and far more knowledgeable. We both lived in Lowell House, then brand new. Garrett impressed on me that, to succeed as a mathematician, one had to be utterly dedicated to the subject. I had no reservations about my commitment to mathematics, even though I had as yet little understanding of what would be involved in making a successful career as a mathematician on a university faculty, other than to be a scholar and teach well. I was sure I could do both.

As I continued my studies I always had regular members of the faculty as mathematics teachers —never a graduate student, and only one teacher (Stone) of rank below that of full professor. Joseph Walsh was then an associate professor and David Widder was an assistant professor, but I didn't get to know either of them until some years later. Stone went to Yale as an associate professor the year after I took a class under him. I got acquainted with Sud (J. S.) Frame, who was an instructor and resident tutor in Lowell House. G. Baley Price was at Harvard then, as an instructor and tutor, but I didn't meet him until years later, when we were both operations analysts with the Eighth Air Force, in England.

The other faculty members under whom I studied were Osgood, E. V. Huntington (an excellent teacher), and Julian Lowell Coolidge, my tutor and Master of Lowell House.

In those days theoretical mechanics (often called rational mechanics) had an important place in the Harvard mathematics curriculum. There were sections on the dynamics of a mass particle in Osgood's elementary calculus text. There I learned about the velocity needed to project an object entirely away from the earth (neglecting the effect of air resistance). In my sophomore year I took from Huntington a whole year course in statics and in the mechanics of systems of particles and rigid bodies in planar motion. There I learned how to deduce Kepler's laws from Newton's laws of motion and gravitational attraction. In my junior year I had a year course in dynamics that was to have been taught by O. D. Kellogg (author of a fine book on potential theory), but he died suddenly during the preceding summer. The course was taught by Osgood, who had a great interest in the subject and had published important papers about the motions of gyroscopes and tops. I believe it was in that year that he decided to write his book, *Mechanics*, which appeared in 1937. My experience at Harvard made me an enthusiastic teacher of mechanics at UCLA on several occasions after I went there.

Osgood had more influence on me than any other teacher at Harvard. Nearly all the analysis I learned at Harvard was closely related to his books and his teaching. In my junior year he was my teacher in a course based on his book, *Advanced Calculus*. In my senior year he taught the famous course Mathematics 13, for graduate students and qualified undergraduates. No text was assigned, but in fact the lectures in the course were based on the first volume of his treatise in German, *Lehrbuch der Funktionentheorie*; it was in part rigorous theory of continuous functions, with the necessary supporting portions of general topology (but not called by that name), and in major part the theory of analytic functions of a complex variable, with some things about logarithmic potential and Riemann surfaces. I could read German; I bought the *Lehrbuch* and studied it thoroughly. The inspiration and the influence were lasting.

Osgood was the senior member of the Harvard mathematics faculty at that time. He was born in 1864. After graduating from Harvard with highest honors and second in his class, he had gone to Germany, where he had studied at Göttingen and Erlangen, receiving his Ph.D. at Erlangen in 1890. More about him and the distinguished accomplishments in his career can be found in volume I of the American Mathematical Society Semicentennial Publications (1938), pp. 153–158. He impressed me as a teacher and as a person. He appeared formidable, and he was, in a way, with his ample, unique beard and his dignified bearing. (He removed the beard in 1933, completely altering his appearance.) Once, after class in Sever Hall, I waited in the front of the room to speak with him as students were clustered around. The better to hold my book bag, I had put on my topcoat and perched my hat loosely on the back of my head. When Osgood was free to talk to me he gave me a glance and said sternly, "Taylor, please remove your hat." In fact, Osgood was kindly and considerate of students. He often had a twinkle in his eye. He was fond of uttering advice and opinions while teaching. He would say, "The clearer the teacher makes it, the worse it is for you. You must work things out for yourself and make the ideas your own." About studying something that seemed difficult his dictum was: "Read it over a couple of times and sleep on it. Then, when you think you have grasped the situation, tell yourself the whole story in your own words, perhaps by writing it out on a scrap of paper while on the subway to Boston." In his books he liked to state problems in ways that required the student to enter into the process of formulating the problem sharply and clearly. He regarded this as an important part of the training of a mathematician. Some students, and teachers too, hated his books on that account.

Osgood occasionally made little jokes. In Mathematics 13, after deriving Poisson's integral formula for the value, at a point inside a circle, of the harmonic function with a preassigned continuous function giving its values on the periphery, he made a flourish of the hand and announced: "There you have it, gentlemen; Mr. Fish, his integral!" He inveighed against the notion of "little zeros" whose quotient was supposed to be the limiting value of an "indeterminate form". (I wonder what he would have thought about nonstandard analysis?) He was fond of asserting that Felix Klein and Sophus Lie had divided the universe of groups between them—Klein taking the discrete groups and Lie the continuous ones. Another of his assertions was that geometry is the noblest branch of physics.

Once I had the good fortune to be his guest for dinner at the Harvard Club in Boston—just the two of us; he talked at length about his graduate student days in Germany while smoking a cigar down to a very small stub. He handled it ingeniously, speared on a blade of his penknife, somehow smoking it beyond the point at which I was sure it would set fire to his beard.

For further experience with the application of mathematics to physics I took G. D. Birkhoff's course on Space, Time, and Relativity, and a course, given in the Physics Department, on the partial differential equations of mathematical physics. There I learned the techniques, but not the validating theory, for solving boundary-value problems for Laplace's equation, the wave equation, and the heat equation. The basic technique was one of separating the variables and reducing the problem to the solutions of eigenvalue problems in ordinary differential equations, to be followed by expansions in terms of systems of orthogonal functions. This work fitted in well with things to be found in Osgood's *Advanced Calculus*: sections on Fourier series, Bessel functions, families of orthogonal functions, Bessel's inequality, and least squares approximation to an arbitrary continu-

ous function by a linear combination of mutually orthogonal functions. Nothing, however, about Hilbert space! It was a little early to hear about that in the curriculum. Stone's book on Hilbert space was published in 1932, and it was not addressed to undergraduates.

In order to come to grips with some of the theoretical justifications for these methods in mathematical physics, I determined (with some initial opposition from Professor Coolidge as tutor) to write my senior honors thesis on the theory of Fourier series, with applications to the conduction of heat. Since I had not learned Lebesgue's theory of integration I had to confine myself to the theory of Fourier series in the state to which it had been brought by the end of the nineteenth century, which meant using Riemann integrals. My thesis contained no new results; it was entirely an exposition based on my reading of the literature. I mastered the rigorous convergence theory for a rather general class of functions that are well-behaved in a piecewise fashion.

I have yet to mention a semester course from Huntington called *Fundamental Concepts of Mathematics*. Most of it was carried on by lectures about axiomatic method, with illustrations from algebra and logic. We learned about consistency, independence, and categoricalness of axiom systems, but we didn't go far in studying any particular examples of mathematical structures built up from axioms, definitions, and theorems. The most thorough thing of that kind was our reading of a little book by Huntington: *The Continuum and Other Types of Serial Order*. This included a few aspects of Cantor's general theory of sets and transfinite numbers and the notion of a Dedekind cut.

In my tutorial work with Coolidge the emphasis was on geometry and algebra. I read most of Graustein's book *Introduction to Higher Geometry*, then quite new (1930), nearly all of Maxime Bôcher's *Introduction to Higher Algebra*, and a good deal of L. E. Dickson's *Theory of Equations* (which included nothing about Galois theory). None of these appealed to me as much as what I was learning in analysis. For one thing, the exercises and problems were not as interesting to me. Projective geometry was nice in an aesthetic way. I thought I understood things as I studied them, but I didn't catch fire with enthusiasm. Actually, I didn't fully comprehend all of the underlying ideas, in spite of Coolidge's efforts to be helpful. He interpreted most of the algebra to me geometrically as we talked about my reading. He was nice to me, but probably not demanding enough. Talking to him face to face was disconcerting, because he was cross-eyed. He had his eccentricities. There were many stories about him, the best known, no doubt, being the one about his inability to pronounce the letter R. It came out as a W. While talking to a class about acceleration in circular motion he was twirling his watch in a circle at the end of the watch chain. Suddenly the chain broke and the watch flew across the room. Preserving his aplomb, Coolidge observed to the class: "Gentlemen, you have obsuhwed my watch descwibing a puhfect pawabola."

The theoretical side of higher algebra and geometry seemed nowhere near as coherent and powerful to me as the theory of analytic functions of a complex variable. I didn't really appreciate the structural beauty of algebra until, as a graduate student, I read the abstract theory of groups and rings in Van der Waerden's *Moderne Algebra*. And the theory of matrices and linear equations didn't become attractive to me until, at Cal Tech, I learned to view it from the perspective of linear transformations from one vector space into another. Actually, I learned about linear algebra most effectively as a side issue while I was learning about Hilbert spaces and Banach spaces. The pioneering book on modern algebra by Garrett Birkhoff and Saunders Mac Lane was not published until 1941, seven or eight years after I could have profited from it the most. Perhaps, if I had taken a formal course in algebra at Harvard, I would have had a different experience.

It would have been possible and natural for me to go on to graduate work at Harvard. I was encouraged to do so, and there was even a prospect of a Sheldon Travelling Fellowship. But, for personal reasons, I preferred to return to California. I received appointment as a Teaching Fellow at Cal Tech, where I earned my Ph.D. in three years. Cal Tech was then a very young institution, known primarily for its excellence in physics. The Mathematics Department was small. There

were two men of established reputations in mathematics, Harry Bateman and Eric Temple Bell, and two younger men with growing reputations, A. D. Michal and Morgan Ward. The graduate program in mathematics was very small, with just a few doctoral degree candidates and an extremely limited offering of advanced courses in any one year. There were obvious superiorities of Harvard over Cal Tech as a place to do graduate work in mathematics. Harvard had a richer offering in courses and a greater wealth and diversity of talent in its faculty, and, I've no doubt, a more stimulating and challenging environment of able graduate students. But there were certain advantages in being at Cal Tech. It was small and intimate. The faculty and the graduate students were a community. The graduate classes in mathematics were very small (except for those that catered to the graduate students in physics, chemistry, and engineering). Most of them were run like seminars. We moved quickly to the frontiers of modern mathematics, without a lengthy process of preparatory courses.

I wrote one paper that was accepted for publication during my first year. It was on integral invariants of non-holonomic dynamical systems. I was reading about integral invariants in a book by Elie Cartan suggested to me by Michal. Osgood turned up in Pasadena and I had a meeting with him. It turned out that he was interested in integral invariants in connection with the book he was writing on mechanics. Partly as a result of our conversation, my attention was turned to the case of non-holonomic systems. Cartan had demonstrated that, for holonomic dynamical systems, there is a certain integral invariant of the differential equations, written in Hamiltonian form, that govern the motion. Moreover, the existence of this integral invariant can be used to characterize the Hamiltonian system within a certain general class of first order systems of differential equations. I was able to extend the theory to the case of non-holonomic systems. This was not a very important achievement, but it was my start in research. It was made possible by my study of dynamics under Osgood at Harvard.

In that first year at Cal Tech I was learning Lebesgue's theory of integration, its application to Fourier series, and about the class of L^2 functions and the Riesz-Fischer theorem (from a book by E. C. Titchmarsh) at the same time that, in another course, I was reading Banach's book, *Opérations Linéaires*, and the book by M. H. Stone on *Linear Transformations in Hilbert Space*. I never took a formal course in topology at Cal Tech. I had to learn about metric spaces and Hausdorff spaces on the side, mainly from Hausdorff's books on *Mengenlehre* (of which the first edition, of 1914, is the best). It was indeed fortunate that I had become proficient in both French and German as an undergraduate. The courses I have mentioned were taught by A. D. Michal. What often happened in class was that Michal asked the students to be prepared to present a certain portion of the material to the rest of us. He would then offer supplementary remarks and explanations, or help the student speaker out, if need be.

I have already mentioned that I learned the abstract algebra of groups and rings at Cal Tech, studying from a book by Van der Waerden. The teacher was E. T. Bell. In some ways he was not a good teacher, but in other ways he was. He was a stimulating person, given to expressing strong opinions. I don't think he spent much time preparing what he was going to say in class. He rarely lectured. His classes were seminars. He tried to steer us through Galois theory as expounded in a book by L. E. Dickson called *Modern Algebraic Theories*. We all (including Bell) found this rather unsatisfactory, and switched to more abstract accounts in German (by Hasse, I believe, and perhaps Van der Waerden, also).

Harry Bateman was the first distinguished mathematician on the Cal Tech faculty. Actually, he came to Pasadena from Johns Hopkins in 1917, to teach at Throop College, an institutional ancestor of Cal Tech. In those early days he was considered locally as a mathematical physicist. He had a phenomenal knowledge of differential equations, both ordinary and partial, as well as a deep knowledge of special functions. He was English, educated at Cambridge, where he made an outstanding record. He was never my teacher, but I came to know and like him in the four years I was at Cal Tech. One of my early encounters with him occurred when I had begun to suspect that there must be solutions of the functional equation $f(x + y) = f(x) + f(y)$ (in real variable

theory) other than the well known $f(x) = Ax$. I knew that any such other solution must be discontinuous, and I could see that it would have to be very badly discontinuous. Professor Michal told me to see if Bateman could help me. He at once gave me the reference to the paper by G. Hamel (Math. Ann., vol. 60, 1905) in which the matter is fully disposed of by means of what has come to be known as a Hamel basis for the real numbers. Bateman was a mild, shy man with a kind and generous heart; to shake his hand was like grasping a leaf of wilted lettuce. Two stories about him: Once, lecturing on classical differential geometry, he put on the board a diagram dealing with curvatures of a surface; putting his finger on a point of the surface where a special situation obtained, he addressed the class in a bird-like voice. "This point, gentlemen, rejoices in the name of the umbilical!" On another occasion, at an open-book examination in a course based on the text, *Modern Analysis*, by Whittaker and Watson, a friend of mine had taken with him a briefcase loaded with reference books. As a practical joke, some of his student colleagues had placed a spring-loaded percussion cap inside one of the books. When my friend pulled the book out to consult it, the cap exploded loudly. Bateman, sitting at his desk at the front of the room, commented cheerfully: "My! That was exciting, wasn't it!"

My mentor, A. D. Michal, whose full given names were Aristotle Demosthenes, was a small, dark-haired man with a ready smile and very white skin. He was very proud of his Greek origin. A few years later, when I was at UCLA, he organized an evening monthly seminar that was attended by people from Cal Tech, UCLA, USC, and some mathematicians from industry. We moved around in rotation between the three educational institutions; we called our meetings THE PERIPATETIC SEMINAR because we were, literally, followers of Aristotle. In 1936, when my wife and I accompanied Aris (as we called him) and his wife on a motor trip to attend the Harvard Tercentenary and the summer meeting of the American Mathematical Society in Cambridge, we made a side trip to Buffalo and Niagara Falls. We crossed into Canada to see the impressive falls on the Canadian side. As we started back across the international bridge we were stopped by a U.S. official who asked where we were born. Aris, who was driving, replied: "On the island of Lesbos in the Aegean Sea." Because he didn't have his passport with him it took us more than a moment to be cleared for re-entry into the U.S.

E. T. Bell is probably best known in recent decades by his two books, *Men of Mathematics* (1937) and *The Development of Mathematics* (1940). But he had a distinguished career; his special research interests were in algebra and number theory. He was one of two Colloquium Lecturers at the meeting of the American Mathematical Society in 1927; his subject was algebraic arithmetic. He was Vice President of the Society in the year 1926–27 and President of the MAA in 1931. He shared the Bôcher prize with Solomon Lefschetz in 1924. He wrote the lead article, *Fifty Years of Algebra in America, 1888–1938*, in Volume 2 of the Society Semicentennial Publications (1938). Bell, born in Scotland, was rather imposing in appearance and manner. Handsome, quick and forceful in speech, with a nose that gave a patrician quality to his profile, he threw off sparks in conversation. He was fond of making bold assertions, usually buttressed by a basis of factual truth, but sometimes more extreme than was warranted. His writings on the history of mathematics are interesting, often vividly so, but not always entirely accurate, according to some professional historians of science and mathematics.

I had a number of interesting conversations with Bell about abstraction in mathematics. I remember one in October of 1935 while I was still a graduate student. He expressed the firm opinion that generalization by abstraction in analysis "does nothing really new and finds no new results." (I wrote these words in my diary at the time.) I disagreed, although I felt too inexperienced and respectful to get into an argument with him. I thought abstract method had certainly clarified, simplified, and generalized in important ways the work of Fredholm and Hilbert in the theory of integral equations. I thought especially of what I knew of the work of F. Riesz, M. H. Stone, and von Neumann. I think I know some of the reason for Bell's attitude. I think he disapproved of some of what Michal was doing in attempting to create a theory of differential geometry for manifolds in spaces of an infinite number of dimensions. Bell's assertion

was extreme, but behind it there was a valid note of caution. He was not alone among well-established mathematicians in being skeptical about abstraction in analysis. I recall very well, a few years later, hearing Professor Einar Hille say, at a meeting of the American Mathematical Society at Stanford, that there was a danger that young mathematicians might be tempted to succumb to the lure of generalization by abstraction in ways that would be easy, sterile, and lacking in substantial content, with too few points of contact with the ongoing body of significant work in mathematics. Hille did not utterly disdain abstraction in analysis. In fact, he used abstraction quite effectively, as may be seen in his book on *Functional Analysis and Semi-Groups.*

In my three years as a Teaching Fellow at Cal Tech I was on what was often referred to as "a breadline fellowship". I received no pay in cash, merely board and room for ten months a year, and was sent no bills of any kind—not even for tuition. In return I taught two classes each quarter, mainly freshman or sophomore calculus (with some analytic geometry). Several times I taught a one-quarter course in differential equations, quite elementary, with applications to geometry, physics, chemistry, and electrical engineering. After I got my Ph.D. I remained at Cal Tech for a year as a full-time instructor. That year I taught an honors course in calculus for sophomores. One of by bright students was Edmund Pinney, now on the mathematics faculty at UC Berkeley. I also taught a full year course in differential equations from an excellent book by L. R. Ford. From that experience I learned a great deal that I had never learned before.

I found that I greatly enjoyed teaching mathematics. I felt elated after a class meeting at which I sensed my success as a teacher. It was fun to see students catching on to the ideas and techniques of calculus. There was deep satisfaction when I encountered students who began to go ahead rapidly after a doubtful start. And I enjoyed the much deeper understanding I gained as I prepared to teach and worked out my own ideas for the exposition of the material to my classes. I read all the homework and examinations of my students. This was not all pleasure, but it was instructive, for I learned where the students were deficient in understanding and technique, and this helped me to be more effective as a teacher. It was in those years that, without quite realizing where my efforts were taking me, I began to lay the groundwork for my future writing of books on calculus. I did not go overboard in trying to teach elementary calculus as though it were a course in "baby real variable theory", but I did begin to think about the theoretical issues in a more critical way than I had ever done before. I think that the shortcoming I minded the most in the most commonly used calculus texts of that day was the failure to be clear in a distinction between the definite integral (as the limit of a sum) and the indefinite integral, or antiderivative (the latter a term not used then much, if at all). There were books that *defined* the definite integral of $f(x)$ from a to b as $F(b) - F(a)$, where $F(x)$ is any function whose derivative is $f(x)$. Osgood's books were not used at Cal Tech. A commonly used book was by Woods and Bailey. A very widely used text of that time was by Granville, Smith, and Longley. It had some merits in problems and narrative, but it had numerous weaknesses in the correct presentation of fundamental concepts and principles. I began to formulate in my mind the notion that, in teaching calculus, the relevant concepts and principles that are needed for a correct and clear treatment should be given recognition at the appropriate places, although not necessarily with the detail and logical completeness needed for full rigor.

My doctoral dissertation (1936) was entitled *Analytic Functions in General Analysis.* It dealt with functions from open sets in the complex plane to a complex Banach space (a subject opened up by Norbert Wiener) and with functions from one complex Banach space to another. A function of the latter sort, defined on an open set, is called locally analytic if it has a Fréchet differential at each point, or, alternatively and equivalently, if it is continuous and has a Gateaux differential at each point. I have written expository historical articles about analyticity in functional analysis and operator theory, one in the volume, *Problems in Analysis* (*A Symposium in Honor of Salomon Bochner*, Princeton University Press 1970), and one in this MONTHLY, vol. 78, 1971. My interest in analyticity, particularly in connection with the spectral theory of operators, has remained strong all my life.

After leaving Cal Tech I spent a postdoctoral year at Princeton University as a National Research Fellow, with a grant of $2000 from the National Research Council. My instructor's salary at Cal Tech had been $1800. At Princeton I had a working relationship with Professor Bochner, some of whose work I had read and admired. We wrote two joint papers. At that time the Princeton Mathematics Department and the mathematicians of the Institute for Advanced Study were all housed together in old Fine Hall. That year I saw a good deal of Ralph Boas, Charles B. Morrey, Anthony P. Morse, Frank Smithies (from England), John Tukey (then a graduate student), and Aurel Wintner (on leave from Johns Hopkins). Bochner was very helpful to me. At first I wasn't sure what to make of him. He seemed rather cynical, and put on a bantering manner with me. At times he was abrupt and sharply critical. It was sometimes hard to interpret him, to distinguish between what was intended as a kind of mocking humor and what was serious criticism, often deserved. There was one time when he at first refused to believe that I was right when I told him that I had discovered that, for an operator-valued function from an open set in the complex plane to the space of bounded linear operators on a complex Banach space, to be strongly analytic (that is, to be analytic as a vector function upon application to each arbitrary vector) is equivalent to being analytic in the uniform topology of the space of operators. But I *did* convince him. This result of mine was independent of, but roughly contemporaneous with, Nelson Dunford's discovery that, for functions from an open set in the complex plane to a complex Banach space, weak analyticity is equivalent to strong analyticity.

During that year at Princeton I first heard of Bourbaki. A great many interesting things were being done by Bourbaki and others in the realm of Banach spaces and topological linear spaces. Boas, Smithies, Tukey, and I used to visit together in old Fine Hall at tea time, talking about what was going on. Much that was new was in the air: various weak topologies, weak compactness of the unit ball in a Banach space, reflexivity, ...

In the spring of 1938 I was appointed to a second year of the National Research Fellowship. I had barely accepted when I was invited to go to UCLA as an instructor at $2000. I tried unsuccessfully to get the offer held open until the following year. What to do? There were clear advantages in having another year free for research. But I really wanted to have a permanent job in California. This seemed like a fine opportunity, I thought, for although UCLA was not then a prestigious university, its future was bound to be good. I talked to Professors Eisenhart and Veblen. They assured me that the bulk of the best places for work in mathematics in America were on the Atlantic coast, and advised me to keep the fellowship another year. It was a hard choice. I decided on UCLA; everything thereafter vindicated my action, so far as my own life has been concerned.

T. Y. Thomas, a differential geometer at Princeton University, moved from there to UCLA at the same time I did. UCLA, as it existed in 1938, was just beginning to be a full-fledged university. it had developed out of what was called the University of California—Southern Branch, created in 1919 with initial facilities, staff, and administration taken over from the Los Angeles State Normal School. Offering of the Ph.D. at UCLA was first authorized in 1936, but very few departments awarded any doctorates until after 1940. When Thomas and I arrived, the graduate course offerings in mathematics were extremely limited and no students were far advanced in work toward the Ph.D. The main thrust of the department at that time was at the undergraduate level and the guidance of a few students toward the M.A. The onset of U.S. involvement in World War II drained away the potential Ph.D. candidates (several went into meteorology), with the result that the first doctorate in mathematics was not awarded until 1947 (to William Gustin, working under Edwin Beckenbach). But the tempo of graduate work picked up right after World War II. Campus enrollment was under 9000 in 1938. In 1940–41 it was just over 10,000. It dropped to about 6500 in 1943–44, and then grew steadily after 1945, except for a slight dip during the Korean war, to over 26,000 in my last full year of duty at UCLA (1964–65). By that time the Mathematics Department was awarding about twenty Ph.D. degrees per year.

Our graduate program was broadly diversified and the quality of our graduate students was a source of pleasure, especially to those of us who had clear memories of the days prior to World War II.

In my early days at UCLA the regular teaching load of the mathematics faculty—all ranks with two individual exceptions—was four courses, each meeting three hours a week. This was a tradition carried over from the days when the school was predominantly undergraduate. Only a few of the faculty in my first year were trying to carry on a program of research. My most stimulating colleague was Max Zorn. Another colleague was Glenn James, an early editor of the Mathematics Magazine (and later, with his son Robert C. James, co-author of the Mathematics Dictionary). In charge of the UCLA campus at that time (with the title of Provost) was Earle R. Hedrick, the first President of the M.A.A. (1916).

Among my earliest superior students were Robert C. James (in a class of Mechanics) and Richard F. Arens (in my first class in Advanced Calculus). Both went on to fine careers as professors. In 1942 Herman Kahn (known much later as a theorist on the subject of nuclear war) was an A student in my class in potential theory. He thought and talked very rapidly; the rush of ideas was sometimes faster than his speech could manage without incoherence.

Not until after World War II was the policy on teaching loads modified, making the normal teaching assignment three regular courses in one term. A year or so after I became chairman of the department in 1958, I found that I could not recruit the most promising young people as new junior members of the faculty unless we could assure them a two-course teaching load (six hours a week). I managed to obtain the resources (mainly an increased number of teaching assistants) to make this possible. The price paid, in educational policy, was a change in our way of teaching freshmen and sophomore courses. We had been teaching nearly all the regular calculus classes in groups of from twenty-five to forty, using mostly full-time faculty members and a few of the most experienced teaching assistants. We shifted to a system of using regular faculty to teach calculus in large lecture classes supplemented by fairly small discussion and problem sections in charge of teaching assistants. How this affected the quality and effectiveness of the teaching, I'm not sure. I would have preferred to continue with our previous method, but budgetary limitations forced my hand.

On the matter of teaching load it is interesting to note what was said by George D. Birkhoff in 1938 in an address, *Fifty Years of American Mathematics* (Vol. 2 of the Semicentennial Publications of the American Mathematical Society). I quote from page 278 of that publication, where Birkhoff stated that at Harvard they had been able to reduce the teaching and tutorial load of the regular staff to six hours a week, of which only three hours were at the more or less elementary level; he then continued: "Such a schedule gives to all concerned a notable opportunity to carry on mathematical research, and would be socially unjustifiable unless the highest standards of achievement were being attained. Although such ideal conditions are impracticable at present except in a few fortunate institutions, it should be strongly emphasized that twelve hours of instruction a week (including at least one course of advanced grade) is about all that can be required if the best standards of scholarship are expected. Indeed, wherever possible, the hours of instruction should be reduced to not more than nine, and if there are heavy outside duties there should be a compensating diminution in teaching." In the next paragraph Birkhoff added the following: "It is our duty to take an active and thoughtful part in the elementary mathematical instruction in our colleges, universities, and technical schools. To these tasks we must bring a broad mathematical point of view and a finite enthusiasm."

Much has changed in the condition of college and university mathematics and its teaching since Birkhoff's time. The volume of published mathematical research has expanded enormously. The pressure on teachers to produce publishable research has greatly increased. The share of their time devoted to undergraduate teaching, and especially to teaching freshmen and sophomores, by university professors has greatly diminished in many institutions. In the case of those who are clearly able to make the most outstanding contributions in their research, I don't question the

appropriateness of their choice if they prefer to concentrate on research and teaching closely related to it. But I do wonder about the wisdom and the justifiability of the general patterns of teaching assignments that have developed and spread so widely in America since the mid-1950's.

In my own teaching at UCLA I was often dissatisfied with the available teaching materials. The result was that I wrote a number of books. When Professor G. E. F. Sherwood and I contracted with Prentice-Hall to write our *Calculus*, I was in my second year as an instructor. I recall that Professor Thomas disapproved of my undertaking. He told me it would never get me anywhere professionally. Perhaps he thought it would finish me as a research mathematician. My motivation for writing that book was to present the subject matter more clearly, effectively, and correctly than in available books. My senior colleague, Professor Sherwood, made a strong contribution of problems to the first edition of our book. He was a fine teacher. I wrote all the expository material. I believe that our *Calculus* (the first edition in 1942) set a new standard of accuracy, clarity and correctness in setting forth the basic principles of calculus, both differential and integral, for functions of one variable. It was also, I think, the first American introductory text in calculus to define the differential of a function of several variables in the manner that makes possible a natural and rigorous proof of the chain rule without resort to any requirement of continuity of the partial derivatives. This definition was first introduced into textbook literature by O. Stolz in 1893. He was a professor at Innsbruck. I first learned this definition from the third edition of de la Vallée Poussin's *Cours d'Analyse Infinitésimale* (1914). It was Zorn who told me about Stolz. I have written a paper on the history of the differential in the 19th and 20th centuries, in *Archive for History of the Exact Sciences*, vol. 12 (1974), pp. 355–383. This connects with Fréchet's definition (1911) of the differential for vector-valued functions in functional analysis.

My *Advanced Calculus* (first edition 1955) was the fruit of a sustained effort to develop a full year course that would provide a maturing experience in analysis for undergraduate mathematics majors and be useful to mathematically minded students in other majors. I taught more semesters of advanced calculus at UCLA than of any other subject above the level of elementary calculus. In that book I covered at least all of the point set topology and real variable function theory that I had learned from Osgood in Mathematics 13 at Harvard. I required students to learn to write verbal arguments in correct English, with complete sentences, in proving theorems about set theory and continuous functions. I had a reputation as a difficult taskmaster in these things, but my course was popular. One of my excellent students was Robert Phelps, now a professor at the University of Washington. The book, revised and modernized with the help of Professor W. Robert Mann, is still in print after almost thirty years.

Many mathematicians were drawn into special roles during World War II. At UCLA we had contingents of Army and Navy cadets on campus for technical training. I taught one highly specialized course for gun-laying in anti-aircraft artillery. (This was before automatic gun control with digital computers.) Outside of regular duties, I taught evening courses in the theory of structures, elasticity, and other engineering mathematics for University Extension courses for employees of aircraft companies. In the summer of 1944 I went to England, for the duration of the war in Europe, as an operations analyst in the Operational Research Section of the U.S. Army Eighth Air Force, which was conducting heavy bomber operations over German and German-occupied territory. In that service I found a number of American mathematicians, including W. L. Ayres, J. A. Clarkson, Robert Dilworth, George Mackey, G. Baley Price, and J. W. T. Youngs, and narrowly missed meeting Edwin Hewitt. The English had introduced the idea of operations analysis as an aid in military operations. Our usefulness stemmed more from our common sense and general reasoning skills than from sophisticated mathematics. I worked on bombing accuracy in connection with the use of radar. One of the interesting problems worked on by others was to teach airmen the principles of machine gunnery in defending the bombers from attack by enemy fighters. Shooting from a moving platform at a moving target, with the various angles and velocities involved, called for aiming that gunners often found contrary to their intuition. Computer control of machine guns was not available.

I had never been to Europe before. I took advantage of my location to explore Cambridge, London, and Oxford when I could take a day or two of leave. In this way I met Titchmarsh in Oxford and Hardy, along with others, including John Todd and Olga Taussky, at a meeting of the London Mathemàtical Society. I was so impressed with England, and Cambridge in particular, in spite of the deprivations and shabbiness of war time, that I determined to spend my first sabbatical (1947–48) at Cambridge University. It was a fruitful year, during which I saw a great deal of A. S. Besicovitch (who often invited me to dine with him in Trinity College), Hans Hamburger, Sir Harold and Lady Jeffreys, J. E. Littlewood, and Frank Smithies, as well as having pleasant acquaintance with a number of others. Hamburger, formerly a professor at Cologne, had been in England all during the war. He and Margaret Grimshaw had finished the manuscript of a little book (published soon after) on finite-dimensional vector space, as an introduction to Hilbert space theory. Hamburger liked American coffee, of which my wife and I had a good supply sent over to us. He discovered that fact and was a frequent visitor to our flat. Unlike the English, he eagerly talked shop on social occasions. He was then much interested in Hermitian operators in Hilbert space and in self-adjoint differential operators. The next time I saw Hamburger after that year in England was at the Symposium on Spectral Theory and Differential Problems held at Oklahoma State University in the summer of 1951. On that occasion we took an afternoon off to attend a spectacle of ceremonial American Indian dances at a big get-together of Indians from all over the southwest. The master of ceremonies learned that our group of mathematicians, including a number from Europe, was in the grandstand; he announced over a loud-speaker that any of us who wished to could join the dancers on the football field. He might not have expected any takers, but Hamburger happily went down and could soon be seen gyrating with Indian maidens clad in fringed buckskin.

During one term in Cambridge I attended a regular course of lectures by J. E. Littlewood. One of the minor points I recall is that he was lecturing on a subject that frequently involved the factor $1/2\pi$ in front of an integral sign. Littlewood stated that, to simplify his chores in writing on the blackboard, he was going to make the convention that $2\pi = 1$.

My next sabbatical (1954–55) was spent in Europe, partly at Geneva and partly at Mainz, on the Rhine in Germany, where I held a Fulbright Research Fellowship. The first big event of that year abroad was the International Congress of Mathematicians in Amsterdam in September. It was exciting and inspiring to see, meet, or hear so many mathematicians with international reputations. For me the high point of the Congress was the wonderful lecture by Paul Alexandrov on the development of point set topology. I chose to spend my Fulbright in Mainz in order to be near Gottfried Köthe. He was at that time Rektor of the Johannes Gutenberg University. He allowed me to use his professorial office because he had another office for his administrative duties. Originally from Austria (Graz), Köthe was a student of O. Toeplitz. A rektorship is usually held by an individual for just a year or two. Köthe seems to have had a good reputation as a Rektor, for he subsequently held that position at Heidelberg, and I believe also at Frankfurt. He is a genial man. In February we journeyed together to Göttingen for a meeting commemorating the 100th anniversary of the death of Gauss. The Germans were all dressed in black. I felt embarrassed because my vest-sweater, which I needed for warmth, was of Argyle design, making me very conspicuous among all the somberly dressed people. The main speaker was Courant. After the program and the mid-day dinner, Köthe and I drifted with others over to the Kron und Lanz for talk and coffee. There we met Professor Arnaud Denjoy, and had a pleasant conversation. The longevity of French mathematicians used to be a notable thing. Denjoy was one of the six mathematicians who at that time filled the roster of the Section de Géométrie of the Paris Académie des Sciences, but at age 70 he was among the younger members, along with R. Garnier (67) and G. Julia (61). The three older members were J. Hadamard (88), E. Borel (83), and P. Montel (78). When Borel died in 1956, his replacement was M. Fréchet at age 77! Hadamard lived to almost 98, and Montel died three months short of 99.

During that year I had pleasant experiences as an invited lecturer, giving lectures in German at

Göttingen, where I was hosted by F. Rellich, then the Director of the Mathematical Institute, and at Tübingen, where my hosts were H. Wielandt and Karl Zeller. For help in putting the text of my lecture into correct German I was indebted to G. Neubauer, then a student in Mainz. It was a fine experience to be in Mainz during the pre-Lenten festivities of Fastnacht (Mardi Gras). While at Mainz I wrote most of the first draft of my book, *Introduction to Functional Analysis*. I undertook that task in order to broaden and deepen my knowledge and help me do a better job with my teaching and research.

All of my doctoral students (about fifteen) investigated problems in operator theory. One of them, Seymour Goldberg, has written a book, *Unbounded Linear Operators*, that springs in part from his dissertation, but contains many things from his own studies and accomplishments in subsequent years. Another doctoral student, Edward O. Thorp, whom, I like to think, I won over from physics to mathematics as a result of having him in some of my courses, had his career fundamentally changed a few years after he earned the Ph.D. when he discovered how to win consistently at the game of Blackjack. His short paper about the odds in favor of a knowledgeable player created a sensation when he presented it at the meeting of the American Mathematical Society in Washington, D. C. in January, 1961. Another of my students, David C. Lay, has done a fine job in revising and modernizing the *Introduction to Functional Analysis* in a second edition (1980). One of the earliest of my doctoral students, William Bade, now my colleague at Berkeley, got off to an interesting start on his career by becoming a member of the team working with Nelson Dunford on the production of that very important three-volume work, *Linear Operators*, by Dunford and Schwartz.

Among my great pleasures in life there have been opportunities to travel and meet mathematicians in other countries. I have already mentioned experiences in England and Germany. In 1960 and 1961 I accepted invitations that took me to Jerusalem, Warsaw, and Prague for relatively small international conferences, and in 1962 I attended the International Congress of Mathematicians in Stockholm, followed by a colloquium in Helsinki. On my way to the conference in Jerusalem in July of 1960, my first experience in Israel was the drive from the Tel Aviv airport to Jerusalem long after midnight. Abraham Robinson was the driver, and in the car with me were W. A. Luxemburg, A. C. Zaanen, W. Orlicz, and Jan Mikusinski. On the way Abby Robinson pointed out to us a gas station called Samson's Place! That was my first acquaintance with Abby, who was just a few years later to be my cherished colleague at UCLA, and who died prematurely some years after moving to Yale.

The Jerusalem conference, a week long, was small enough to be intimate. I found it enormously satisfying, both mathematically and socially, and also as a cultural experience. It was in Jerusalem that I first met Jean Dieudonné. At the closing dinner there were a number of speeches, the principal one, by Dieudonné, being eloquent in appreciation of the accomplishments of the Israeli people in the past twelve years. Dieudonné has always been cordial to me, and helpful, especially in connection with my need, in later years, to work at the Archives of the Académie des Sciences in Paris while studying the letters and other material there relating to Fréchet.

Later that same summer I attended a conference in Poland and had a very interesting time. We were housed in a country estate near Warsaw belonging to the Polish Academy of Sciences. The lovely palace on the estate had been reconstructed after World War II. The first session of the conference was devoted to the memory of Banach; there were speeches by Mazur, Sobolev, M. H. Stone, Nagy, and Steinhaus. Among the older Poles there were Sierpinski and Kuratowski. The up and coming younger generation of Polish analysts and topologists were well represented. I found Steinhaus to be a very charming and interesting man. This was my first meeting of Sobolev. He visited UCLA later, in 1964, while I was still chairman of the Mathematics Department. I found him to be as expansive and extroverted as a Texan. My wife and I were with him for an exhausting day during which we took him to Marineland, then to Disneyland, and finally to a party given by Professor and Mrs. Ivan Sokolnikoff. Politics were buried in favor of mathematics. (Sokolnikoff had fled Russia by way of Siberia after the Revolution.)

The Prague symposium in September, 1961, occurred during my Swiss sabbatical year, during which I lived in a small village on Lake Geneva, very near Lausanne. The symposium was enjoyable, but the atmosphere, because of the international situation at the time, was less easy and comfortable than all of us wished. It opened with remarks in honor of Eduard Čech, who didn't live to attend the conference of which he was the initiator. Speeches were made by M. Katětov, M. H. Stone, K. Kuratowski, and P. S. Alexandrov. At this conference I especially enjoyed talking with V. Pták and Ivo Marek. The latter was the instigator of action leading to the publication of a Czech translation of my *Introduction to Functional Analysis*.

My main occupation during that year in Switzerland was the writing of the first draft of my book *General Theory of Functions and Integration*. It was a somewhat unusual combination of chapters on general topology, general measure theory, and two different approaches to the theory of integration—one via measure, the other by the Daniell method. As with all of my books, it was closely keyed to my teaching program. I sought to connect the general with the particular. My experience with beginning graduate students convinced me that, while they would need to progress rapidly to a grasp of the general and abstract in analysis, they would be in danger of failing to understand how the general is linked to the particular, both in the topology of Euclidean space and in the "classical" Lebesgue theory of measure and integration. The book went out of print, but I am happy that it is to be reissued by Dover Publications.

In Lausanne I had made the acquaintance of Professor de Rham, who had sponsored me for membership in the Swiss Alpine Club. (Over the years I climbed a good many of the great peaks in the Alps, but I am not at all in de Rham's class as an alpinist.) At a dinner party given by de Rham I was interested to learn that his residence is in part of an 18th century house that once belonged to Necker, the father of Madame de Staël and for a time Minister of Finance for the French King Louis XVI. It was at this dinner that I first met Alexandre Chorin, now my colleague in Berkeley.

A few mathematicians excel in making fine and interesting oral presentations to an audience. Among the best in my professional experience have been Lipman Bers, H. F. Bohnenblust, E. F. Beckenbach, Einar Hille, and Raymond Redheffer. Among undergraduates I taught who subsequently became academic mathematicians, by far the most impressive (in chronological order) were Robert C. James, Richard Arens, William Gustin, Edward Thorp, Robert Phelps, and David Lay.

One of the trends in the period 1955–1965 that bothered me as a faculty member in mathematics was the separation between pure and applied mathematics, not merely in the research interests of the oncoming generation, but in the attitudes toward teaching. As chairman at UCLA I was once told by a newly recruited assistant professor that he didn't want to teach calculus to engineers. I also knew of cases in which professors wanted the undergraduate year course in differential equations to be devoted entirely to theory, with no applications to the sciences. We had an internal tug of war in the Mathematics Department at UCLA that was related to the issue of applied mathematics as it affected our hiring policy. The main focus of this difference of opinion was on numerical analysis and the software side of computer science. That was at an early stage in the development of computers, before there were such things as departments of computer science. Toward the end of my chairmanship (1964) and in the immediately following years the attitude toward applied mathematics began to change for the better.

As I contemplate my life as a mathematician, I view it as the life of a scholar-teacher, with the two roles firmly conjoined. I derived great satisfaction from teaching. In order to teach well I had to be striving for thorough understanding of the material, for clarity and insight in presenting it, for sympathy and empathy for the attitudes and interests of my students, and for ways of enhancing student interest and enthusiasm. My own visible delight in the subject and in talking about it in the classroom were, I am convinced, essential to the maximal achievement of an hour

of successful rapport with a class. I have experienced many such hours, at all levels of teaching, but perhaps most notably (1) with the first year of calculus, (2) with the more mature approach to the rigors of analysis as presented in my *Advanced Calculus*, and (3) with graduate students on the threshold of the wider vista of analysis made possible by abstract linear algebra, general topology, and the perspective of functional analysis. I was fortunate to have experienced a nice balance in my duties and opportunities, and to have achieved satisfaction from a modest accomplishment in research along with success as a scholar-teacher.

DIFFERENTIABLE, NOWHERE ANALYTIC FUNCTIONS

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1. Let F denote the set of all infinitely many times differentiable real valued functions on the closed unit interval $[0,1]$. We say that a function f is analytic in the interval $(x_0 - d, x_0 + d)$ if f is the limit of a power series in $x - x_0$ with constant coefficients in this interval. We say that f is nowhere analytic if it is not analytic in any interval. Let F_0 denote the set of nowhere analytic functions in F. In this paper we study functions in F_0.

The functions in F form a real vector space under pointwise addition and scalar multiplication. We will construct a subspace of F of dimension c, the cardinality of the continuum, all of whose nonzero vectors are functions in F_0.

Now suppose f_1 and f_2 are in F_0 and $f_1^{(k)}(\frac{1}{2}) = f_2^{(k)}(\frac{1}{2}) = 0$ for all $k = 0,1,2,\ldots$. Let $f_3(x) = f_1(x)$ for $0 \leqslant x \leqslant \frac{1}{2}$, and $f_3(x) = f_2(x)$ for $\frac{1}{2} < x \leqslant 1$. Let $f_4(x) = -3f_1(x)$ for $0 \leqslant x \leqslant 1$. Then f_3 and f_4 are in F_0, but they do not convey any information not already provided by f_1 and f_2. On the other hand, the vectors in any basis of the subspace we construct will be essentially distinct from each other, and there are c of these functions. Thus we will construct a large number of functions in F_0 that are essentially distinct from each other.

The set of continuous functions on $[0,1]$ forms a complete metric space under the sup metric, and the set of functions that have derivatives at one or more points is a first category subset of this space [3]. In this sense "most" of the continuous functions are nowhere differentiable. We would like to find that "most" functions in F are in F_0, but the sup metric on F is not complete. (Note that the polynomials are dense in the set of all continuous functions by Weierstrass' Theorem [3].) We remedy this by defining another metric on F that is complete, and discover that $F \setminus F_0$ is in fact a first category subset of F using this metric. Finally, we construct a vector subspace of F of dimension c, and dense in F, all of whose nonzero vectors are in F_0.

The theorem that $F \setminus F_0$ is a first category set is already known. Two references in English are [1] and [2]. However [2] is brief and omits some details, and [1] is hard to locate and involves a number of other issues. I did not find any work in the literature on the vector spaces of functions in $F_0 \cup \{0\}$.

2. It is convenient to define the metric d on F first. We put

$$d(f_1,f_2) = \sum_{k=0}^{\infty} 2^{-k} \sup_{0 \leqslant x \leqslant 1} \left(|f_1^{(k)}(x) - f_2^{(k)}(x)|\right)\left[1 + |f_1^{(k)}(x) - f_2^{(k)}(x)|\right]^{-1}$$

F. S. Cater: I received my Ph.D. from the University of Southern California in 1960. Most of my academic career since then has been at Portland State University, where I was promoted to professor in 1970. Since 1961 I have published at various times on topics ranging from analysis to algebra and linear algebra, and even a little topology, geometry and number theory. However, I do not claim to be a specialist in any area. Recently I published on real functions, a subject that has interested me since my undergraduate days.

and verify that d is a complete metric on F. (We understand that $f^{(0)}$ means f.) Plainly $d(f_1, f_2) \geqslant 0$ and $d(f_1, f_2) = 0$ if and only if $f_1 = f_2$. Moreover $d(f_1, f_2) = d(f_2, f_1)$. For positive numbers a and b, the inequality

$$a(1 + a)^{-1} + b(1 + b)^{-1} \geqslant (a + b)(1 + a + b)^{-1}$$

holds. This can be verified by clearing fractions. Moreover, the function $u \mapsto u(1 + u)^{-1} = 1 - (1 + u)^{-1}$ increases with u for u positive. Consequently

$$|f_1^{(k)}(x) - f_2^{(k)}(x)| + |f_2^{(k)}(x) - f_3^{(k)}(x)| \geqslant |f_1^{(k)}(x) - f_3^{(k)}(x)|$$

and

$$\begin{aligned}
&|f_1^{(k)}(x) - f_2^{(k)}(x)|\big(1 + |f_1^{(k)}(x) - f_2^{(k)}(x)|\big)^{-1} \\
&\quad + |f_2^{(k)}(x) - f_3^{(k)}(x)|\big(1 + |f_2^{(k)}(x) - f_3^{(k)}(x)|\big)^{-1} \\
&\geqslant \big(|f_1^{(k)}(x) - f_2^{(k)}(x)| + |f_2^{(k)}(x) - f_3^{(k)}(x)|\big) \\
&\quad \times \big(1 + |f_1^{(k)}(x) - f_2^{(k)}(x)| + |f_2^{(k)}(x) - f_3^{(k)}(x)|\big)^{-1} \\
&\geqslant |f_1^{(k)}(x) - f_3^{(k)}(x)|\big(1 + |f_1^{(k)}(x) - f_3^{(k)}(x)|\big)^{-1},
\end{aligned}$$

and the triangle inequality $d(f_1, f_2) + d(f_2, f_3) \geqslant d(f_1, f_3)$ holds.

It remains to prove that d is complete on F. Let (f_n) be a Cauchy sequence of functions in F; that is, $\lim_{n,m\to\infty} d(f_n, f_m) = 0$. Then

$$\sup_{0\leqslant x\leqslant 1}\big(|f_n^{(k)}(x) - f_m^{(k)}(x)|\big[1 + |f_n^{(k)}(x) - f_m^{(k)}(x)|\big]\big)^{-1} \leqslant d(f_n, f_m)$$

for each k, so the sequence of continuous functions $f_n^{(k)}$ converges uniformly on $[0, 1]$, say to the continuous function g_k. Then

$$\lim_{n\to\infty} \sup_{0\leqslant x\leqslant 1} |f_n^{(k)}(x) - g_k(x)| = 0,$$

$$\lim_{n\to\infty} \sup_{0\leqslant x\leqslant 1} |f_n^{(k)}(x) - g_k(x)|\big[1 + |f_n^{(k)}(x) - g_k(x)|\big]^{-1} = 0$$

for each $k \geqslant 0$, and hence

$$(*) \qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} 2^{-k} \sup_{0\leqslant x\leqslant 1} |f_n^{(k)}(x) - g_k(x)|\big[1 + |f_n^{(k)}(x) - g_k(x)|\big]^{-1} = 0.$$

It suffices to prove that $g_0 \in F$ and $g_0^{(k)} = g_k$ for all $k = 0, 1, 2, \ldots$, by induction on k. For $k = 0$, there is nothing to prove. We assume that $g_0^{(N-1)} = g_{N-1}$ for fixed $N \geqslant 1$ in all such situations. Clearly the sequence (f_n') is also a Cauchy sequence in F because $d(f_n', f_m') \leqslant 2d(f_n, f_m)$ by the definition of d. Moreover $f_n^{(i)}$ converges uniformly to g_{i+1} for $i \geqslant 0$. By the induction hypothesis, $g_0^{(N)} = g_0'^{(N-1)} = g_N$. This completes the induction. Thus $g_0 \in F$ and $\lim_{n\to\infty} d(f_n, g_0) = 0$ follows from $(*)$.

We now show that the complete metric space (F, d) is also separable. This means that there is a countable dense subset of F. In particular, we show that the set of polynomials in x with rational coefficients is dense in F.

Take any $\varepsilon > 0$ and any $f \in F$. We prove by induction on j that for each $j \geqslant 0$ there is a polynomial p with rational coefficients such that

$$\sum_{i=0}^{j} |f^{(i)}(x) - p^{(i)}(x)| < \varepsilon \text{ for } 0 \leqslant x \leqslant 1.$$

For $j = 0$ we use Weierstrass' approximation theorem [3] to find a polynomial $q(x)$ such that

$$|f(x) - q(x)| < \frac{1}{4}\varepsilon \text{ for } 0 \leqslant x \leqslant 1.$$

We approximate the coefficients of q with rational numbers to find the appropriate polynomial p such that

$$|p(x) - q(x)| < \frac{1}{4}\varepsilon$$

and hence

$$|f(x) - p(x)| < \frac{1}{2}\varepsilon \text{ for } 0 \leqslant x \leqslant 1.$$

Now assume that p can be found whenever $j = N$. But $f' \in F$, so there is a polynomial $p_0(x)$ with rational coefficients such that

$$\sum_{i=0}^{N} |f^{(i+1)}(x) - p_0^{(i)}(x)| < \frac{1}{4}\varepsilon \text{ for } 0 \leqslant x \leqslant 1.$$

Let a be a rational number with $|f(\frac{1}{2}) - a| < \frac{1}{4}\varepsilon$. Put

$$p(x) = a + \int_{\frac{1}{2}}^{x} p_0(t)\,dt \text{ for } 0 \leqslant x \leqslant 1.$$

Then

$$|p(x) - f(x)| = \left| a + \int_{\frac{1}{2}}^{x} p_0(t)\,dt - f\left(\frac{1}{2}\right) - \int_{\frac{1}{2}}^{x} f'(t)\,dt \right|$$

$$\leqslant \left| a - f\left(\frac{1}{2}\right) \right| + \left| \int_{\frac{1}{2}}^{x} |p_0(t) - f'(t)|\,dt \right| < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon$$

for $0 \leqslant x \leqslant 1$.

So $p' = p_0$, $p^{(i+1)} = p_0^{(i)}$ and

$$\sum_{i=0}^{N+1} |f^{(i)}(x) - p^{(i)}(x)| = |f(x) - p(x)| + \sum_{i=0}^{N} |f^{(i+1)}(x) - p^{(i+1)}(x)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon,$$

for all $0 \leqslant x \leqslant 1$. This completes the induction on j.

Let j be chosen so large that $2^{-j} < \varepsilon$. Then

$$d(f,p) = \sum_{i=0}^{J} 2^{-i} \sup_{0 \leqslant x \leqslant 1} |f^{(i)}(x) - p^{(i)}(x)| \left[1 + |f^{(i)}(x) - p^{(i)}(x)|\right]^{-1}$$

$$+ \sum_{i=j+1}^{\infty} 2^{-i} \sup_{0 \leqslant x \leqslant 1} |f^{(i)}(x) - p^{(i)}(x)| \left[1 + |f^{(i)}(x) - p^{(i)}(x)|\right]^{-1}$$

$$< \varepsilon + \varepsilon = 2\varepsilon.$$

Let (a, b) be any interval. By the *spike* on (a, b) we mean the function $h(x) = e^{-(x-a)^{-2}(x-b)^{-2}}$ for $a \leqslant x \leqslant b$ and $h(x) = 0$ for all other x. Obviously h is differentiable at any x other than a and b. Moreover all derivatives of h exist at a and b and are 0. Note that any derivative of h is of the form $h(x)p(x)/q(x)$ for some polynomials p and q, and $h(x)$ dominates $q(x)$ around a and b and makes $h^{(k)}(a) = h^{(k)}(b) = 0$ for all $k \geqslant 0$. Thus h is not analytic on any interval centered at a or b; for if it were, it would be the limit of its Taylor series expansion about a or b for some $x \in (a, b)$, and this is impossible.

We say that a subset of a complete metric space is *residual* if its complement is a first category set in the metric space.

With these spikes we prove our first theorem.

THEOREM 1. *The set F_0 of nowhere analytic functions forms a residual subset of the complete metric space (F, d).*

Proof. By analytic function theory [4] f is analytic on any subinterval of any interval on which f is analytic. It suffices to prove that the set of functions that are the limit of their Taylor series in $x - x_0$ on an interval I centered at x_0, with rational endpoints, is a first category set in F. Fix an interval I with rational endpoints and center x_0.

For each positive integer j, let

$$U_j = \left\{ f \in F: \left| \sum_{i=0}^{j} f^{(i)}(x_0)(x - x_0)^i/i! - f(x) \right| > 1 \text{ for some } x \in I \right\}.$$

Then U_j is an open subset of F, for if $f \in U_j$ and

$$\varepsilon = -1 + \left| \sum_{i=0}^{j} f^{(i)}(x_0)(x - x_0)^i/i! - f(x) \right|$$

and $d(f_1, f) < \frac{1}{2}\varepsilon/(1 + \frac{1}{2}\varepsilon)$, then it follows easily from the definition of d that

$$\left| \sum_{i=0}^{j} \left[f^{(i)}(x_0)(x - x_0)^i/i! - f_1^{(i)}(x_0)(x - x_0)^i/i! \right] \right|$$

$$\leqslant \sum_{i=0}^{j} |f^{(i)}(x_0) - f_1^{(i)}(x_0)||x - x_0|^i/i! < \frac{1}{2}\varepsilon, \qquad |f(x) - f_1(x)| < \frac{1}{2}\varepsilon,$$

for $0 \leqslant x \leqslant 1$ and $f_1 \in U_j$.

For any index J, $\cup_{j=J}^{\infty} U_j$ is open in F. We next show that $\cup_{j=J}^{\infty} U_j$ is dense in F. Take any $f \in F$ and $\varepsilon_1 > 0$. Let g be the spike on an interval $(x_0, x_0 + \delta) \subset I$. Let c be a number such that $cg(x_0 + \frac{1}{2}\delta) > 2$. Let $p(x)$ be one of the polynomials in the countable dense subset constructed before such that $d(p, cg) < \varepsilon_1$. Put $h(x) = f(x) + cg(x) - p(x)$ for $0 \leqslant x \leqslant 1$. Then

$$d(f, h) = d(0, cg - p) = d(cg, p) < \varepsilon_1.$$

Let $T(G, j, x) = \sum_{i=0}^{j} G^{(i)}(x_0)(x - x_0)^i/i!$ for any function G. For large enough $j \geqslant J$, $p^{(j)} = 0$ and $T(p, j, x) = p(x)$ for all x. For such $j, T(cg, j, x) = 0$ because $g^{(i)}(x_0) = 0$ for all i, and

$$T\left(h, j, x_0 + \frac{1}{2}\delta\right) = T\left(f, j, x_0 + \frac{1}{2}\delta\right) + p\left(x_0 + \frac{1}{2}\delta\right).$$

Thus for such j,

$$T\left(h, j, x_0 + \frac{1}{2}\delta\right) - h\left(x_0 + \frac{1}{2}\delta\right) = T\left(f, j, x_0 + \frac{1}{2}\delta\right) - f\left(x_0 + \frac{1}{2}\delta\right) - cg\left(x_0 + \frac{1}{2}\delta\right).$$

Now either

$$\left| T\left(f, j, x_0 + \frac{1}{2}\delta\right) - f\left(x_0 + \frac{1}{2}\delta\right) \right| > 1$$

or

$$\left| T\left(h, j, x_0 + \frac{1}{2}\delta\right) - h\left(x_0 + \frac{1}{2}\delta\right) \right| > 1,$$

because $cg(x_0 + \frac{1}{2}\delta) > 2$. Thus either $f \in \cup_{j=J}^{\infty} U_j$ or $h \in \cup_{j=J}^{\infty} U_j$. Hence $\cup_{j=J}^{\infty} U_j$ is an open dense subset of F, and $\cap_{J=1}^{\infty} \cup_{j=J}^{\infty} U_j$ is a residual set. Finally, the set of all functions in F analytic on I is disjoint from $\cap_{J=1}^{\infty} \cup_{j=J}^{\infty} U_j$ and must be a first category set. Since there are only

countably many such intervals I, it follows that $F \setminus F_0$ is a first category subset of F, and F_0 is a residual subset of F.

3. Before we construct the promised nowhere analytic functions, we prove a lemma about subintervals on $(0,1)$. Let $\{r_1, r_2, r_3, \ldots, r_n, \ldots\}$ be the set of rational numbers in $(0,1)$ enumerated.

LEMMA 1. *There exists a sequence of irrational positive numbers (v_n), such that $v_n < n^{-1}$, $[r_n - v_n, r_n + v_n] \subset (0,1)$ for all n, and none of the points $r_i, r_i + \frac{1}{2}v_i, r_i + v_i (i = 1, \ldots, j-1)$ lie in the closed interval $[r_j - v_j, r_j + v_j]$ for any $j \geqslant 2$.*

Proof is by induction on n. For $n = 1$, let v_1 be any irrational number such that $[r_1 - v_1, r_1 + v_1] \subset (0,1)$ and hence $v_1 < 1$. Now suppose that $v_1, \ldots, v_{N-1}$ have been suitably selected for some $N > 1$. Then r_N is not any of the rational numbers $r_1, \ldots, r_{N-1}$ and is certainly not any of the irrational numbers $r_1 + \frac{1}{2}v_1, \ldots, r_{N-1} + \frac{1}{2}v_{N-1}, r_1 + v_1, \ldots, r_{N-1} + v_{N-1}$. So let v_N be a positive irrational number such that $v_N < N^{-1}, [r_N - v_N, r_N + v_N] \subset (0,1)$, and such that the closed interval $[r_N - v_N, r_N + v_N]$ excludes all the points $r_1, \ldots, r_{N-1}, r_1 + \frac{1}{2}v_1, \ldots, r_{N-1} + \frac{1}{2}v_{N-1}, r_1 + v_1, \ldots, r_{N-1} + v_{N-1}$. This completes the induction.

Now for each n, let c_n be a positive number so small that $d(0, c_n h_n) < 2^{-n}$, where h_n is the spike on the interval $(r_n, r_n + v_n)$. We construct a number of nowhere analytic functions in F.

LEMMA 2. *Let (a_n) be any bounded sequence of real numbers, and let c_n and h_n be as given before. Let $w_n = 1$ if $a_n \neq 0$ and $w_n = 0$ if $a_n = 0$. Then the series $\sum_{n=1}^{\infty} a_n c_n h_n$ converges to a nowhere analytic function in F if the sequence $(w_n r_n)$ is dense in $[0,1]$.*

Proof. Let $(w_n r_n)$ be dense in $[0,1]$. Since $d(0, c_n h_n) < 2^{-n}$ and $|a_n| \leqslant M$ for fixed M, it follows that

$$d(0, a_n c_n h_n) < (M+1)2^{-n}$$

from the definition of d. Thus

$$d\left(0, \sum_{i=N}^{N+j} a_i c_i h_i\right) < (M+1)2^{1-N}$$

and $\sum_{i=1}^{\infty} a_i c_i h_i$ must converge in the complete metric space (F, d), say to f.

Now fix any index $N > 1$ for which $w_N = 1$ and $a_N \neq 0$. By Lemma 1, for each $i = 1, \ldots, N-1$, the interval $[r_N - v_N, r_N + v_n]$ is a subinterval of $(-\infty, r_i)$ or $(r_i + v_i, \infty)$ or $(r_i, r_i + v_i)$. Each such h_i is analytic in $(r_N - v_N, r_N + v_N)$ and so is $\sum_{i=1}^{N-1} a_i c_i h_i$. Note also that $r_N \notin (r_i, r_i + v_i)$ and $r_N + \frac{1}{2}v_N \notin (r_i, r_i + v_i)$ for $i > N$. Thus

$$h_i^{(k)}(r_N) = h_i^{(k)}\left(r_N + \frac{1}{2}v_N\right) = 0 \text{ for } i > N \text{ and all } k \geqslant 0.$$

Of course $h_N^{(k)}(r_N) = 0$ for all $k \geqslant 0$.

For any function G, point x, and index j, let $T(G, j, x)$ denote the sum $\sum_{i=0}^{j} G^{(i)}(r_N)(x - r_N)^i / i!$. It follows that $\lim_{j\to\infty} \sum_{i=1}^{j} a_i c_i h_i^{(k)} = f^{(k)}$ uniformly for $k \geqslant 0$, and

$$\sum_{i=1}^{N} a_i c_i h_i\left(r_N + \frac{1}{2}v_N\right) = f\left(r_N + \frac{1}{2}v_N\right),$$

so

$$\lim_{j\to\infty} T\left(f, j, r_N + \frac{1}{2}v_N\right) = \lim_{j\to\infty} \sum_{i=1}^{N-1} T\left(a_i c_i h_i, j, r_N + \frac{1}{2}v_N\right)$$

$$+ \lim_{j\to\infty} \sum_{i=N}^{\infty} T\left(a_i c_i h_i, j, r_N + \frac{1}{2}v_N\right)$$

$$= \lim_{j\to\infty} \sum_{i=1}^{N-1} T\left(a_i c_i h_i, j, r_N + \frac{1}{2}v_N\right)$$

$$= \sum_{i=1}^{N-1} a_i c_i h_i\left(r_N + \frac{1}{2}v_N\right).$$

But

$$f\left(r_N + \frac{1}{2}v_N\right) = \sum_{i=1}^{N-1} a_i c_i h_i\left(r_N + \frac{1}{2}v_N\right) + a_N c_N h_N\left(r_N + \frac{1}{2}v_N\right).$$

Since $h_N(r_N + \frac{1}{2}v_N) > 0$, and $c_N \neq 0, a_N \neq 0$, we conclude that

$$\lim_{j\to\infty} T\left(f, j, r_N + \frac{1}{2}v_N\right) \neq f\left(r_N + \frac{1}{2}v_N\right)$$

and f is not analytic on the interval $(r_N - v_N, r_N + v_N)$. Finally, if I is any subinterval of $[0,1]$, then by hypothesis there is some index $N > 1$ for which $a_N \neq 0$ and $[r_N - v_N, r_N + v_N] \subset I$, so f is not analytic on I. Consequently f is nowhere analytic.

Next we produce the required vector space.

THEOREM 2. *There exists a real vector space of functions on* $[0,1]$ *under pointwise addition and scalar multiplication, of dimension* c, *in which all the nonzero vectors are in* F_0.

Proof. Let V denote the real vector space of bounded infinite sequences of real numbers under coordinatewise addition and scalar multiplication. Let $p_1 < p_2 < p_3 < \cdots < p_n < \cdots$ denote the prime integers in increasing order. For each n, let J_n denote the set of all indices j for which the denominator of r_j (in lowest terms) is a power of p_n.

For any bounded sequence of real numbers (a_n), not the zero sequence, the series

$$\sum_{k=1}^{\infty} a_k\left(\sum_{j\in J_k} c_j h_j\right)$$

satisfies the hypothesis of Lemma 2 because the sets $J_k (k = 1, 2, 3, \ldots)$ are mutually disjoint and each set $\{r_j\colon j \in J_k\}$ is dense in $(0,1)$. Let $S(a_n)$ be the sum of this series in F_0. Then S is clearly a homomorphism of the real vector space V into F with zero kernel. Thus S is an isomorphism.

It remains to prove that V has dimension c. The way we prove this is by comparing V with the vector space V_0, the space of all bounded sequences of rational numbers over the field Q of rational numbers. We first plan to show that any linearly independent set of vectors in V_0 is also linearly independent in V. It suffices to let this set be finite.

Let $v_1, \ldots, v_n$ be linearly independent vectors in V_0. Let $v_i = (a_{ij})_{j=1}^{\infty}$ for each $i = 1, \ldots, n$. Let Q^n denote n-space over Q. For each integer $m > 1$, let U_m denote the vector subspace of Q^n, composed of all vectors $(x_1, \ldots, x_n)$ such that $\sum_{i=1}^{n} x_i a_{ij} = 0$ for each $j = 1, \ldots, m$. Then $U_1 \supset U_2 \supset U_3 \supset \cdots$ and $\cap_{m=1}^{\infty} U_m = (0)$, since $v_1, \ldots, v_n$ are linearly independent. But $\dim U_1$ is finite, so $U_k = (0)$ for some k. Let A be the n by k matrix whose ith row is $(a_{i1}, \ldots, a_{ik})$. Since the rows of A are linearly independent, there are n linearly independent columns $C_1, \ldots, C_n$, of A. Thus $\det(C_1, \ldots, C_n) \neq 0$ and $C_1, \ldots, C_n$ are linearly independent over the real field. It follows that there are no real $y_1, \ldots, y_n$, not all 0, such that $\sum_{i=1}^{n} y_i a_{ij} = 0$ for each $j = 1, \ldots, k$. Hence $v_1, \ldots, v_n$ are also linearly independent vectors in V. It follows that $\dim V \geqslant \dim V_0$.

Finally, the cardinality of V_0 is c and Q is countable, so the dimension of V_0 is c. The cardinality of V is c, so the dimension of V is also c. This completes the proof.

In conclusion we produce a subspace with these properties that is dense in (F, d). The metric d on F used here is, of course, the complete metric defined in §2.

THEOREM 3. *There is a vector subspace of* F, *satisfying the hypothesis of Theorem* 2, *that is dense in the metric space* (F, d).

Proof. Let Z be a basis of the vector subspace $S(V)$ of F constructed in the proof of Theorem 2. The dimension of $S(V)$ is c. Let $z_1, z_2, z_3, \ldots$ be countably infinitely many members of Z and let $Z_1 = Z \setminus \{z_1, z_2, z_3, \ldots, \}$. Let b_n be a positive number so small that $d(0, b_n z_n) < n^{-1}$ for each n. Let $\{p_1, p_2, p_2, \ldots\}$ be the set of polynomials in F with rational coefficients, enumerated. Let W_1 denote the subspace of F generated by all the functions $p_n + b_n z_n (n = 1, 2, 3, \ldots)$ together with all the functions in Z_1.

We claim that any linear combination of these generators is in F_0. Suppose that $\Sigma_n a_n (p_n + b_n z_n) + \Sigma_\alpha a_\alpha z_\alpha = f$ is analytic on some interval. Then

$$\sum_n a_n b_n z_n + \sum_\alpha a_\alpha z_\alpha = f - \sum_n a_n p_n$$

is analytic on the same interval, so all the coefficients a_n and a_α are 0. It follows that W_1 has dimension c and all the nonzero vectors in W_1 are nowhere analytic.

It remains only to show that W_1 is dense in F. Take any $f \in F$ and any $\varepsilon > 0$. Fix an integer N so large that $N^{-1} < \frac{1}{2}\varepsilon$. There is an $n \geqslant N$ such that $d(f, p_n) < \frac{1}{2}\varepsilon$ because the set $\{p_n\}$ is dense in F (see §2). So

$$d(f, p_n + b_n z_n) \leqslant d(f, p_n) + d(p_n, p_n + b_n z_n) < \frac{1}{2}\varepsilon + n^{-1} < \varepsilon.$$

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UNSOLVED PROBLEMS

EDITED BY RICHARD GUY

In this department the MONTHLY *presents easily stated unsolved problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.*

A COUPLE OF CUBIC CONUNDRUMS

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Here is an editorial collage of comments from referees and other correspondents concerning two submissions to this section, the first by

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who asks

How Few Incomparable Cuboids Will Tile the Cube?

He asks the question in any number of dimensions, but let us start in two. Two rectangles are

called **incomparable** if neither will fit inside the other (with sides parallel; a 1×6 rectangle will fit inside a 5×5 square, but we count them as incomparable). That is, if we can assign their dimensions $a_1 \times a_2$ and $b_1 \times b_2$ so that $a_1 < b_1 \leqslant b_2 < a_2$. The problem of tiling a rectangle with (at least two) incomparable rectangles (every pair of tiles is to be incomparable) was posed by Reingold [5] and solved by P. van Emde Boas, H. W. Lenstra, W. A. A. Nuij, and others [2].

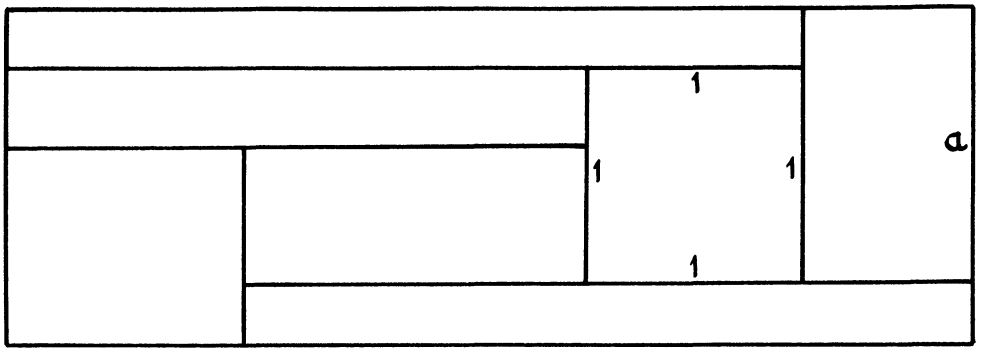


FIG. 1

They showed that no rectangle can be incomparably tiled with 6 or fewer tiles. Nuij used Fig. 1, in which each of the 7 tiles has unit area and $a = (2 + \sqrt{19})/5$, to show that a rectangle can be tiled with 7 incomparable rectangles. In fact all but a finite number of shapes of rectangle can be so tiled, since an affine transformation will not affect the ratios of the areas. The exceptions occur when the transformation makes two tiles equal (but with different orientation). Then make the width a little less and add an eighth tile of smaller area, but with length equal to that of the whole rectangle.

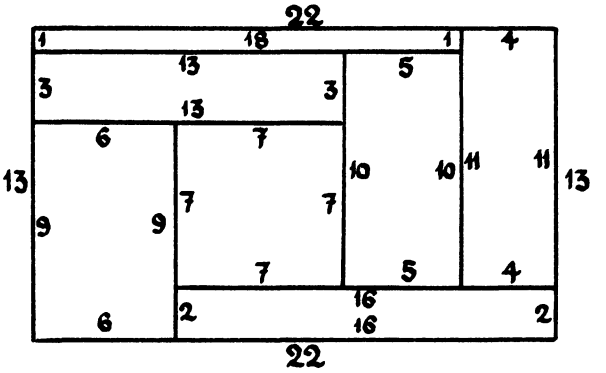


FIG. 2

The definitive work on the subject is by Yao, Reingold, and Sands [9], with help from Sheldon Akers and Wilfred Hansen. They confine their attention to tiles with integer sides. This loses very little generality and poses some further problems. They show, *inter alia*, that there are only a finite number of integer pairs $n \geqslant m \geqslant 10$ such that an $m \times n$ rectangle *cannot* be tiled with 7 incomparable rectangles, and that for each $t \geqslant 7$ there are only a finite number of integers n such that an $n \times n$ square cannot be tiled with t incomparable rectangles. They show that Fig. 2 is the smallest (measured by perimeter or by area) integer rectangle that can be tiled with 7 pieces, and that Fig. 3 is the smallest integer square with 7 incomparable tiles, while Fig. 4 shows a smaller square, tiled with 8 tiles; it is not known if this is optimal.

When we move to higher dimensions, the definition of incomparable is harder to see, but the

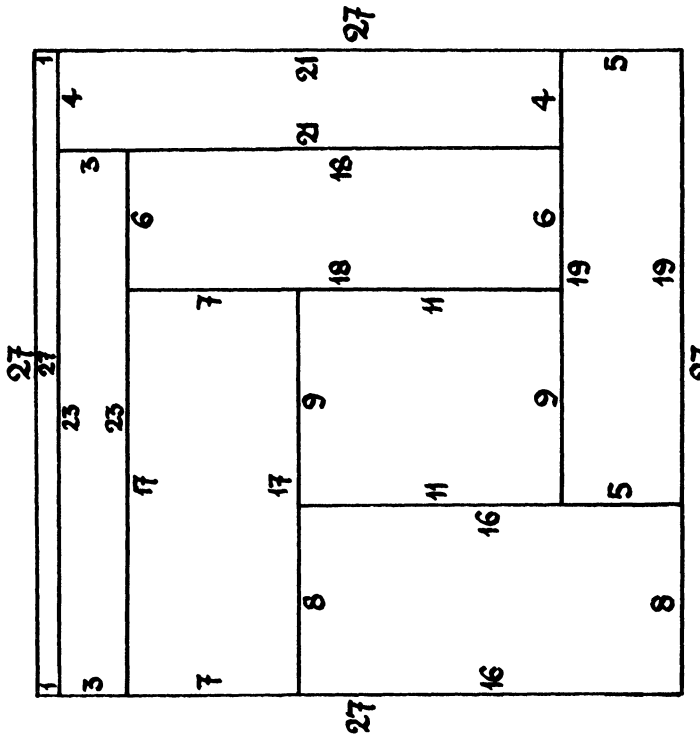

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FIG. 4

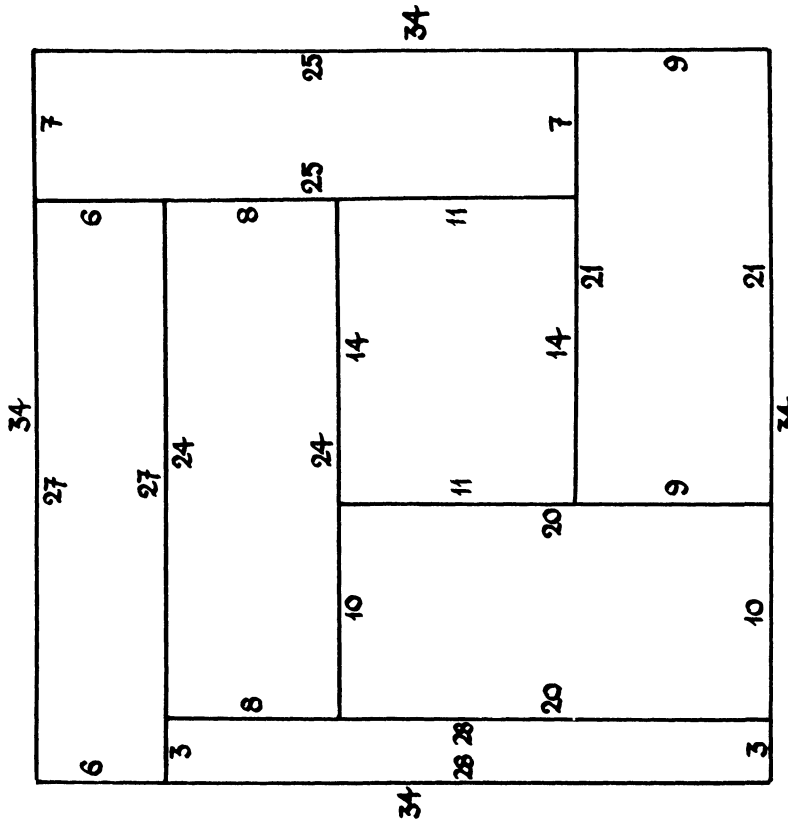

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FIG. 3

situation is clarified by Guysand's lemma, whose proof is left to the reader:

If two cuboids (in any number of dimensions) are comparable, then they may be placed one inside the other with corresponding dimensions in increasing order.

To compare two d-dimensional cuboids, write their dimensions in increasing order:

$$a_1 \leqslant a_2 \leqslant a_3 \leqslant \cdots \leqslant a_d, \qquad b_1 \leqslant b_2 \leqslant b_3 \leqslant \cdots \leqslant b_d.$$

The cuboids are **incomparable** if there is an i and a j such that $a_i < b_i$ and $a_j > b_j$.

A **trivial** incomparable tiling of a higher-dimensional cuboid can be obtained by erecting prisms with a lower-dimensional incomparable tiling as base. For example, cuboids of height 1 on Fig. 2, or of heights 34 and 27 on Figs. 3 and 4, give what might be thought to be optimal tilings of a cuboid or cube in three dimensions. But the first of these is not optimal, neither in the sense of minimum integer volume, nor in its number of pieces! Sands has found the nontrivial incomparable tiling shown in Fig. 5 which involves only six pieces $1 \times 1 \times 15$, $1 \times 2 \times 11$, $1 \times 3 \times 10$, $2 \times 3 \times 6$, $2 \times 4 \times 4$ and $3 \times 3 \times 5$ of total volume only 180.

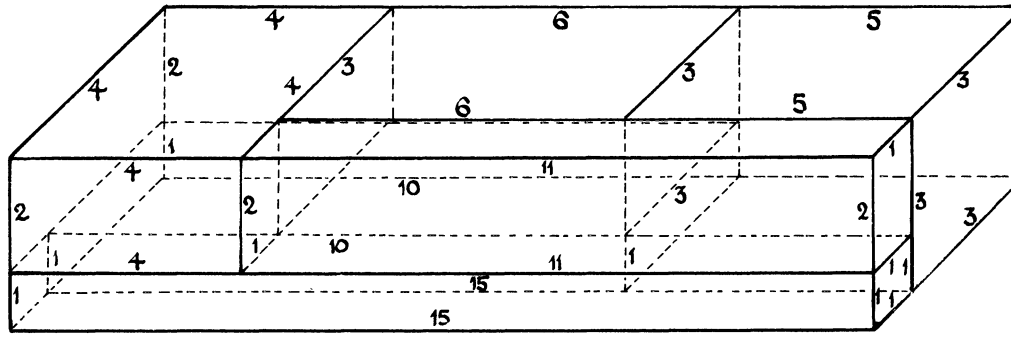


FIG. 5

Although Nuij used an affine transformation to tile various shapes of rectangle, it is not in general possible to do this, even in two dimensions, without affecting the comparability, because of the possibility of changing the orientation (permuting the dimensions) of a tile before attempting to fit it into another. In fact it has been shown that the pattern of Fig. 5 cannot be used to produce a six-piece non-trivial incomparable tiling of a cube, as opposed to a cuboid. Indeed, Grošek and Ondrejička claim that the least order for the incomparable tiling of a cube is seven, though a proof would seemingly be at least tedious. The following problems arise:

1. Give a neat proof that there is no 6-piece incomparable tiling of the cube.
2. Is there a smaller integer tiling of the cuboid than that of Sands (Fig. 5)?
3. Is there a nontrivial (not all pieces have the edge of the cube as a dimension) 7-piece incomparable tiling of the cube? If not, what is the least order of such a nontrivial tiling?
4. Fig. 5 implies that there is a (trivial) 6-piece incomparable tiling of a 4-dimensional cuboid. What is the optimal nontrivial incomparable tiling of a 4-dimensional cuboid, where optimal is interpreted as (a) having minimum number of pieces, or (b) having minimum integer dimensions, measured by volume?
5. Answer the same questions with cube in place of cuboid.
6. Continue into more dimensions.
7. Return to two dimensions. Is Fig. 4 the smallest incomparable integer tiling of a square, regardless of the number of pieces? This question is asked on page 152 of [**6**]. In fact the number of pieces in the smallest such tiling is 8 or 9.
8. Are there incomparable tilings (with arbitrarily many pieces) in which the pieces are all arbitrarily close to being square? Alternatively, among the n-piece incomparable tilings of a

rectangle (square), what is the minimax of b_i/a_i where the tiles measure $a_i \times b_i$, $a_i \leqslant b_i$, the maximum is taken over all tiles, $1 \leqslant i \leqslant n$, and the minimum over all n-piece tilings? There are two answers, one for the rectangle and one for the square, each functions of n. For example, Fig. 3 shows that $r(7)$ and $s(7)$ are each at most $28/3$, while Fig. 6 (Bill Sands) gives $r(8) \leqslant s(8) \leqslant 15/2$. Are there values of n for which $r(n) < s(n)$?

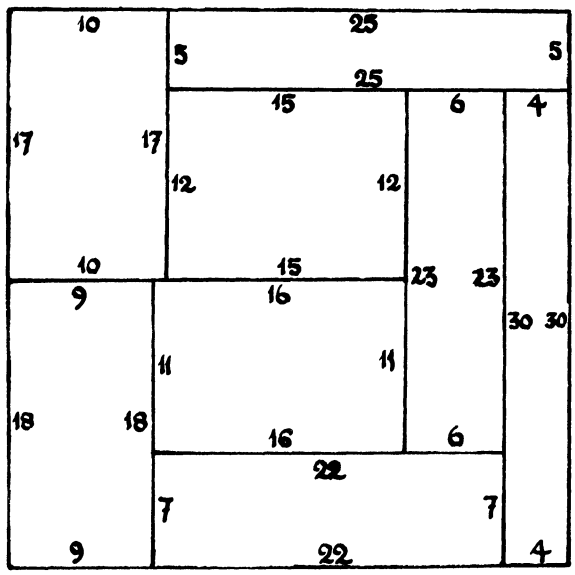


FIG. 6

The second problem comes from

John F. Sallee, The Applied Mathematics Group, University of Washington, 3024 N.E. Blakeley, Seattle, WA 98105. He asks

What Is the Simplexity of the *d*-Cube?

The word "simplicity" is too simple to alert the reader to what is wanted. As in the first problem, we wish to dissect a d-dimensional cube into polytopes, but this time the pieces are to be simplexes. We use **simplicate** to generalize "triangulate" to more than two dimensions. What is the least number, $S(d)$, of simplexes in a simplication of the d-dimensional cube?

A good first guess is $d!$ The reader will easily verify inductively that this is a candidate by joining one vertex of the d-cube to a simplication of each of the d $(d-1)$-dimensional faces which do not contain that vertex. But even for $d = 3$ we can do better. The well-known (to Kepler and to the Greeks) inscription of a regular tetrahedron in a cube (Fig. 7) shows that $S(3) \leqslant 5$; in fact equality holds.

This, and the argument already given, show that $S(4) \leqslant 20$. In fact Cottle [1], Mara [3], [4] and Sallee [7], [8] have shown that $S(4) = 16$. Sallee has also shown that $60 \leqslant S(5) \leqslant 67$ and $223 \leqslant S(6) \leqslant 344$. Can these rather rapidly widening bounds be improved?

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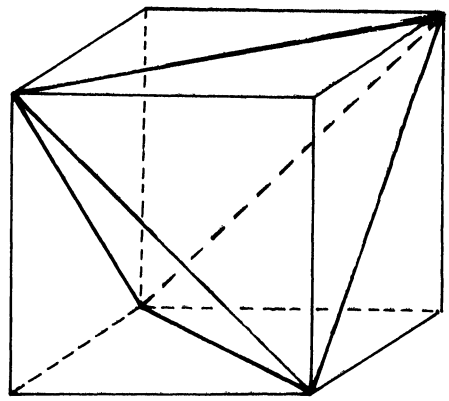


FIG. 7

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NOTES

EDITED BY SABRA S. ANDERSON, SHELDON AXLER, AND J. ARTHUR SEEBACH, JR.

For instructions about submitting Notes for publication in this department see the inside front cover.

SUMMATION OF DIVERGENT SERIES BY COMPUTER

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It is well known, at least in a qualitative way, that if the terms of certain divergent series, such as the harmonic series, are summed on a computer, the accumulated sum will eventually stop increasing. This happens because the terms eventually become so small compared to the sum that the computer, because of the limited number of digits it carries, does not recognize that the sum is increased by the addition of these small terms. In this note we give a detailed analysis of this phenomenon and apply the results of this analysis to some divergent series with, in at least one case, a surprising result.

Suppose we are computing in b bit arithmetic; i.e., given two numbers which have mantissas of b or fewer bits, we perform arithmetic operations by first performing them correctly, then rounding the results to b bits, rounding up in the ambiguous case. Let $\sum_1^\infty a_n$ be a divergent series with $a_n > 0$ for all n and decreasing monotonically to 0. Let $S(n) = \sum_1^n a_k$. Instead of $S(n)$ our computer gives us $\hat{S}(n; b)$ defined as follows:

$$\hat{S}(1;b) = \mathrm{rd}_b a_1; \quad \hat{S}(j+1;b) = \mathrm{rd}_b\big(\hat{S}(j;b) + \mathrm{rd}_b a_{j+1}\big), \quad j = 1,2,3,\ldots$$

where $\mathrm{rd}_b x$ is the number obtained by rounding x to b bits. For n sufficiently large $\hat{S}(n;b) = \hat{S}(n-1;b)$; i.e., our computer no longer recognizes that addition of the nth term increases the partial sum. The largest value of n such that $\hat{S}(n;b) > \hat{S}(n-1;b)$ is the number of terms our computer sums and it is this number that we seek. Two approximations will be made in finding this number. First the condition $\hat{S}(n;b) > \hat{S}(n-1;b)$ will be replaced by

$$\mathrm{rd}_b(\mathrm{rd}_b S(n-1) + \mathrm{rd}_b a_n) > \mathrm{rd}_b S(n-1). \tag{1}$$

This approximation is equivalent to ignoring round-off errors in the accumulation of the partial sums until this error causes a term (and therefore all subsequent terms) to have no effect on the partial sums.

Let $\mathrm{rd}_b S(n-1)$ have mantissa M and exponent ε in normalized floating point representation so that $\frac{1}{2} \leqslant M < 1$ and $\mathrm{rd}_b S(n-1) = 2^{\varepsilon} M$. It is easy to show that

$$\varepsilon - 1 = \left[\frac{\log \mathrm{rd}_b S(n-1)}{\log 2}\right], \tag{2}$$

where $[x]$ means the greatest integer not greater than x. Let $\mathrm{rd}_b a_n = 2^{\varepsilon} z$. Then

$$\mathrm{rd}_b(\mathrm{rd}_b S(n-1) + \mathrm{rd}_b a_n) = \mathrm{rd}_b 2^{\varepsilon}(M+z) = 2^{\varepsilon}\mathrm{rd}_b(M+z)$$

and inequality (1) is equivalent to $\mathrm{rd}_b(M+z) > M$. Since M is a b bit number, the last inequality is satisfied if and only if $z \geqslant 2^{-(1+b)}$ or, using (2),

$$b \geqslant \left[\frac{\log \mathrm{rd}_b S(n-1)}{\log 2}\right] - \frac{\log \mathrm{rd}_b a_n}{\log 2}. \tag{3}$$

The next approximation we make is to replace this inequality with $b \geqslant D(n-1)$, where

$$D(n) = \left[\frac{\log S(n)}{\log 2}\right] - \frac{\log a_{n+1}}{\log 2}. \tag{4}$$

Note that the right side of (3) is close to $D(n-1)$ unless $\mathrm{rd}_b S(n-1)$ has a different exponent in its normalized floating point representation than does $S(n-1)$. This only happens for special values of $S(n-1)$ which we may suppose do not occur frequently, therefore our approximation is a reasonable one.

Subject to the above approximations the number we seek, namely, the largest number of terms which can be summed in b bit arithmetic is characterized by

$$D(n-1) \leqslant b < D(n). \tag{5}$$

The unique integer n satisfying (5) will be called the bth *critical value*. By examining equation (4) we can see that critical values can occur in two ways. The integer n is a critical value for some b if and only if one of the following two conditions is satisfied:

(i) $$\left[\frac{\log S(n-1)}{\log 2}\right] < \left[\frac{\log S(n)}{\log 2}\right],$$

(ii) $$a_n \geqslant 2^{-l} > a_{n+1} \text{ for some integer } l.$$

In the first case, which is equivalent to the condition that $S(n)$ have a larger exponent than $S(n-1)$, we will call n a *sum critical value* and in the second case we will call n a *term critical value*.

We can find small critical values (if any exist) by examining values of $S(n)$ and a_n. In the following examples we find large critical values in certain series where for some decreasing function f on $(0,\infty)$, $a_n = f(n)$ and for some increasing function F on $(0,\infty)$, which we can easily compute, $F(n)$ approximates $S(n)$ for large n. We find term critical values by setting $f(x) = 2^{-l}$, where l is a nonnegative integer. Let ξ_l be the solution to this equation. Then

$$f([\xi_l]) \geqslant f(\xi_l) = 2^{-l} > f([\xi_l]+1);$$

therefore $[\xi_l]$ is a term critical value. Similarly, to find term critical values we set $F(x) = 2^k$. Let x_k be the solution to this equation. Then

$$F([x_k]) \leqslant F(x_k) = z^k < F([x_k] + 1),$$

which implies $[x] + 1$ is a sum critical value.

For our first example we take the harmonic series $\Sigma 1/k$. As is well known, $S(n) - \log n \to \gamma$ where $\gamma = .5772156649\ldots$ is Euler's constant; therefore for large n we have the approximation $S(n) \approx \log n + \gamma$. The term critical values are clearly the positive powers of 2. By examining the numerical values of $S(n)$ for $n = 1$ to 31 we find that the first two sum critical values are 4 and 31. If we let $F(x) = \log x + \gamma$, we get, using the method described above, the sum critical values $[\exp(2^k - \gamma)] + 1$. The first few sum critical values are

$$4, 31, 1674, 4.9891 \times 10^8, 4.4335 \times 10^{13} \text{ and } 3.5008 \times 10^{27}.$$

(These values do not change if we use a more accurate approximation for $S(n)$, derived from the Euler-Maclaurin sum formula.) Results for this example are summarized in Table 1.

TABLE 1

Bits(b)	Terms	Sum
1	2	1.5
2	4*	2.08
3	2^2	2.08
4	2^3	2.72
5	2^4	3.38
6	31*	4.03
7	2^5	4.06
.		
12	2^{10}	7.51
13	1674*	8.00

Asterisks denote sum critical values. The table can be extended to 97 bits using the sum critical values given above. Note that in the intervals between sum critical values the sum is linear in the number of bits.

The next example we discuss is $\Sigma 1/\sqrt{j}$. Since the terms approach zero more slowly than in the harmonic series, we would expect that more terms can be summed for a given number of bits and this expectation will be fulfilled. By calculating $S(1), S(2), \ldots, S(100)$ we find that the first few sum critical values are 3, 7, 22, and 76. To get more sum critical values we use the approximate formula $S(n) \approx F(n)$, where

$$F(x) = 2\sqrt{x} + \frac{1}{2\sqrt{x}}\left(1 - \frac{1}{12x}\right) - 1.460354509$$

obtained from the Euler-Maclaurin sum formula [1]. (This approximation is accurate to seven digits when n is greater than 10.) By solving numerically for x in $F(x) = 2^k$, $k = 5, 6, 7, \ldots$ and adding 1 to the integral part of the solution, we get the sum critical values 280, 1071, 4190, 16571, 65910, ... The term critical values are easily seen to be the positive integral powers of 4. The results for up to 10 bits are given in Table 2.

TABLE 2

Bits	Terms	Sum
1	3*	2.28
2	4	2.78
3	7*	4.02
4	16	6.67
5	22*	8.03
6	64	14.50
7	76*	16.03
8	256	30.57
9	280*	32.04
10	1024	62.55

Asterisks denote sum critical values. This table may be extended to 17 bits by using the sum critical values given above. From the table we see that sum critical and term critical values alternate. Another observation one can make from the table is that increasing the number of bits from b to $b + 1$ has a small effect on the sum if b is even and nearly doubles the sum if b is odd. Both these phenomena are not limited to the few values on this table. Let x_k be the solution of $F(x) = 2^k$. It is not hard to show that

$$F(4^{k-1}) < 2^k < F(4^k - 1);$$

therefore

$$4^{k-1} < [x_k] + 1 < 4^k.$$

This shows that sum and term critical values continue to alternate. Furthermore

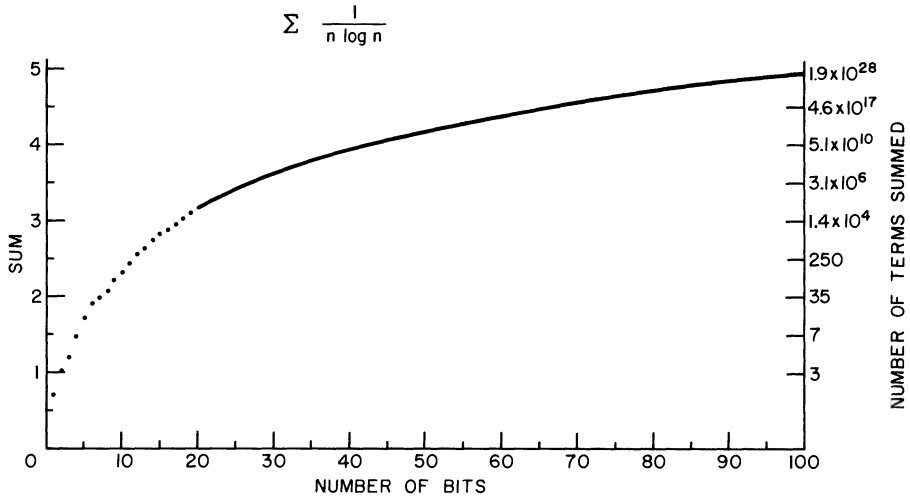
$$\frac{F([x_k] + 1)}{F(4^{k-1})} \to 1 \text{ and } \frac{F(4^k)}{F([x_k] + 1)} \to 2,$$

which verifies the remarks made above about the effect of increasing the number of bits by 1.

The last example we offer is $\Sigma_2^\infty (j \log j)^{-1}$. This series diverges much more slowly than either of the other two examples. Again we use the Euler-Maclauren sum formula to find an approximation $S(n) \approx F(n)$ where

$$F(x) = \log\log x + \frac{1}{2x \log x}\left(1 - \frac{1 + \log x}{6x \log x}\right) + 0.7946786454.$$

(This approximation is accurate to 7 digits when n is greater than 10.) This example differs from the two preceding ones in that it is not possible to write down the term critical values immediately. To find them we must solve numerically the equation $x \log x = 2^l$. If we denote the solution by ξ_l, it is easily verified that $\xi_{l+1}/\xi_l \to 2$ and this fact facilitates the numerical solution, which was found for $l = 1$ through 100. Since $F([\xi_{100}]) \approx 4.95$, there are only 3 sum critical values less then $[\xi_{100}]$, namely, those values of n for which the sum first exceeds 1, 2 and 4. (These values are 3, 38, and 5.1427×10^{10}, respectively!) Results are shown in the accompanying graph.



Reference

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He can tell something that remains fixed from something that does not. (See p. 648.)

THEOREM. *If the nonstationary Poisson postulates are satisfied, then for every* $s \geq 0$ *and* $t > 0$ *the random variable* $N(s+t) - N(s)$ *has a Poisson distribution with parameter equal to* $\int_s^{s+t} \lambda(u)\,du$.

From this theorem students will see that the Poisson distribution with parameter λt remains appropriate for the distribution of $N(s+t) - N(s)$ even in nonstationary examples provided $\lambda = (1/t)\int_s^{s+t} \lambda(u)\,du$, which is the rate function averaged over the time interval of interest. Thus, for modeling the probability distribution of $N(s+t) - N(s)$ it is sufficient to have a good estimate of the average rate over the time period $[s, s+t]$; a detailed knowledge of how $\lambda(u)$ varies over this time interval is not required.

Proof of Theorem. Fix $s \geq 0$ and $t > 0$ arbitrarily. Divide the time interval $[s, s+t]$ into n subintervals of equal length and label the subinterval end-points as follows:

$$t_0 = s, \qquad t_1 = s + \frac{t}{n}, \qquad t_2 = s + \frac{2t}{n}, \ldots, t_n = s + t.$$

Let $\lambda_i = \inf\{\lambda(u)\colon t_{i-1} \leq u \leq t_i\}$, $i = 1, 2, \ldots, n$, and let $X_1, X_2, \ldots, X_n$ be independent random variables such that X_i has a Poisson distribution with parameter $\lambda_i\, t/n$ for $i = 1, 2, \ldots, n$. Now we use the nonstationary Poisson postulates and Lemma 2 to conclude that

$$N(s+t) - N(s) = \sum_{i=1}^{n} \{N(t_i) - N(t_{i-1})\}$$

is stochastically larger than $S_n = X_1 + X_2 + \cdots + X_n$. From theory we know that the sum S_n must have a Poisson distribution with parameter equal to $\sum_{i=1}^{n} \lambda_i\, t/n$. Since, by lower Riemann sum analysis, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i\, t/n = \int_s^{s+t} \lambda(u)\,du,$$

it follows that S_n converges in distribution to the Poisson distribution with parameter equal to $\int_s^{s+t} \lambda(u)\,du$. Hence, by Lemma 1, we see that $N(s+t) - N(s)$ is stochastically larger than the Poisson distribution with parameter $\int_s^{s+t} \lambda(u)\,du$. But, from upper Riemann sum analysis it follows similarly that $N(s+t) - N(s)$ is stochastically smaller than the Poisson distribution with parameter $\int_s^{s+t} \lambda(u)\,du$. Therefore, the desired result is established. □

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ANSWER TO PHOTO ON PAGE 633

Donald Ornstein has made deep contributions to ergodic theory. He discovered, in particular, a transformation with no invariant measure, and later he proved that for many transformations entropy constitutes a complete set of invariants.

BERNSTEIN POLYNOMIALS VIA THE SHIFTING OPERATOR

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Let $f(x)$ be defined in $[0,1]$. The nth Bernstein polynomial for $f(x)$ is given by

$$B_n(f;x) = \sum_{i=0}^{n} f\left(\frac{i}{n}\right) J_{n,i}(x), \tag{1}$$

where

$$J_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}, \quad i = 0,1,\ldots,n. \tag{2}$$

The $J_{n,i}$ are called the nth Bernstein basis polynomials, since they form a basis of P_n, the collection of all polynomials with degree $\leqslant n$. In a first course in real variables the classical Weierstrass approximation theorem is often proved by showing that the $B_n(f;x)$ converge uniformly to $f(x)$ in $[0,1]$ for any $f(x)$ which is continuous on $[0,1]$.

Section 1 of this note shows that the shifting operator and the difference operator make derivations for some basic results of the Bernstein polynomials much simpler than the typical approaches. The rest of the paper provides two applications of the Bernstein polynomials. The first application is to find the formula for the sum of pth powers of the first n integers. The other gives an explicit expression for the cardinal basis functions for two-point Hermite interpolation. (A cardinal basis in this context is a basis for the space of interpolating polynomials consisting of polynomials for each of which the values and the values of its derivatives up to certain order at two interpolation points are all zero except for one value which is 1.)

1. Throughout the paper n is a fixed positive integer. We set $f_i = f(i/n)$ for simplicity; then $B_n(f;x)$ is called the Bernstein polynomial determined by the coefficient sequence $(f_0, f_1, \ldots, f_n)$. If a line segment is drawn between each two points $((i-1)/n, f_{i-1})$ and $(i/n, f_i)$, $i = 1,2,\ldots,n$, the graph of a piecewise linear function $\hat{f}_n(x)$ is obtained. This graph of $y = \hat{f}_n(x)$ is called the characteristic polygon associated with $B_n(f;x)$ by the French engineer P. Bézier [1] who is one of the pioneers in Computer Aided Geometric Design (CAGD).

If the shifting operator E is defined by $Ef_i = f_{i+1}$, then $f_i = E^i f_0$. Hence by the binomial theorem (1) can be rewritten as

$$B_n(f;x) = [(1-x)I + xE]^n f_0, \tag{3}$$

where I denotes the identity operator. Let $\Delta = E - I$, then $\Delta f_i = f_{i+1} - f_i$, so the operator Δ is just the forward difference operator. Because $(1-x)I + xE = I + x\Delta$, (3) is equivalent to

$$B_n(f;x) = (I + x\Delta)^n f_0. \tag{4}$$

By the binomial theorem again we get from (4) that

$$B_n(f;x) = \sum_{i=0}^{n} \binom{n}{i} x^i \Delta^i f_0. \tag{5}$$

Since

$$\frac{d^k}{dx^k}(I + x\Delta)^n = \frac{n!}{(n-k)!}(I + x\Delta)^{n-k}\Delta^k,$$

we have

$$\frac{d^k}{dx^k} B_n(f;x) = \frac{n!}{(n-k)!}(I + x\Delta)^{n-k}\Delta^k f_0 \tag{6}$$

for $k = 0, 1, \ldots, n$. Note that $(I + x\Delta)^{n-k}\Delta^k f_0$ is the $(n-k)$th Bernstein polynomial determined by the coefficient sequence $(\Delta^k f_0, \Delta^k f_1, \ldots, \Delta^k f_{n-k})$. Hence, we see that if $\hat{f}_n(x)$ is positive in $[0,1]$, then so is $B_n(f; x)$; and that if $\hat{f}_n(x)$ is increasing in $[0,1]$, then so is $B_n(f; x)$. The same is true of convexity. Substitution of $x = 0$ and $x = 1$ into both sides of (6) shows that

$$\frac{d^k}{dx^k} B_n(f; 0) = \frac{n!}{(n-k)!}\Delta^k f_0 \tag{7}$$

and

$$\frac{d^k}{dx^k} B_n(f; 1) = \frac{n!}{(n-k)!}\Delta^k f_{n-k}, \tag{8}$$

respectively, for $k = 0, 1, \ldots, n$. Notice that commutativity of E and Δ has been used in the derivation of (8). The binomial theorem gives

$$\Delta^k = (E - I)^k = \sum_{i=0}^{k} (-1)^{k-i}\binom{k}{i} E^i;$$

thus (7) and (8) can be rewritten, respectively, as

$$\frac{d^k}{dx^k} B_n(f; 0) = \frac{n!}{(n-k)!} \sum_{i=0}^{k} (-1)^{k+i}\binom{k}{i} f_i, \tag{9}$$

and

$$\frac{d^k}{dx^k} B_n(f; 1) = \frac{n!}{(n-k)!} \sum_{j=0}^{k} (-1)^{j}\binom{k}{j} f_{n-j}. \tag{10}$$

Consider two special cases $k = 0$ and $k = 1$ in which (9) and (10) become

$$B_n(f; 0) = f_0, \quad B_n'(f; 0) = n(f_1 - f_0), \tag{11}$$

$$B_n(f; 1) = f_1, \quad B_n'(f; 1) = n(f_n - f_{n-1}). \tag{12}$$

We know from (11) and (12) that $B_n(f; x)$ interpolates to its characteristic polygon at the first and the last vertices and that the curve $y = B_n(f; x)$ is tangential to the first side and the last side of the polygon at its starting point and ending point. All the results above can be found in P. J. Davis [**2**], but the derivations there are more complicated than ours.

2. Bézier Basis Polynomials. Consider the following $n + 1$ polynomials

$$f_{n,i}(x) = J_{n,i}(x) + J_{n,i+1}(x) + \cdots + J_{n,n}(x), \text{ for } i = 0, 1, \ldots, n. \tag{13}$$

Obviously they form another basis of P_n. These polynomials, in quite different forms, were used by Bézier to represent parametric curves. The representation, now called the Bézier curve in CAGD, is substantially the vector-valued Bernstein polynomial (e.g., see [**1**]). From (13) it is clear that $f_{n,0}(x) \equiv 1$. We need only study $f_{n,i}(x)$ for $i = 1, 2, \ldots, n$. By definition (13), $f_{n,i}$ is the Bernstein polynomial determined by the sequence $(0, \ldots, 0, 1, \ldots, 1)$, i.e., the sequence whose elements are all 1 except i zeros in first places $(1 \leqslant i \leqslant n)$. Hence by (11) and (12) we have

$$f_{n,i}(0) = 0, \quad f_{n,i}(1) = 1, \quad \text{for } i = 1, \ldots, n. \tag{14}$$

The difference sequence of $(0, \ldots, 0, 1, \ldots, 1)$ is the sequence having n elements with a 1 in the ith position and 0 everywhere else. Then by (6) we have shown that

$$f_{n,i}'(x) = nJ_{n-1,i-1}(x), \quad \text{for } i = 1, 2, \ldots, n. \tag{15}$$

We conclude by (14) and (15) that $f_{n,i}(x)$ $(1 \leqslant i \leqslant n)$ is strictly increasing from 0 to 1 in $[0,1]$. For further properties of these functions, a recent paper [**3**] by this author can be consulted. Since we have $J_{n,i} = f_{n,i} - f_{n,i+1}$ from (13), the Bernstein polynomial (1) can be expressed in terms of

the Bézier polynomials

$$B_n(f;x) = f_0 + \sum_{i=1}^{n} f_{n,i}(x)\,\Delta f_{i-1}. \tag{16}$$

Equating the right-hand side of (16) and (5), we get

$$\sum_{i=1}^{n} f_{n,i}(x)\,\Delta f_{i-1} = \sum_{i=1}^{n} \binom{n}{i} x^i \Delta^i f_0. \tag{17}$$

Now set $a_i = \Delta f_{i-1}$, for $i = 1,2,\ldots,n$. Then (17) becomes

$$\sum_{i=1}^{n} a_i f_{n,i}(x) = \sum_{i=1}^{n} \binom{n}{i} x^i \Delta^{i-1} a_1. \tag{18}$$

Substituting $x = 1$ into both sides of (18) yields a well-known identity of combinatorics

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} \binom{n}{i} \Delta^{i-1} a_1. \tag{19}$$

Assume $a_i = i^p$. By (18) we then have a series of identities involving the Bézier polynomials,

$$\sum_{i=1}^{n} f_{n,i}(x) = nx,$$

$$\sum_{i=1}^{n} i f_{n,i}(x) = nx + n(n-1)x^2/2,$$

$$\sum_{i=1}^{n} i^2 f_{n,i}(x) = nx + 3n(n-1)x^2/2 + n(n-1)(n-2)x^3/3,$$

$$\cdots,$$

for $p = 0,1,2,\ldots$. Putting $x = 1$ into both sides of the above identities immediately yields the formulas of the sum of pth powers of the first n integers. For other derivations of the formula, see [4] and [5].

3. Cardinal Basis for Two-Point Hermite Interpolation. The properties of the Bernstein polynomials and their derivatives provide an easy method for finding an explicit expression for a polynomial characterized by its values and the values of its derivatives at two points. Let us begin with a simple example to illustrate the ideas involved in this section. The cubic cardinal basis functions for the Hermite interpolation at $x = 0$ and $x = 1$ are uniquely determined by the following cardinal properties

$$\begin{bmatrix} \phi_0(0) & \phi_0(1) & \phi_0'(0) & \phi_0'(1) \\ \psi_0(0) & \psi_0(1) & \psi_0'(0) & \psi_0'(1) \\ \phi_1(0) & \phi_1(1) & \phi_1'(0) & \phi_1'(1) \\ \psi_1(0) & \psi_1(1) & \psi_1'(0) & \psi_1'(1) \end{bmatrix} = I_4,$$

where I_4 denotes the 4×4 identity matrix.

Let (a_0, a_1, a_2, a_3) be the coefficient sequence associated with the cubic ϕ_0; $\phi_0(0) = 1$ and $\phi_0(1) = 0$ imply $a_0 = 1$ and $a_3 = 0$, respectively. The first and the last side of the characteristic 3-gon of ϕ_0 should be parallel to x-axis since $\phi_0'(0) = \phi_0'(1) = 0$. Hence we must have $a_1 = 1$, $a_2 = 0$ and then

$$\phi_0(x) = J_{3,0}(x) + J_{3,1}(x) = 1 - 3x^2 + 2x^3.$$

Similar considerations show that $(0,0,1,1)$, $(0,1/3,0,0)$ and $(0,0,-1/3,0)$ are the coefficient

sequences for $\psi_0(x)$, $\phi_1(x)$ and $\psi_1(x)$, respectively. Thus

$$\psi_0(x) = J_{3,2}(x) + J_{3,3}(x) = 3x^2 - 2x^3,$$
$$\phi_1(x) = J_{3,1}(x)/3 = x(1-x)^2,$$
$$\psi_1(x) = -J_{3,2}(x)/3 = x^2(x-1).$$

Let us generalize the above problem. Let m and n be nonnegative integers and $N \equiv m + n - 1$. We seek $m + n$ polynomials $\phi_k(x)$ $(0 \leqslant k \leqslant m-1)$ and $\psi_t(x)$ $(0 \leqslant t \leqslant n-1)$, each of which is of degree N, having the following cardinal properties:

$$\phi_k^{(i)}(0) = \delta_{ik}, \quad \phi_k^{(j)}(1) = 0; \tag{20}$$

$$\psi_t^{(i)}(0) = 0, \quad \psi_t^{(j)}(1) = \delta_{jt}, \qquad 0 \leqslant i,k \leqslant m-1, \quad 0 \leqslant j,t \leqslant n-1. \tag{21}$$

Hosaka [**6**] has given explicit expressions for these functions without proof, and what he discusses is the special case where $m = n$. Derivations of these functions for general m and n are given in this section. Let $(a_0, a_1, \ldots, a_N)$ be the coefficient sequence for $\phi_k(x)$. From the second part of (20) we see clearly that

$$a_m = a_{m+1} = \cdots = a_N = 0. \tag{22}$$

By (9) we see that $a_0, a_1, \ldots, a_{m-1}$ can be determined by the system of linear equations:

$$M\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix} = \frac{(N-k)!}{N!}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \ldots (k+1)\text{st row},$$

where the $m \times m$-matrix M is $\left[(-1)^{i+j}\binom{i}{j}\right]$, $0 \leqslant i, j \leqslant m-1$. It is understood that $\binom{i}{j} = 0$ for $i < j$; so M is a lower triangular matrix. It is easy to verify that $M^{-1} = \left[\binom{i}{j}\right]$, $0 \leqslant i, j \leqslant m-1$.

Hence we get

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{m-1} \end{bmatrix} = \frac{(N-k)!}{N!}M^{-1}\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = \frac{(N-k)!}{N!} \times \text{the } (k+1)\text{st column of } M^{-1}.$$

Thus we have

$$a_i = \begin{cases} 0, & 0 \leqslant i < k, \\ \dfrac{(N-k)!}{N!}\binom{i}{k}, & k \leqslant i \leqslant m-1, \end{cases}$$

and then

$$\phi_k(x) = \frac{(N-k)!}{N!}\sum_{i=k}^{m-1}\binom{i}{k}J_{N,i}(x), \quad \text{for } 0 \leqslant k \leqslant m-1. \tag{23}$$

If we substitute $1 - x$ for x, and n for m in the right-hand side of (23), then $(-1)^k\psi_k(x)$ is obtained, i.e.,

$$(-1)^k \psi_k(x) = \frac{(N-k)!}{N!} \sum_{i=k}^{n-1} \binom{i}{k} J_{N,i}(1-x).$$

Since $J_{N,i}(1-x) = J_{N,N-i}(x)$, we have at last

$$\psi_l(x) = (-1)^l \frac{(N-l)!}{N!} \sum_{j=l}^{n-1} \binom{j}{l} J_{N,N-j}(x), \quad 0 \leqslant l \leqslant n-1. \tag{24}$$

We point out that when $n = 0$ ($m = 0$), (23) [(24)] becomes

$$\phi_k(x) = x^k/k!\left[\psi_k(x) = (x-1)^k/k!\right],$$

which is the cardinal basis for Taylor interpolation at $x = 0$ ($x = 1$).

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A THEOREM CONCERNING LORENTZ TRANSFORMATIONS

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This note presents an outline of a theorem in Linear Algebra relating to Lorentz Transformations in Special Relativity. Although only the statement of the theorem is of direct interest to physics students, students in my upper level algebra courses have found the proof interesting since it shows a direct application of abstract Linear Algebra. Variations of the theorem can be found in the literature: [5] (pp. 69–75), presents a straightforward derivation based on sophisticated physical postulates; [3] (pp. 100–7), uses group structure ideas and the accounting for independent parameters; [1] (pp. 491–5), and [7] (pp. 146–9), are fairly sophisticated in the mathematical treatment; finally, [4] (p. 161), poses as a problem the corresponding result in two dimensions. The significance of our presentation lies certainly not with the statement of the result, but the algorithmic nature of the proof—unlike the above, our proof implicitly shows how to calculate the relevant physical quantities. The presentation is given here in outline form and relies on ideas with which students in an upper level Linear Algebra course are likely to be acquainted.

We give only the most minimal background in Special Relativity; interested readers can find an elementary, but mathematically respectable and very well written treatment in [6].

An event occurring at time x_0 and spatial location (x_1, x_2, x_3) is represented by the vector $u = (x_0, x_1, x_2, x_3)$ in Euclidean space R^4. Vectors with 0th component 0 are called spatial and the vector $(1, 0, 0, 0)$ pointing towards increasing time is denoted e_0. Assuming throughout that units have been chosen so that c = speed of light is 1, the Spacetime metric between $u = (x_0, x_1, x_2, x_3)$ and $v = (y_0, y_1, y_2, y_3)$ is $x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3$ which can be written as the standard inner product, (Gu, v), between Gu and v where

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \tag{1}$$

But words such as "norm," "orthonormal," and the like will refer to the standard inner product

$(\,,\,)$ rather than the Relativistic one.

A *Lorentz Transformation* is defined to be a linear map $L: R^4 \to R^4$ which preserves the Spacetime metric: for all u in R^4

$$(GLu, Lu) = (Gu, u). \tag{2}$$

Given a real number α let

$$A = \begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \sinh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

A is called a *boost* in the x direction; it is straightforward to show that A is in fact a Lorentz Transformation. Without going into many of the details (which are explained in [**6**]) we note in passing that A relates coordinates measured relative to "lab frame" axes with coordinates measured relative to "rocket frame" axes. Both frames are assumed to be inertial, that is, equations of motion of objects undergoing no force are straight line equations. Now, *if* the spatial axes of the two frames are coincident at a time which is labeled 0 by observers in each frame and also *if* the rocket is moving at speed $v = \tanh\alpha$ in the direction of positive x as seen by observers in each frame, *then* the equation $v = Au$ is the relationship between the coordinates u in the rocket frame and v in the lab frame of the same event. Now, noting that boosts are Lorentz and noting that the Spacetime metric is the Euclidean metric when restricted to the last three coordinates, any boost composed with a rotation in the last three coordinates is also Lorentz.

The converse is

THEOREM. *Suppose that L is Lorentz. Then $L = AR$, where A is a boost and R is an orthogonal operator in the spatial dimensions (with possible spatial and temporal reversals of orientation). More specifically, there is an orthonomal basis in which the first vector is e_0 and in terms of which A has the form* (3) *and*

$$R = \begin{bmatrix} \pm 1 & 0 \\ 0 & S \end{bmatrix},$$

where S is an orthogonal 3×3 matrix ($SS^t = I$).

Two fundamental properties of operators are used in the proof ([**2**], pp. 341–3): (a) A nonnegative operator B (self-adjoint with $(Bv, v) \geqslant 0$ for all v) has a unique nonnegative square root A, that is, $A^2 = B$. A nonnegative B will be denoted $B \geqslant 0$. (b) (Polar Decomposition) Any $L: R^4 \to R^4$ can be decomposed as

$$L = AR, \tag{4}$$

where $A \geqslant 0$ and R is orthogonal and therefore $A = A^t$, $RR^t = I$ in an orthonormal basis. Furthermore A is unique in this decomposition; R is also if L is nonsingular.

Outline of Proof. Assume that L satisfies (2) for all u.

Step 1. Show that $(GLu, Lv) = (Gu, v)$ for all u, v in R^4. Hence

$$L^tGL = G \tag{5}$$

and, in particular, L is nonsingular. Square (5) to show that

$$LL^tGLL^t = G. \tag{6}$$

Step 2. Use Polar Decomposition to write $L = AR$ uniquely. Let $B = A^2 \geqslant 0$. Show that (6) implies

$$BGB = G. \tag{7}$$

Step 3. Since B is self-adjoint, it has four real eigenvalues λ_i; the associated eigenvectors v_i

can be taken to be orthonormal Since $B \geqslant 0$, all $\lambda_i \geqslant 0$. Show that (7) implies

$$\lambda_1\lambda_2\lambda_3\lambda_4 = \det B = +1. \tag{8}$$

Step 4. Use (7) to show that

$$\lambda_i\lambda_j(Gv_i, v_j) = (Gv_i, v_j).$$

Hence for all i, j

$$\lambda_i\lambda_j = 1 \quad \text{or} \quad (Gv_i, v_j) = 0. \tag{9}$$

Step 5. Give labels to the components: $v_i = (a_0^i, a_1^i, a_2^i, a_3^i)$. Using orthonormality and (1) show that

$$\delta_{ij} + (Gv_i, v_j) = 2a_0^i a_0^j \tag{10}$$

for all i, j, where $\delta_{ij} = 1, 0$ if $i = , \neq j$, respectively.

Step 6. Assume that $B \neq I$; then there is an eigenvalue—call it λ_1—not equal to 1. By (8) there is another one not equal to 1—call it λ_2. Show that (9), (10) imply $2(a_0^i)^2 = 1$ for $i = 1, 2$. Without loss of generality choose v_1, v_2 so that

$$a_0^i = 1/\sqrt{2} \tag{11}$$

for $i = 1, 2$. Show that (9), (10), (11) imply $\lambda_1\lambda_2 = 1$. Assume

$$\lambda_1 > 1 > \lambda_2, \quad \lambda_2 = 1/\lambda_1.$$

Step 7. Expand the inner product to show that

$$\begin{aligned} &(v_1 - e_0/\sqrt{2}, v_2 - e_0/\sqrt{2}) = -1/2, \\ &\|v_i - e_0/\sqrt{2}\| = 1/\sqrt{2}, \end{aligned} \tag{12}$$

for $i = 1, 2$ where $\| \ \|$ indicates the Euclidean norm. Use the Schwarz inequality to show that (12) implies the two vectors entering into the first equation in (12) are linearly dependent. Hence there exists a unit spatial vector $n = (0, x, y, z)$ with

$$\begin{aligned} v_1 &= (e_0 + n)/\sqrt{2}, \\ v_2 &= (e_0 - n)/\sqrt{2}. \end{aligned}$$

Step 8. Set $\mu_1, \mu_2 = (\lambda_1 + \lambda_2)/2, (\lambda_1 - \lambda_2)/2$, respectively. Show that

$$\begin{aligned} Be_0 &= \mu_1 e_0 + \mu_2 n, \\ Bn &= \mu_2 e_0 + \mu_1 n, \end{aligned} \tag{13}$$

and that there is a real number α (noting $\lambda_2 = 1/\lambda_1$) with

$$\begin{aligned} \sinh\alpha &= \mu_2, \\ \cosh\alpha &= \mu_1. \end{aligned} \tag{14}$$

Step 9. B has two other eigenvectors v_3, v_4 perpendicular to v_1, v_2 and hence to e_0. Since v_3, v_4 are spatial, use (1), (7) to show that $\lambda_i = +1$ for $i = 3, 4$.

Step 10. Combining the previous two steps, it is straightforward to see that B has the form (3) and hence that

$$A = \begin{bmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

in the orthonormal basis e_0, n, v_3, v_4 where $\theta = \alpha/2$.

Step 11. Since $L = AR$, show that (5) implies $RGR^t = AGA = G$. Equating components in $RG = GR$, noting that $RR^t = I$, show that

$$R = \begin{bmatrix} \pm 1 & 0 \\ 0 & S \end{bmatrix},$$

where S is 3×3 orthogonal in the standard basis. Hence R has the same form in the basis e_0, n, v_3, v_4. This concludes the proof outline.

The physical interpretation of the theorem is worth noting: The theorem claims that if L is linear and satisfies (2) and if R is orientation preserving in the sense that $R_{11} = +1$ and $\det S = +1$, then L is the transformation of coordinates from a rocket to a lab frame in which case n points in the direction of motion of the rocket. Finally note that L may be decomposed as $L = \tilde{R}\tilde{A}$. The reader will find that the relationships among $R, A, \tilde{R}, \tilde{A}$ have interesting computational details.

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THE JORDAN CURVE THEOREM VIA THE BROUWER FIXED POINT THEOREM

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A homeomorphic image of a closed interval $[a, b]$ $(a < b)$ is called an arc and a homeomorphic image of a circle is called a Jordan curve. One of the most classical theorems in topology is

THEOREM (Jordan Curve Theorem). *The complement in the plane R^2 of a Jordan curve J consists of two components, each of which has J as its boundary.*

Since the first rigorous proof given by Veblen [**4**] in 1905, a variety of elementary (and lengthy) proofs have been provided by many authors. Among them, the one given by Moise [**3**] is intuitive and transparent yet lengthy. The purpose of this note is to provide a short proof by modifying Moise's method. In order to avoid the tedious arguments, we will use the following celebrated theorem of Brouwer (for an elementary proof, for example, see [**1**]).

THEOREM (Brouwer Fixed Point Theorem). *Every continuous map from a disk into itself has a fixed point.*

To begin with, we note two simple facts concerning the components of $R^2 - J$, where J is a Jordan curve: (a) $R^2 - J$ has exactly one unbounded component, and (b) each component of $R^2 - J$ is path connected and open. The assertion (a) follows from the boundedness of J, and (b) from the local path-connectedness of R^2 and the closedness of J.

LEMMA 1. *If $R^2 - J$ is not connected, then each component has J as its boundary.*

Proof. By assumption, $R^2 - J$ has at least two components. Let U be an arbitrary component. Since any other component W is disjoint from U and open, W contains no point of the closure $\overline{U}$ and hence no point of the boundary $\overline{U} \cap U^c$ of U. Thus $\overline{U} \cap U^c \subset J$. Suppose $\overline{U} \cap U^c \neq J$. Then there exists an arc $A \subset J$ such that

$$(\#) \qquad \overline{U} \cap U^c \subset A.$$

We will show that this leads to a contradiction. By the preceding remark (a), $R^2 - J$ has at least one bounded component. Let o be a point in a bounded component; if U itself is bounded we choose o in U. Let D be a large disk with center o such that its interior contains J. Then the boundary S of D is contained in the unbounded component of $R^2 - J$. Since arc A is homeomorphic to the interval $[0,1]$, the identity map $A \to A$ has a continuous extension $r: D \to A$ by the Tietze Extension Theorem (see, for example, [2]). We define a map $q: D \to D - \{o\}$, according as U is bounded or not, by

$$q(z) = \begin{cases} r(z) & \text{for } z \in \overline{U}, \\ z & \text{for } z \in U^c, \end{cases} \quad \text{or } q(z) = \begin{cases} z & \text{for } z \in \overline{U}, \\ r(z) & \text{for } z \in U^c, \end{cases}$$

respectively. By (#), the intersection of the two closed sets $\overline{U}$ and U^c lies in A on which r is the identity map. Thus q is well defined and continuous. Note that $q(z) = z$ if $z \in S$. Let $p: D - \{o\} \to S$ be the natural projection and let $t: S \to S$ be the antipodal map. Then the composition $t \cdot p \cdot q: D \to S \subset D$ has no fixed point. This contradicts the Brouwer fixed point theorem.

Note that the preceding proof implicitly contains a proof that no arc separates R^2, which is often a lemma to the Jordan curve theorem.

We need another lemma for our purpose. Let $E(a, b; c, d)$ denote the rectangular set $\{(x, y) \mid a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$ in the plane R^2, where $a < b$ and $c < d$.

LEMMA 2. *Let $h(t) = (h_1(t), h_2(t))$ and $v(t) = (v_1(t), v_2(t))$ $(-1 \leqslant t \leqslant 1)$ be continuous paths in $E(a, b; c, d)$ satisfying*

$$(\#\#) \qquad h_1(-1) = a, \quad h_1(1) = b, \quad v_2(-1) = c, \quad v_2(1) = d.$$

Then the two paths meet, i.e., $h(s) = v(t)$ for some s, t in $[-1,1]$.

Proof. Suppose $h(s) \neq v(t)$ for all s, t. Let $N(s,t)$ denote the maximum-norm of $h(s) - v(t)$, i.e.,

$$N(s,t) = \operatorname{Max}\{\, |h_1(s) - v_1(t)|, \quad |h_2(s) - v_2(t)| \,\}.$$

Then $N(s,t) \neq 0$ for all s, t. We define a continuous map F from $E(-1,1; -1,1)$ into itself by

$$F(s,t) = \left(\frac{v_1(t) - h_1(s)}{N(s,t)}, \frac{h_2(s) - v_2(t)}{N(s,t)} \right).$$

Note that the image of F is in the boundary of $E(-1,1; -1,1)$. To see that F has no fixed point, assume $F(s_0, t_0) = (s_0, t_0)$. By the above remark, we have $|s_0| = 1$ or $|t_0| = 1$. Suppose, for example, $s_0 = -1$. Then by (##), the first coordinate of $F(-1, t_0)$, $(v_1(t_0) - h_1(-1))/N(-1, t_0)$, is nonnegative and hence cannot equal $s_0 (= -1)$. Similarly, the other possibilities of $|s_0| = 1$ or $|t_0| = 1$ cannot occur. This contradicts the Brouwer fixed point theorem since $E(-1,1; -1,1)$ is homeomorphic to a disk.

We are now ready to prove the Jordan curve theorem. By Lemma 1, we need only show that $R^2 - J$ has one and only one bounded component. The proof will consist of the following three steps: Establishing the notation and defining a point z_0 in $R^2 - J$; proving that the component U containing z_0 is bounded; and proving that there is no bounded component other than U.

Since J is compact, there exist points a, b in J such that the distance $\|a - b\|$ is the largest. We may assume that $a = (-1, 0)$ and $b = (1, 0)$. Then the rectangular set $E(-1,1; -2,2)$ contains J, and its boundary Γ meets J at exactly two points a and b. Let n be the middle point of the top side of $E(-1,1; -2,2)$, and s the middle point of the bottom side; i.e., $n = (0,2)$ and $s = (0, -2)$. The segment $\overline{ns}$ meets J by Lemma 2. Let l be the y-maximal point (that means the point $(0, y)$ with maximal y) in $J \cap \overline{ns}$. Points a and b divide J into two arcs; we denote the one containing l by J_n and the other by J_s. Let m be the y-minimal point in $J_n \cap \overline{ns}$ (possibly, $l = m$). Then the segment $\overline{ms}$ meets J_s; otherwise, the path $\overline{nl} + \widehat{lm} + \overline{ms}$ (where $\widehat{lm}$ denotes the subarc of J_n with end points l and m) could not meet J_s, contradicting Lemma 2. Let p and q

denote the y-maximal point and the y-minimal point in $J_s \cap \overline{ms}$, respectively. Finally, let z_0 be the middle point of the segment $\overline{mp}$. (see Fig. 1).

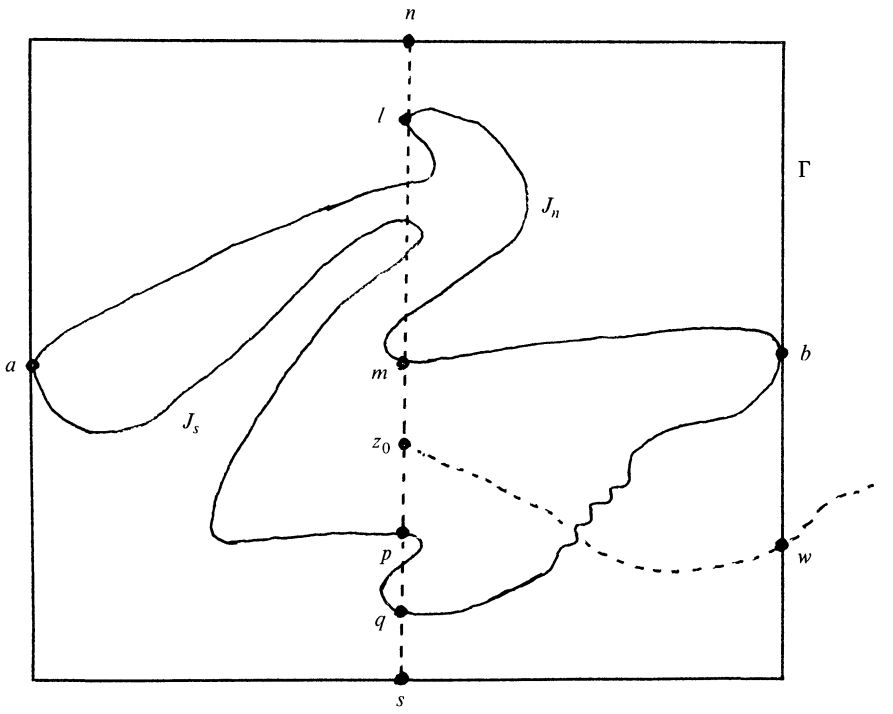


FIG. 1

Now we show that U, the component of $R^2 - J$ which contains z_0, is bounded. Suppose that U is unbounded. Since U is path connected, there exists a path α in U from z_0 to a point outside $E(-1,1;-2,2)$. Let w be the first point at which α meets the boundary Γ of $E(-1,1;-2,2)$. Denote by α_w the part of α from z_0 to w. If w is on the lower half of Γ, we can find a path $\widehat{ws}$ in Γ from w to s which contains neither a nor b. Now consider the path $\overline{nl} + \widehat{lm} + \overline{mz_0} + \alpha_w + \widehat{ws}$. This path does not meet J_s, contradicting Lemma 2. Similarly, if w is on the upper half of Γ, the path $\overline{sz_0} + \alpha_w + \widehat{wn}$ fails to meet J_n, where $\widehat{wn}$ is the shortest path in Γ from w to n. The contradiction shows that U is a bounded component.

Finally suppose that there exists another bounded component $W(\neq U)$ of $R^2 - J$. Clearly $W \subset E(-1,1;-2,2)$. We denote by β the path $\overline{nl} + \widehat{lm} + \overline{mp} + \widehat{pq} + \overline{qs}$, where $\widehat{pq}$ is the subarc of J_s from p to q. As seen easily, β has no point of W. Since a and b are not on β, there are circular neighborhoods V_a, V_b of a, b, respectively, such that each of them contains no point of β. By Lemma 1, a and b are in the closure $\overline{W}$. Hence, there exist $a_1 \in W \cap V_a$ and $b_1 \in W \cap V_b$. Let $\widehat{a_1b_1}$ be a path in W from a_1 to b_1. Then the path $\overline{aa_1} + \widehat{a_1b_1} + \overline{b_1b}$ fails to meet β. This contradicts Lemma 2 and completes our proof.

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C E N T E R S E C T I O N
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Telegraphic Reviews

Edited by Lynn Arthur Steen, with the assistance of the Mathematics Departments of Carleton, Macalester, and St. Olaf Colleges. Books submitted for review should be sent to Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, Minnesota 55057.

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General, T(16: 1), L*. Concepts of Mathematical Modeling. Walter J. Meyer. McGraw-Hill, 1984, xi + 440 pp, $33.95. [ISBN: 0-07-041747-4] A textbook for an introductory course in mathematical modeling. An unusual feature of this text is its emphasis on the modeling process. Includes both analytical and computer exercises. AO

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Elementary, T?(13-14). Theory and Applications of Mathematics for Elementary School Teachers. Joseph Hashisaki. Wiley, 1984, xxxv + 402 pp, $23.95. [ISBN: 0-471-09637-7] Ambitious text tries to cover everything, consequently elucidates nothing. Some sections (e.g., p. 220) consist entirely of one sentence echoing the section title, with no examples or explanation. Foreward gives promise of microcomputer orientation with three articles on computers in mathematics reprinted, yet computers are mentioned only peripherally thereafter (and not at all in the index). MW

Elementary, T(13: 1, 2). Topics in Contemporary Mathematics, Third Edition. Jack R. Britton, Ignacio Bello. Harper & Row, 1984, xi + 683 pp, $25.50. [ISBN: 0-06-040992-4] Changes from the Second Edition (First Edition, TR, August-September 1975; Second Edition, TR, June-July 1979) include: new chapters on consumer mathematics, counting techniques; revision of chapter on geometry, new examples, and 700 additional exercises. JRG

Elementary, T(13). Mathematics for Computer Programmers. Christine Benedyk Kay. Prentice-Hall, 1984, xvi + 393 pp, $24.95. [ISBN: 0-13-562140-2] Essentially college algebra oriented towards the needs of computer programmers. Emphasis on how to interpret problems and construct algorithms. Topics include arithmetic, algebraic inequalities, sequences, number systems, and Boolean algebra. RM

Elementary, T(13). Fear No More: An Adult Approach to Mathematics. Peter Hilton, Jean Pedersen. Addison-Wesley, 1983, viii + 281 pp, $18.25. [ISBN: 0-201-05713-1] First of three (projected) volumes designed to teach mathematics to intelligent yet reluctant adults. Concepts are motivated by problems, explained through relevant examples, then studied abstractly. Covers decimals, fractions, integers, factorization, scientific notation, elementary ideas of probability. Decimals precede fractions, are motivated by monetary and metric calculations. MW

Precalculus, T(13: 1). College Algebra and Trigonometry, Second Edition. Robert Ellis, Denny Gulick. HarBrace J, 1984, viii + 631 pp, $24.95. [ISBN: 0-15-507909-3] Changes from the First Edition

(TR, August-September 1981) seem quite minor, most noteworthy being a right triangle introduction to trigonometry. JS

Precalculus, T*(13: 1). College Algebra and Trigonometry. John R. Durbin. Wiley, 1984, x + 674 pp, $23.95. [ISBN: 0-471-03367-7] Standard topics. Prerequisites are one year of high school geometry and two years of high school algebra. No previous experience with trigonometry is assumed. Larger than usual number of examples. Well-graded exercises. Answers to odd numbered problems and to review exercises. Calculator use encouraged. Summaries of formulas inside front and back covers. Carefully done. JK

Precalculus, T(13: 1). College Algebra. William Bosch. Brooks/Cole, 1983, xiii + 432 pp, $23.95. [ISBN: 0-534-02866-7] Focus on calculus preparation, with problems and examples more like actual calculus (and science) problems. Example: simplify $[(a^2-x^2)^{1/2} - (a^2+x^2)(a^2-x^2)^{-1/2}]/(a^2-x^2)$, as well as: simplify $(a^{-1}b^{-1}/c^{-2})^3$. Deemphasis on definitions and terms in favor of examples. RB

Education, T(16-17: 1). Calculators in the Classroom: With Applications for Elementary and Middle School Teachers. David G. Moursund. Wiley, 1981, vi + 202 pp, $13.95 (P). [ISBN: 0-471-08113-2] Most suitable for inservice courses. Capabilities and limitations of nonprogrammable calculators. Problem solving, games, motivational uses. Fairly standard fare enhanced by plenty of exercises and suggestions for reports and opinion papers. MW

Education, P. Microcomputers in Education. Ed: I.C.H. Smith. Ser. in Computers & Their Applic. Halsted Pr, 1982, 212 pp, $34.95 (P). [ISBN: 0-470-27319-4] Principles of development and distribution of educational software (primarily CAI) described in Part I. Part II contains information on computer languages: COMAL, a Basic-Pascal cross; PETCAI, an authoring language; and graphics routines. Part III describes applications in education, including computer-assisted experimentation. MW

Education, P, L. Second International Mathematics Study: Summary Report for the United States. F. Joe Crosswhite, et al; Ed: Ken Travers. US National Coordinating Center (U. of Illinois at Urbana-Champaign), 1984, ix + 98 pp, (P). An overly-simplified summary of data from a 1981-82 international assessment of mathematics achievement and classroom processes in grades 8 and 12 in 21 countries. This widely-reported study suggests that U.S. achievement is generally below international norms. Even the achievement of our best students (the 2% who take high school calculus) is only slightly better than the mean performance of senior high mathematics students in other countries. The report is flawed by lack of confidence intervals for its summary figures, and by an inexplicable lack of correction for guessing in the analysis of the raw data. Nevertheless, the conclusions appear inescapable. LAS

Education, T(14-16: 1). Helping Children Learn Mathematics. Robert E. Reys, Marilyn N. Suydam, Mary M. Lindquist. Prentice-Hall, 1984, x + 262 pp, $24.95 (P). [ISBN: 0-13-387027-8] General methods content as well as ideas for teaching specific mathematical topics in the elementary school. Much attention to results from research, both in teaching suggestions and in reference lists. Emphasis on problem solving, estimation, drill that is meaningful. "Lesson snapshots," lesson cards, and activity cards provide hints for teaching. MW

Education, S(17-18). Make It Simpler: A Practical Guide to Problem Solving in Mathematics. Carol Meyer, Tom Sallee. Addison-Wesley, 1983, xiii + 282 pp, $17.60 (P). [ISBN: 0-201-20036-8] Detailed plans for teaching problem solving to middle school students. Mandates student groups of four, suggests a timeline for the year involving, on average, 15 minutes per day. Strategies include using subproblems, diagrams, smaller numbers and patterns, and working backwards. Sixty problems, each presented at three levels of difficulty. Best for inservice course or individual teacher use. MW

History, S*(15), P*, L*. Ptolemy's Almagest. Transl. & Annotated: G.J. Toomer. Springer-Verlag, 1984, ix + 693 pp, $64. [ISBN: 0-387-91220-7] Translation based on and keyed to the Teubner text of Heiberg with corrections derived from the medieval Arabic translations. Profusely footnoted. Worked examples in one appendix to ease understanding of technical aspects. Corrections to Heiberg's text in another. Excellent reference list and index. JK

History, S*(13-18), P*, L.** Compendium Euclidis Curiosi: Or Geometrical Operations. Georg Mohr. Introduction by: Henrik Meyer. Georg Mohr Foundation (University Library, Copenhagen, Denmark), 1982, xiii + 29 pp, (P). [ISBN: 87-7421-337-7] Photographic reproduction of the anonymously published Dutch original published in Amsterdam in 1673 and of Joseph Moxon's English translation printed in London in 1677. In the preface, Mohr tells the reader that he had read in Peter Ramus' Dutch Geometry (Amsterdam, 1622) that one John Baptista had performed all of Euclid's constructions with an unmarked straightedge and a pair of compasses with a fixed opening but that he had not been able to find a copy of the work of Baptista. The exposition consists of problems followed by elegant solutions. A copy should be in every college and university library. JK

Discrete Mathematics, S*(13), L*. Recursion Sequences. A.I. Markushevich. Transl: V. Zhitomirsky. Little Math. Lib. MIR, 1983, 48 pp, (P). A brief monograph devoted to the study of those sequences which can be expressed as finite recurrance relations. An interesting look at familiar objects from a fresh perspective. (First Printing, TR, August-September 1977.) CEC

Number Theory, S(15-18), P, L. Zahlen. H.-D. Ebbinghaus, et al. Grund. Math, No. 1. Springer-Verlag, 1983, xii + 288 pp, $17.90 (P). [ISBN: 0-387-12666-X] First in the new Grundwissen series of texts. Offers a unifying thread in the concept of "number" from historical definitions of integers

through the reals and complexes to (much) more advanced topics on quaternion and composition algebras, stressing connections throughout with various mathematical disciplines. Many excellent theorems (nicely proved) not usually found in any single text, together with historical notes and good chapter-by-chapter bibliographies. DD

Linear Algebra, T*(13-14: 1), S, L. Introductory Linear Algebra with Applications, Third Edition. Bernard Kolman. Macmillan, 1984, xvii + 455 pp, $27.95. [ISBN: 0-02-366020-1] The format is different from the Second Edition (First Edition, TR, May 1976; Second Edition, TR, August-September 1980). Includes larger pages and a two-color printing process. Chapter reviews and chapter tests have been added. Includes more exercises, examples and figures. The table of contents is the same. CEC

Linear Algebra, T(13-14: 1). Elementary Linear Algebra. John A. Thorpe, Paul G. Kumpel. Saunders Coll, 1984, xiv + 433 pp, $26.95. [ISBN: 0-03-061249-7] A down-to-earth geometric presentation beginning with Gaussian elimination and concluding with abstract vector spaces. Instead of having numerous "teasing references" to applications, chapters end with a single extensively-discussed application (e.g., linear programming, Markov processes, least square approximation). A brief appendix treats computer-related topics. LAS

Algebra, T(18: 1), S, P. Theory of Mathematical Structures. Jiří Adámek. D Reidel, 1983, x + 317 pp, $59.50. [ISBN: 90-277-1459-2] An attempt to bridge the gap between the original Bourbaki approach to mathematical structures in Theory of Sets and contemporary category theory by way of considering a formal system called a construct. Part I deals with basic notions of constructs; Part II develops categories and functors; Part III considers concrete categories and embedding of categories into special constructs. Exercises, references. JS

Finite Mathematics, T*(13: 1). Finite Mathematics for Management, Life, and Social Sciences, Third Edition. Raymond A. Barnett, Michael R. Ziegler. Dellen, 1984, xvi + 621 pp, $28.95. [ISBN: 0-89517-059-0] The chapter on linear programming has been rewritten. The chapters on probability have been reorganized. More examples and exercises have been added. Emphasis on applications remains. (Second Edition, TR, May 1982.) JK

Calculus, S?(13-14). Bigeometric Calculus: A System with a Scale-Free Derivative. Michael Grossman. Archimedes Foundation, 1983, vii + 100 pp, $3 (P). A sequel to the 1972 volume Non-Newtonian Calculus (TR, May 1973). The "bigeometric calculus" is one of the non-Newtonian calculi introduced by Grossman in which ratios are substituted for differences in the definitions of derivatives and integrals. LAS

Calculus, T(13: 1-3). College Mathematics for Management, Life, and Social Sciences, Third Edition. Raymond A. Barnett, Michael R. Ziegler. Dellen, 1984, xvi + 1085 pp, $29.95. [ISBN: 0-89517-058-2] Suitable for one-year course beyond intermediate algebra on topics most useful to students in disciplines given in title. Additional examples and exercises. Chapter on linear programming has been rewritten. Material on probability has been rearranged. Some topics in calculus have been added. (First Edition, TR, August-September 1978; Second Edition, TR, May 1982.) JK

Calculus, T*(13-14: 2, 3). Calculus. Dennis D. Berkey. Saunders Coll, 1984, xviii + 1194 pp, $39.95. [ISBN: 0-03-059522-3] Unusually clear, very detailed, rather verbose treatment of the standard set of topics. Exercise sets are very complete and varied, including routine drill, theoretical problems, calculator and computer exercises. Early review of trigonometric functions. Differential equations are treated at several places in the text. Good graphics aid exposition; few exercises are graphical. PZ

Complex Analysis, T(15-16: 1), S*, L.** Complex Analysis and Applications, Second Edition. William R. Derrick. Wadsworth, 1984, xiii + 347 pp, $31.95. [ISBN: 0-534-02853-5] Detailed and relatively rigorous treatment of standard topics, with many worked examples. Less standard is a chapter on integral transforms. Unusually many and varied applications to physics and engineering. Integration is introduced early. Includes a useful appendix on real vector calculus. An excellent choice for well-prepared science and mathematics students. PZ

Complex Analysis, P. Lecture Notes in Mathematics-1039: Analytic Functions, Błażejewko 1982. Ed: J. Ławrynowicz. Springer-Verlag, 1983, x + 494 pp, $23 (P). [ISBN: 0-387-12712-7] Papers from the 8th Conference on Analytic Functions, held in Poland in August 1982. Many concern extremal methods. Final sections on open problems in extremal problems in one variable, quasiconformal analysis, and analysis on complex manifolds. PZ

Differential Equations, T(14-15: 2), S.** Differential Equations, Third Edition. Shepley L. Ross. Wiley, 1984, vii + 807 pp, $32.95. [ISBN: 0-471-03294-8] Several sections rewritten for improved clarity and readability. Numerous worked-out examples and good exercise sets are positive features of this excellent textbook. Chapters 1-9 are available separately for use in a one-semester introductory course. Chapters 10-14 contain fundamental theory, nonlinear differential equations, Fourier series and boundary value problems. (Second Edition, TR, May 1974.) JK

Differential Equations, P. Approximation of Nonlinear Evolution Systems. Joseph W. Jerome. Math. in Sci. & Eng., V. 164. Academic Pr, 1983, xx + 280 pp, $46. [ISBN: 0-12-384680-3] Physical, biological, and engineering scientists are listed among the primary audience but the reader should be well grounded in functional analysis, operator theory, and partial differential equations. Discussion of global weak solutions and local smooth solutions for various boundary value problems.

Models include phase transition, diffusion in porous media, fluid flow, and general relativity. References, index. JS

Numerical Analysis, T?(18), P. Introduction to Interval Computations. Götz Alefeld, Jürgen Herzberger. Transl: Jon Rokne. Computer Sci. & Appl. Math. Academic Pr, 1983, xviii + 333 pp, $49.50. [ISBN: 0-12-049820-0] Mathematical treatment of the fundamentals of numerical solutions to algebraic problems (zeroes of polynomials, systems of equations, eigenvalues, etc.) using interval computations; problems with both exact and inexact data are considered, with inclusions sets and convergent sequences of bounds developed as solutions. Translated from the German edition. RM

Functional Analysis, S(16-17), L. Fixed Point Theorems. D.R. Smart. Tracts in Math., No. 66. Cambridge U Pr, 1980, viii + 93 pp, $14.95 (P). [ISBN: 0-521-29833-4] Set in the context of Banach spaces, most results are derived from Brouwer's fixed-point theorem for the ball Bn using proofs that seem most natural to the functional analyst. (1974 edition, TR, January 1975.) AWR

Functional Analysis, T(17: 1), S, P, L. Introduction to the Theory of Distributions. F.G. Friedlander. Cambridge U Pr, 1982, vii + 157 pp, $34.50. [ISBN: 0-521-24300-9] A straightforward, readable introduction to the theory of distributions, assuming relatively modest background. First eight chapters are basic theory; Chapters 9-10 deal with Fourier and Fourier-Laplace transforms. Exercises, appendix, index. JS

Functional Analysis, S(18), P. Topological Vector Spaces, Second Edition. A.P. and Wendy Robertson. Tracts in Math. & Math. Physics, No. 53. Cambridge U Pr, 1980, viii + 172 pp, $11.95 (P). [ISBN: 0-521-29882-2] The only significant change from the First Edition is the inclusion of an appendix on spaces with webs and a more general ver version of the closed graph theorem. (1973 hardcover copy of the Second Edition, TR, August-September 1973.) JS

Analysis, P. Les Inégalités de Markoff et de Bernstein. Qazi Ibadur Rahman, Gerhard Schmeisser. Pr U Montreal, 1983, 173 pp, $15 (P). [ISBN: 2-7606-0605-8] Indicates some directions of research inspired by Markoff and Bernstein inequalities (relating size of derivatives of polynomials to size of polynomials), useful in various approximation theories and in applied problems (not covered). Easy reading (in French). Decent bibliography with cross references to text. DD

Analysis, P. Lecture Notes in Mathematics-1043: Linear and Complex Analysis: Problem Book. Ed: V.P. Havin, S.V. Hruscëv, N.K. Nikol'skii. Springer-Verlag, 1984, xviii + 721 pp, $25.50 (P). [ISBN: 0-387-12869-7] A sequel to Volume 81 of Zapiski Nauchnyh Seminarov LOMI published in 1978, this book contains 199 research problems concerning analytic functions, linear function spaces, and linear operators and includes all of the problems of its predecessor. Each chapter of problems has a brief introduction. BH

Differential Geometry, S(17-18), P. The Variational Theory of Geodesics. M.M. Postnikov. Transl: Scripta Technica, Inc. Dover, 1983, viii + 200 pp, $6 (P). [ISBN: 0-486-63166-4] Unabridged and unaltered republication of the 1967 edition, by W.B. Saunders, of Scripta Technica's translation of Variatsionnaya Teoriya Geodezicheskikh. Fundamentals of differential geometry and the calculus of variations. Basic tools needed for Morse theory. Manifold theory, Riemannian geometry and variational properties of geodesics. Emphasis is on the analytical rather than the topological aspects. Sophisticated. No exercises. JK

Geometry, S, P, L. Dual Models. Magnus J. Wenninger. Cambridge U Pr, 1983, xii + 156 pp, $19.95. [ISBN: 0-521-24524-9] In this successor to his Polyhedron Models (TR, February 1972) and Spherical Models (TR, April 1980), Wenninger presents patterns for, and photographs of, models of the duals of not only all the convex but also all the non-convex uniform polyhedra. The impressive nature and beauty of these solids are certain to appeal to mathematicians and non-mathematicians alike. JNC

Topology, T(16-18: 1, 2), S, L*.** Topology. Klaus Jänich. Transl: Silvio Levy. Undergrad. Texts in Math. Springer-Verlag, 1984, ix + 192 pp, $24. [ISBN: 0-387-90892-7] If Kelley's "Topology" was "What every young analyst should know," this book is "What every young mathematician should know." Although it is in the Undergraduate Texts in Mathematics Series, the relative sophistication of the material and the lack of any problems or exercises probably will make this text more useful as supplemental reading. For this latter it appears to be outstanding! It is highly readable, full of beautifully presented intuition, and lively reading. This is what topology--all of topology--is all about: fundamental concepts of spaces, topological vector spaces, quotients, metrics, homotopy, CW-complexes, covering spaces, and a few sections on major theorems. JAS

Topology, P. Topology and Order Structures, Part 2. Ed: H.R. Bennett, D.J. Lutzer. Math Centre Tracts, No. 169. Math Centrum, 1983, iii + 107 pp, Dfl. 15,50 (P). [ISBN: 90-6196-266-8] Proceedings of an August 1981 workshop at the Free University of Amsterdam, a sequel to a similar August 1980 workshop at Texas Tech. (Math. Centre Tracts, No. 142, TR, November 1982.) LAS

Operations Research, T(16-17). Elements of Queueing Theory With Applications. Thomas L. Saaty. Dover, 1983, xv + 423 pp, $8 (P). [ISBN: 0-486-64553-3] A classic from a recognized expert, first printed in 1961, now available as a paperback. AWR

Operations Research, S(14-17), L. Lecture Notes in Economics and Mathematical Systems-222: Energy Systems Analysis for Developing Countries. Peter Meier. Springer-Verlag, 1984, vi + 344 pp, $18.50 (P). [ISBN: 0-387-12879-4] A self-contained monograph describing the basic principles of systems analysis with applications to the study of energy systems. Several case studies are included. AO

Optimization, S(17-18), P. Markov Decision Processes with Continuous Time Parameter. F.A. van der Duyn Schouten. Math. Centre Tracts, No. 164. Math Centrum, 1983, iv + 194 pp, Dfl. 26,40 (P). [ISBN: 90-6196-261-7] An introduction to Markov processes on the space $J[0,\infty)$ which consists of finite sequences of time-state pairs. This serves as an appropriate mathematical model for queue and inventory models where continuous processes are subject to irregular discrete "shocks." This monograph contains introductory theory and applications to economic models. LAS

Optimization, S*, L*. The Shortest Lines: Variational Problems. L.A. Lyusternik. Transl: Yuri Ermolyev. Little Math. Lib. MIR, 1983, 104 pp, (P). Reprint of the original 1976 edition (TR, August-September 1977). LCL

Probability, P. On Point Processes. P.C.T. van der Hoeven. Math. Centre Tracts, No. 165. Math Centrum, 1983, iii + 126 pp, Dfl. 17,60 (P). [ISBN: 90-6196-262-5] A point process is a probabilistic mechanism by which one can choose a locally finite subset of a locally compact space with a countable base. This monograph, the author's dissertation, consists of an analysis of the probability distributions of the locally finite subsets chosen by point processes. LAS

Probability, T(18: 1, 2), S, P. Ergodic Theory. Karl Petersen. Stud. in Adv. Math., No. 2. Cambridge U Pr, 1983, xi + 329 pp, $49.50. [ISBN: 0-521-23632-0] An exposition of selected parts of ergodic theory--convergence theorems, recurrence properties, entropy--with the intent of bringing the reader quickly from introductory material to the research frontier in those select areas. Prerequisite: measure theory and functional analysis. LAS

Probability, P. Seminar on Stochastic Processes, 1982. Ed: E. Cinlar, K.L. Chung, R.K. Getoor. Progress in Prob. & Stat., V. 5. Birkhauser Boston, 1983, 302 pp, $24.95. [ISBN: 0-8176-3131-3] Partial proceedings ("about half of the papers") of a March 1982 seminar at Northwestern University. LAS

Probability, T(18: 1), P. Random Linear Operators. A.V. Skorohod. Math. & Its Applic. D Reidel, 1984, xvi + 199 pp, $43. [ISBN: 90-277-1669-2] Random linear operators represent a natural extension of "random analysis," one of the fastest growing areas of mathematics. Random operators arise in stochastic differential equations, in control theory applications, and in theoretical physics. This monograph introduces the field, beginning with random operators on Hilbert space, and including operator-valued martingales, stochastic integrals equations, and linear stochastic operator equations. LAS

Probability, T(14-15: 1), S, L. A First Course in Probability, Second Edition. Sheldon Ross. Macmillan, 1984, xii + 392 pp, $29.95. [ISBN: 0-02-403910-1] The author has deleted sections on the t-distribution and the F-distribution, and added sections on conditional variance and the Poisson process. Most of the expansion (an increase of about 100 pages) is aimed at making pedagogical improvements (motivation, organization within sections, additional examples and problems). (First Edition, TR, May 1976; Extended Review, February 1979.) LCL

Probability, T(18), P. Semimartingales: A Course on Stochastic Processes. Michel Métivier. Stud. in Math., No. 2. Walter de Gruyter, 1982, xi + 287 pp, $40. [ISBN: 3-11-008674-3] A course in martingales, quasimartingales, semimartingales, and stochastic calculus, presented "in their full force in a rather concise way." Intended to present basic tools for the description and study of stochastic processes. Based on courses given by the author in Erlangen, Berkeley, and Munich. LAS

Probability, T(18), P. Foundations of Stochastic Analysis. M.M. Rao. Prob. & Math. Stat. Academic Pr, 1981, xi + 295 pp, $30.50. [ISBN: 0-12-580850-X] Exposition of the Kolmogorov-Bochner fundamental existence theory of stochastic analysis in which the probability spaces supporting stochastic processes are defined as projective limits of their finite dimensional distributions. Since each such projective system is uniquely associated with a set martingale, the theory leads naturally to an abstract theory of martingales. LAS

Probability, P. Branching Processes. S. Asmussen, H. Hering. Prog. in Prob. & Stat., V. 3. Birkhauser Boston, 1983, x + 461 pp, $34.95. [ISBN: 3-7643-3122-4] A self-contained treatment of recent developments in branching processes, emphasizing the classical limit theory, Markov branching processes, and extensions to unbounded domains, age-dependent processes, and two-sex models. LAS

Probability, P. Lecture Notes in Mathematics-1042: Amarts and Set Function Processes. Allan Gut, Klaus D. Schmidt. Springer-Verlag, 1983, 258 pp, $12.50 (P). [ISBN: 0-387-12867-0] Amarts are asymptotic martingales. Notes from a series of lectures at Uppsala University in 1979. Concludes with an extensive bibliography. LAS

Probability, T(17-18: 1), P. Introduction to Stochastic Integration. K.L. Chung, R.J. Williams. Progress in Prob. & Stat., V. 4. Birkhauser Boston, 1983, xiii + 191 pp, $19.95. [ISBN: 3-7643-3117-8] Presentation of the modern theory of stochastic integration, emphasizing integration with respect to a local martingale with continuous paths. Based on a 1981 lecture series at Stanford. Typeset in TeX. LAS

Statistics, S(15-18), P, L.** The Chronological Annotated Bibliography of Order Statistics, Volume II: 1950-1959. H. Leon Harter. Ser. in Math. & Management Sci., V. 8. American Sci Pr, 1983, vi + 993 pp, $149.50 (P). [ISBN: 0-935950-05-2] Continuation of Volume I: Pre-1950 (TR, April 1984) summarizing 1056 further articles dealing with order statistics (but not rank statistics) from the years 1950-59. Includes a listing of citations of pre-1950 articles by the 1950-59 authors, so the

two volumes form a closed set. Again, note price! RSK

Statistics, P*. Generalized Linear Models. P. McCullagh, J.A. Nelder. Mono. on Stat. & Appl. Prob. Chapman & Hall, 1983, xiii + 261 pp, $29.95. [ISBN: 0-412-23850-0] Unified study of a class of models that generalizes classical linear models to include, in addition to linear regression and analysis of variance models, log-linear models for count data, probit and logit models, models for continuous data with constant proportional standard error, and certain models for survival analysis. Good set of references. RSK

Statistics, T*(13: 1). Elementary Statistics, Fourth Edition. Robert Johnson. Duxbury Pr, 1984, xv + 557 pp [ISBN: 0-87150-406-5]; Study Guide with Self-Correcting Exercises, xii + 346 pp, (P) [ISBN: 0-87150-793-5]; Solutions Manual, 205 pp [ISBN: 0-87150-794-3]; Test Item Card File [ISBN: 0-87150-795-1]; MINITAB Student Supplement, James P. Scott, Kenneth M. Bond, 112 pp. [ISBN: 0-87150-821-4] Revision of the 1980 Third Edition of this popular text. Major changes include reorganization and rewriting of parts of the text, substantial changes in the exercise sets, and some new material, including MINITAB computer output for some examples, p-values, and one-way analysis of variance with unequal numbers of observations. The Study Guide is an aid for students; the Solutions Manual gives solutions to all exercises; the Test Item Card File gives 240 sample test questions on cards; the MINITAB Student Supplement is a new booklet designed specifically for use with the text and provides material for an integrated computer approach to statistics. (First Edition, TR, June-July 1974; Second Edition, October 1976.) RSK

Statistics, T(16-18), S, P. Introduction to Statistical Modelling. Annette J. Dobson. Chapman & Hall, 1983, viii + 125 pp, $33; $14.95 (P). [ISBN: 0-412-24850-6; 0-412-24860-3] Generalized linear models provide a practical, unified approach for the most commonly used statistical methods: regression, analysis of variance and covariance, log-linear models for contingency tables. Includes the theoretical background as well as illustrative numerical examples. LCL

Statistics, P. Bayesian Estimation and Experimental Design in Linear Regression Models. Jürgen Pilz. Teubner-Texte zur Math., B. 55. BG Teubner, 1983, 216 pp, 20M (P). Technical monograph dealing with "the design and analysis of linear regression experiments in the presence of prior knowledge about the model parameters." Divided into three parts: Bayesian regression and prior distributions, Bayesian estimation, and Bayesian experimental design. Extensive references. RSK

Statistics, S(16-18), P, L. Statistics on Spheres. Geoffrey S. Watson. U. of Arkansas Lect. Notes, V. 6. Wiley, 1983, x + 238 pp, $21.50 (P). [ISBN: 0-471-88866-4] Exposition of probability distributions and statistical methods for directions (unit vectors) in n-dimensional Euclidean space. Covers a selection of mathematical topics originally motivated by investigations of earth magnetism, continental drift and homing pigeons, but now fully mature as a separate mathematical discipline. Topics include symmetry considerations, center of mass techniques, moment of inertia techniques, and normalizations. LAS

Computer Literacy, T(13). Computers and Data Processing, Second Edition. H.L. Capron, Brian K. Williams. Benjamin/Cummings, 1984, xxvii + 467 pp, $26.95. [ISBN: 0-8053-2214-0] Readable, full of history, lively pictures, lots of vocabulary, and almost without intellectual content. The space devoted to a topic seems to depend on popularity as the sole criterion for significance, e.g., there are five pages (including a two-page program listing) devoted to COBOL, but no mention of C language or Modula II. Couldn't find the word "algorithm" in the book. JAS

Computer Literacy, T*(13: 1), S, L. Understanding Computers and Data Processing: Today and Tomorrow. Charles S. Parker. Holt, Rinehart & Winston, 1984, xix + 555 pp, $22.95 [ISBN: 0-03-063424-5]; Understanding Computers and Data Processing: Today and Tomorrow with BASIC, xix + 650 pp, $24.95 [ISBN: 0-03-063427-X]; Ancillary Sampler, v + 55 pp, (P). [ISBN: 0-03-071496-6] An introduction to computers and data processing. Chapter titles include hardware, software, systems and computers in society. Requires no technical background. Essentially a high-level computer literacy course as there is little said about algorithm design and problem solving. Available in a version which includes an introduction to BASIC in an appendix. CEC

Computer Programming, T(13-18: 1), S. Calcul par l'informatique. Marie José Bertin. Hermann, 1983, 209 pp, 58F (P). [ISBN: 2-7056-1409-5] Presents many of the best known and most useful numerical methods (for changing number bases, solving equations, numerical integration, solving differential equations, etc.) in algorithmic format precisely formulated for execution on hand calculators. Written primarily for French lycée level instruction. Begins with a fine general introduction to the use of hand calculators for solving numerical problems. Many exercises. Appendix. Bibliography. RJA

Computer Programming, T*(15-17: 1, 2), S*, P*, L*. Scientific Pascal. Harley Flanders. Reston, 1984, xii + 563 pp, (P). [ISBN: 0-8359-6931-2] A substantial introduction to scientific computation and the Pascal language, best suited to advanced students or practicing scientists. Language details are covered in the first half of the book, scientific routines (e.g., differential equations, fast Fourier transform, eigenvalues) in the second. The text exudes scholarship (well-referenced, state-of-the-art algorithms) and pedagogical care: all programs are checked with actual runs, and printed directly by a special program that produces uncluttered, well-formatted source code print-out. Includes over 130 pages of solutions to exercises! LAS

Software Systems, P. Lecture Notes in Computer Science-162: Computer Algebra. Ed: J.A. van Hulzen. Springer-Verlag, 1983, xiii + 305 pp, $14 (P). [ISBN: 0-387-12868-9] Papers from EUROCAL '83, the

second annual European Computer Algebra Conference held in March 1983 in London. Covers algorithms, systems, polynomial bases and number theory. LAS

Computer Science, T(16: 1). Microprocessor System Design Concepts. Nikitas A. Alexandridis. Computer Sci Pr, 1984, xvi + 623 pp, $32.95. [ISBN: 0-914894-66-8] A textbook for an advanced undergraduate or graduate course in computer architecture, assuming a background in software design, electrical circuitry, and computer organization. It discusses the topics of semiconductor technology, microprocessor architecture and design, memory system design, buses, I/O, and multiprocessors. The book uses existing microprocessors to illustrate its concepts, especially the Zilog Z80, Intel 8085 and Motorola 68000. MS

Computer Science, T(15: 1). Computability with Pascal. John W. Mallozzi, Nicholas J. De Lillo. Prentice-Hall, 1984, xii + 193 pp, $26.95. [ISBN: 0-13-164443-2] An advanced undergraduate text in theoretical computer science. The topics covered include finite automata, recursive function theory, unsolvable problems, and Turing machines. Wherever possible, the authors have described their algorithms using the Pascal programming language rather than other, more common, mathematical notations. The concepts of computational theory are here imbedded in a more concrete, explicit setting than usually seen in most textbooks. MS

Computer Science, P. Tree Automata. Ferenc Gécseg, Magnus Steinby. Akademiai Kiado, 1984, 235 pp, $26. [ISBN: 963-06-3170-4] Tree automata are mathematical structures that occur frequently in the fields of mathematical linguistics and symbolic logic. This book presents a rigorous presentation of the entire field of tree automata and tree transformations. It is written at an extremely advanced level and assumes that the reader is quite familiar with the fields of universal algebra, finite state automata, and formal languages. It contains an extensive bibliography (over 200 citations) to major research work in this area. MS

Applications (Biology), P. Evolution and the Theory of Games. John Maynard Smith. Cambridge U Pr, 1982, viii + 224 pp, $34.50. [ISBN: 0-521-24673-3] The focus is on biology, with terminology drawn from game theory. Some little mathematical detail appears in the appendix, and it is true that the biologist who reads this book will be greatly helped by reasonable mathematical preparation. AWR

Applications (Economics), T(13-14). College Mathematics with Applications to Management, Economics, and the Social and Natural Sciences. Chester Piascik. Charles E Merrill, 1984, x + 708 pp, $26.95. [ISBN: 0-675-20094-6] Aimed primarily at undergraduates in the business administration curriculum, this text builds background as necessary for chapters on mathematics of finance, linear programming, and optimization (using differential calculus). AWR

Applications (Engineering), T(16-17: 1-3), P. Digital Control System Analysis and Design. Charles L. Phillips, H. Troy Nagle, Jr. Prentice-Hall, 1984, xv + 556 pp, $36.95. [ISBN: 0-13-212043-7] Concerned with the analysis and design of closed loop physical systems (input at least partially determined by output) that contain digital computers, e.g., airliner automatic pilot system. Material suitable for first course in discrete control systems or digital filters, or self-study by a practicing engineer. Numerous exercises. RB

Applications (Engineering), P. Limit Analysis and Concrete Plasticity. M.P. Nielsen. Prentice-Hall, 1984, xii + 420 pp, $42.95. [ISBN: 0-13-536623-2] An introduction to applications of the theory of plasticity to the design of concrete structures. AO

Applications (Physics), P. Noise-Induced Transitions: Theory and Applications in Physics, Chemistry, and Biology. W. Horsthemke, R. Lefever. Ser. in Synergetics, V. 15. Springer-Verlag, 1984, xv + 318 pp, $42. [ISBN: 0-387-11359-2] The notion of a phase transition is extended to a class of transition phenomena occurring in nonequilibrium systems subjected to a randomly fluctuating environment. Results for environments modeled by both white and colored noises are discussed. AO

Reviewers

RJA: Richard J. Allen, St. Olaf; MA: Melissa Anderson, St. Olaf; FA: Fahrad Anklesaria, Macalester; PB: Peder Bolstad, St. Olaf; RB: Richard Brown, Carleton; JNC: Judith N. Cederberg, St. Olaf; CEC: Clifton E. Corzatt, St. Olaf; DD: David Dummit, Macalester; JD-B: John Dyer-Bennet, Carleton; JRG: Jennifer R. Galovich, St. Olaf; SG: Steven Galovich, Carleton; JG: Jack Goldfeather, Carleton; BH: Bruce Hanson; PH: Paul Humke, St. Olaf; BK: Barbara Kaiser, St. Olaf; RBK: Roger B. Kirchner, Carleton; RSK: Richard S. Kleber, St. Olaf; JK: Joseph Konhauser, Macalester; LCL: Loren C. Larson, St. Olaf; GHM: George H. Mills, Carleton; RM: Richard Molnar, Macalester; RWN: Richard W. Nau, Carleton; LN: Linda Ness, Carleton; YN: Yves Nievergelt, St. Olaf; AO: Arnold Ostebee, St. Olaf; TR: Teresa Reardon, St. Olaf; AWR: A. Wayne Roberts, Macalester; MS: Michael Schneider, Macalester; JS: John Schue, Macalester; SS: Seymour Schuster, Carleton; JAS: J. Arthur Seebach, Jr., St. Olaf; KS: Kay Smith, St. Olaf; LAS: Lynn Arthur Steen, St. Olaf; MT: Michael Tveite, St. Olaf; ML: Milton Ulmer, Carleton; CU: Constantino Unguriano, Carleton; TAV: Theodore A. Vessey, St. Olaf; MW: Martha Wallace, St. Olaf; FLW: Frank L. Wolf, Carleton; PZ: Paul Zorn, St. Olaf.

The Mathematical Association of America
The Sixty Fourth Summer Meeting of the Association
University of Oregon, Eugene, Oregon

The Sixty Fourth Summer Meeting of the Association was held from August 16 through August 19, 1984 in conjunction with meetings of the American Mathematical Society and Pi Mu Epsilon. The 33rd Earle Raymond Hedrick Lectures were given by Neil J.A. Sloane of AT&T Bell Laboratories with the following title and abstract:

Lattices, Sphere Packings and Applications

Quite a few problems from the real world translate into geometrical problems of the following type. How to place 100 points "nicely" on the surface of a billiard ball? What is the most efficient way to cover a table-top with overlapping circular disks? Such problems arise in several different ways in designing communications systems. For example, in transforming voice signals so that they can be efficiently transmitted, in designing signalling systems that are resistant to errors, or in the common analog-to-digital conversion process that must take place when real numbers are fed into a digital computer.

A great deal of progress has been made recently on these geometrical problems, and the results have turned out to have some remarkable connections with other branches of mathematics.

Invited Addresses

There were three invited addresses sponsored jointly by the Association and the American Mathematical Society. The speakers and the titles of their addresses were: Philip J. Davis, Brown University, Mathematics and Rhetoric; Judith V. Grabiner, California State University, Dominguez Hills, A Historian's Perspective; Raoul H. Bott, Harvard University, Topology. There were also seven invited addresses, as follows.

Combinatorial Problems with Surprising Solutions, by David P. Roselle, Virginia Tech.

This talk detailed combinatorial problems related to scheduling, arithmetic progressions, lattice paths, and special sequences. The problems were selected as exemplary of the great many combinatorial problems that give rise to seemingly anomalous situations. The talk was intended for a general audience.

Instantons and the Topology of 4-Manifolds, by Ronald J. Stern, University of Utah.

The 1980s have experienced a tremendous growth in the understanding of objects locally modelled on 4-dimensional Euclidean space, i.e., 4-manifolds. Several seemingly distinct areas of mathematics have converged to enhance this understanding. In this talk we outlined and motivated the major problems in the study of smooth 4-manifolds and indicated how the mathematical study of gauge theories in physics (through the work of S. Donaldson and the ancestral work of Atiyah-Hitchin and Singer, Taubes and Uhlenbeck, and later work of Fintushel and Stern) led to solution of these problems.

Mathematicians Without Mathematics, by Constance Reid, San Francisco.

Constance Reid, who is known for her books on the lives of mathematicians, discussed the unique problems in writing a non-technical life of a scientist.

How to Factor an Integer, by Carl Pomerance, University of Georgia, Athens, and Bell Communications Research, Inc.

The RSA cryptosystem (which was described in the talk) bases its security on the supposed intractability of the factoring problem. Since the advent of RSA, several new factoring algorithms have been introduced. One, the quadratic sieve algorithm, could conceivably crack 100 digit numbers in one month on relatively inexpensive special purpose hardware.

How to Fold Regular Polygons and Do Number Theory, Part I, by Jean J. Pedersen, University of Santa Clara.

A simple systematic procedure enables one to fold a straight strip of paper into an arbitrarily close approximation to a regular s-gon for certain folding numbers s. A secondary procedure then leads to the construction of regular N-gons for any $N \geq 3$. The same procedures permit one to fold certain regular star N-gons. The procedures and their convergence properties were discussed.

How to Fold Regular Polygons and Do Number Theory, Part II, by Peter J. Hilton, SUNY, Center at Binghamton.

The set T of folding numbers--that is, numbers s such that we may fold a regular s-gon using the primary folding procedure described in the talk by Jean Pedersen--was generalized to a set T_t, where t is an integer ≥ 2, so that $T = T_2$. Properties of T_t were studied in order to determine the most efficient folding procedure, and to count the number of star polygons which can be folded by a modification of the procedure. Divisibility properties of generalized folding numbers were also elucidated.

Applications of Geometry to the Social Sciences, by Robert I. Jewett, Western Washington State University.

Applications of geometry to economics, politics, and other social sciences were given. The intention was to present a survey of social geometry, stating its main results and clarifying its historical development.

Special Sessions

Minicourses: The Association sponsored eight minicourses. Their titles and organizers were: "The Intersection of Mathematics and Statistics," by Thomas R. Knapp, University of Rochester; "Applications of Probability Theory to the Analysis and Design of Computer Systems," by Robert Geist, Clemson University and Kishor Trivedi, Duke University; "Introductory Computer Science," by J. Arthur Seebach, St. Olaf College; "Teacher In-Service Programs" (A COMET Minicourse), by Eugene A. Maier, Mathematics Learning Center and Portland State University; "Pascal For Mathematicians," by Harley Flanders, Florida Atlantic University; "Introduction to Computer Graphics," by Joan Wyzkoski, Bradley University; "Discrete Algorithmic Mathematics," by Steven B. Maurer, Swarthmore College and Alfred P. Sloan Foundation; "Microcomputer Software in Mathematics Instruction," by Roy E. Myers, Pennsylvania State University, New Kensington.

Film Program: The film program included the following: "M.C. Escher: Geometries and Impossible Worlds," "Accidental Nuclear War," "Regular Homotopies in the Plane, Part I," "Trio for Three Angles," "Area Under a Curve," "Flatland," "Curves of Constant Width," "Newton's Equal Areas," "An Animated Approach to Fractions."

There was also a presentation of "Challenge of the Unknown," an American Association for the Advancement of Science (AAAS) project.

Panel on Curriculum at Two-Year Colleges: This panel included Don Albers of Menlo College, John Bradburn of Elgin Community College, and Ron Davis of Northern Virginia Community College (moderator).

Contributed Paper Sessions

The Association sponsored four contributed paper sessions. The topics, list of presenters and abstracts follow:

Visual Mathematics in the Undergraduate Curriculum. Organizer: Martin E. Flashman, Humboldt State University.

"The Right (and Left) Approach to Teaching Mathematics," by Carlos C. Huerta, Polytechnic Institute of New York, Brooklyn.

With the new psychological and physiological understanding of the right and left hemispheres of the brain, and the recognition that each side contributes to the ease of learning and retention of learned material, new teaching techniques can be brought to bear in the teaching of college level mathematics. Many other professions have used right hemisphere techniques with much success. At the lower levels of mathematics education, i.e., elementary school and high school, right hemisphere techniques are alive and well. As one progresses into college the shift is to predominantly left techniques. It is perhaps this which contributes to math anxiety, newly recognized as a serious impediment to learning but known to educators for a long time as troublesome for many otherwise competent students. Proper balancing of left and right hemisphere teaching techniques at all levels, including graduate, would push the use of mathematics into many non-mathematical professions.

"Calculus Students' Utilization of Graphs as Visual Representations," by Dena Patterson, Northern Arizona University, Flagstaff.

Mathematicians indicate that visual imagery and pictures are essential elements in many forms of mathematical understanding and problem solving. Mathematics teachers often encourage their students to construct a graph or draw a picture in order to solve a mathematics problem. But how are students actually functioning on problems that require the construction and use of graphs for their solution? This question motivated a two-year research project in the Group in Science and Mathematics Education at the University of California in Berkeley. The results of this study indicate that, in general, the students are drawing graphs for appropriate problems; but they draw the graphs inaccurately, tend not to refer to available graphic information, and lack necessary skills in interpreting graphs. Thus it is suggested that the interpretation of graphs, and the interplay between the graph interpretation needs of a specific mathematics problem and the accuracy needed in the graph, be explicitly addressed in the mathematics curriculum.

"Using Sequences of Transparencies to Teach Creating New Theorems by Inversion," by James R. Smart, San Jose State University.

Creating new theorems by inversion is one of the applications of inversive geometry in an upper division course in modern geometries with transformations. This topic can be greatly simplified for students by use of a sequence of transparencies showing what happens to each individual set of

points and each condition as it becomes a part of the inverted figure and the new theorem. This technique will be illustrated for at least two theorems.

"Stereographic Projection--Geological Applications," by Ernest J. Manfred, U.S. Coast Guard Academy, New London.

Stereographic projection is usually encountered in an undergraduate course in complex variables. This projection is introduced as an example of a conformal (angle preserving) mapping of a sphere into a plane. Such mappings are of basic interest because of their applications to the practical problem of constructing maps of the globe. Interesting applications of stereographic projection occur in geology and crystallography. The projection is used to represent half of a sphere on a plane. Once this representation is achieved, the projection is then used as a rapid method for solving problems that are independent of scale graphically. The purpose of this paper is to show how problems in geology and crystallography may be solved using stereographic projection.

"Shall We Show Them Some Geometry?" by Branko Grunbaum, University of Washington.

Utilization of the visual aspects of mathematics should start with the part of mathematics meant to deal with spatial objects and relations--geometry. It is advocated that a course in geometry--devoted to the acquisition of geometric knowledge and understanding, and not as a tool for other subjects--should be developed and taught as a beginning course in college mathematics. This course could be used to provide both the motivation, and the factual background, for many of the following courses as well as leading to a better understanding of the applications of mathematics to sciences, technology and everyday life.

"Computer-Generated Color Graphics for Enhancement of Undergraduate Mathematics Instruction," by E.P. Miles, Jr., Florida State University.

A poster/slide presentation of selections from the author's color graphics related to college mathematics which illustrate many properties of functions difficult to show in normal graphics.

"VISUMAP and the Curriculum," by Solomon Garfunkel, COMAP, Lexington, Massachusetts.

The author discussed recent progress on the VISUMAP Project and its use in the undergraduate curriculum.

"Visualizing Concepts in Modern Algebra," by Doris J. Schattschneider, Moravian College.

Geometric objects, artistic designs, and coloring can all be used to give vivid interpretation to abstract concepts encountered in modern algebra courses. Transformation groups and especially symmetry and color symmetry groups of a variety of objects provide realistic illustration of these concepts.

"The Visual Element in Elementary Group Theory," by Jean J. Pedersen, University of Santa Clara.

One example, among many, of the way in which the visual approach can enrich the student's understanding of a new mathematical concept is provided by group theory. We described particular instances which show how the group elements and the group operation can be given a concrete, or geometrical, interpretation, and how the student thereby develops a real feeling for the true working nature of these important abstract concepts.

Precollege, College, and Remedial Instruction--Common Concerns. Organizer: Anneli Lax, NYU-Courant.

"Some Ideas on the Sequencing of Content the Second Time Around," by Gloria Gilmer, Coppin State College, Baltimore.

Last year the CUPM-CTUM-CTYC Panel on Remediation conducted a limited study of the content and format of remediation programs in mathematics across the country. It indicated that the basic thrust of such programs is to give students technical facility with high school algebra. This is usually achieved by allocating large blocks of time to tedious drill. The most successful programs were dedicated to assisting students in passing minimum competency examinations to satisfy graduation requirements. Few programs provided data to substantiate the success of remedial students in successive courses in mathematics. Thus serious questions arise concerning the value of such remediation programs for successful entry into and pursuit of more advanced studies in mathematics. To achieve this latter goal, the present talk focused on several promising alternatives to the traditional sequence of topics when encountered the second time around.

"Assessing the Effectiveness of Mathematics Instruction with a Computerized Database," by Jerrold W. Grossman, Oakland University, Michigan.

A mathematics department in a large university may deliver 30,000 or more credits per year and yet have very little solid information on how successful its programs and courses are to the students it serves. In particular, few departments could answer such questions as: What fraction of those students who start in remedial elementary algebra successfully complete a year of calculus? How many students must take finite math more than once in order to pass it? Do older, non-

traditional students perform better than younger students? What correlation is there between placement test results, performance in remedial math courses, and performance in regular math courses? Answering such questions by hand-searching records, perhaps scattered in several offices, is disheartening, if not infeasible. We found that the registrar's office and the admininstrative computer staff were willing to provide us with a continually updated database, written to our specifications, stored on the disk of the mainframe computer, containing all desired demographic, enrollment, grade, and placement test information on all current and recently departed students. With this database, we can easily compile the statistics and reports we want, whenever we want them, with just a little programming and a few minutes of CPU time. Using this database, our department prepared a report on our remedial program that resulted in the authorization to hire additional tenure-track faculty; justified and implemented rigorous enforcement of prerequisites; studied the effectiveness of remedial instruction for disadvantaged students; and responded to complaints on grading practices.

"Methods on How to Handle Large Classes: Use Undergraduates to Help!" by David Ballew, South Dakota School of Mines and Technology.

Like so many engineering institutions, the South Dakota School of Mines and Technology has seen massive increases in the number of students with a not so great increase in the number of authorized faculty. Thus, we have been faced with large classes, the resulting increase in workloads, and in our opinion, the danger of a reduction in academic quality. Our Dean agrees that things are bad and advises that we "get creative" and somehow "become more efficient."

Our solution has been to tap our largest unused resource and to use our advanced undergraduates to help us with our classes. We believe we have had many more successes than failures and that there has been an enormous benefit for the students in the classes, the students who help with the classes, and/or the faculty members themselves. Indeed, several undergraduates who have participated have begun graduate school with the goal of becoming college level instructors.

We have used undergraduates in four ways: 1) grading each other's homework; 2) as graders of tests and major assignments; 3) as interns directly helping in the classroom; and 4) as undergraduate teaching assistants.

"Needed: Mathematical Missionaries," by Jo Anne S. Growney, Bloomsburg University, Pennsylvania.

There is a large group of intelligent students with modest mathematical backgrounds and interests who have not been sufficiently considered when mathematics courses have been designed and texts written. True, many books have been written "on their level." However, regardless of their lists of "exciting" topics, because of the attitudes most of these students have toward mathematics, these texts successfully impart only information--and do not involve the students in a "mathematical experience." Mathematical competency is an important trait of a college graduate and that competency requires that a student have confidence in the power of mathematics and his/her ability to draw on that power; thus we need to have students experience this power. It is not enough to fill them with topics that have excited us (but which to them are merely information because they are not engaged in the process of working things out). We need to find new topics (or at least new ways of presenting old ones) that will involve them in the process of mathematics and convince them of its power for their lives, long after their courses have ended. The talk offered examples of materials that have been used by "missionary-teachers" to convert a reluctant audience of non-math-oriented students to mathematical thinking.

Motivating Teaching Ideas that Do Not Compromise Mathematics: Presentations, Examples, or Applications. Organizer: Larry Runyan, Shoreline Community College.

"On Solving Radical Equations," by Bill Bompart, Augusta College, Augusta, Georgia.

College algebra students often have difficulty solving equations such as $x - \sqrt{3x-2} = 4$ and $\sqrt{2x+3} - 2\sqrt{x-7} = 1$. By appropriate substitutions, equations of this type can be replaced by equations of the form $\sqrt{ax+b} = c$ whose roots are easily found. This technique logically eliminates extraneous roots.

"Case Studies in Freshman Mathematics," by Yves Nievergelt, University of Washington.

"I'm an accounting major; where will I ever use this quadratic formula?"--Student.

This talk brings several relevant uses of freshman mathematics back into the classroom, to motivate students and let them exercise their newly acquired skills on specific real cases. The fields of applications covered here include finance, medicine, navigation, and photography, while the level of mathematics ranges from college algebra through calculus. For instance, the quadratic formula allows actuaries to compute the yield to maturity of one-year treasury bills, and techniques of integration enable health scientists to study the diffusion of dopamine in the brain. An annotated bibliography was distributed to the participants, listing original articles in various domains, and cross-indexed by fields of application (cartography, economics, medicine, ...) and by mathematics topics (quadratic formula, power series, ...).

"Exactness and Integrating Factors on Ordinary Differential Equations," by Clement E. Falbo, Sonoma State University, Rohnert Park, California.

We give a brief discussion of exactness in first order equations and derive the partial differential equations whose solutions are integrating factors. For higher order equations, we define "exact" and "recursively exact" ordinary differential equations and derive several general properties, as well as necessary and sufficient conditions for exactness in general second order ordinary differential equations. An algorithm for solving exact and recursively exact second order equations is given and used to solve some examples. All results are derivable from the definitions and fundamental concepts. The algorithm is applied to linear second order equations to get their adjoint equations.

"Differential Equations Derived From Special Properties of Catenary," by David Y. Hsu, California State University, Los Angeles.

It is noted that the following three geometrical quantities of the catenary $y = \cosh(x)$ share the same numerical value: (a) the slope at x, $y'(x)$; (b) the area under y from 0 to x, $\int_0^x y(t)dt$; and (c) the arc length from 0 to x, $\int_0^x \sqrt{1 + (y'(t))^2}\, dt$. Equating them and differentiating, we obtain

(1) $y''(x) = y(x)$

(2) $y''(x) = \sqrt{1+(y'(x))^2}$

(3) $\sqrt{1+(y'(x))^2} = y(x)$.

Also $y'(0) = 0$ from (a), (b); and $y(0) = 1$ from (3) by setting $x = 0$.

Equation (1) is second order and linear, (2) is second order but nonlinear, whereas (3) is first order and nonlinear. With $y(0) = 1$ and $y'(0) = 0$ as initial conditions for each of the above three equations, the solution $y = \cosh(x)$ can be obtained by students in differential equations class.

"Eigenfunctions of an LRC Circuit," by D.G. Hartig, California Polytechnic State University, San Luis Obispo.

The observation that an exponential forcing function passes through an LRC circuit virtually unaffected is the basis for what we call the "method of undetermined coefficients." By exploiting the linearity of the system these exponential eigenfunctions can be used to illustrate the filtering properties of two-element circuits. This leads naturally to Fourier series and can stimulate genuine interest in pointwise vs. uniform convergence, term-by-term integration and differentiation, and "generalized" derivatives.

"A Mathematical Model for Predicting the AL Baseball Standings in 1984," by Stephen Krevisky, University of Wisconsin, Washington Co., West Bend.

This model involves the use of weighted averages, linear functions, and the ideas of sabermetrics, which is the use of mathematics to analyze the performance of baseball players and teams.

Use of Computers in Upper Division Mathematics Courses. Organizer: Ronald H. Wenger, University of Delaware; Presider: Martha J. Siegel, Towson State University.

"A Course in Scientific Programming," by Paul Eenigenburg, Western Michigan University.

"Using the Computer as a Classroom Aide," by Marjorie A. Fitting, San Jose State University.

"A Microcomputer Laboratory in the Theory of Functions of a Complex Variable," by Tyre A. Newton and Leonard G. Hensheid, Washington State University.

"Integration of the Computer into Applied Mathematics Courses," by Sudir Goel and Ron Barnes, University of Houston-Downtown.

"How to Balance a Yardstick on an Apple," by Herbert R. Bailey, Rose-Hulman Institute of Technology.

"The Calculus of Quaternions with Applications to Computer Graphics," by E.L. Perry, Jr., University of Houston at Clear Lake.

"Computer Graphics for the Vibrating String," by Howard L. Penn, United States Naval Academy.

Board and Business Meetings

The Board of Governors met at 9:00 a.m. on August 15, 1984. The major items of business will be announced to the membership in FOCUS.

The Business Meeting was held at 4:35 p.m. on August 17, 1984. It was announced that Gerald L. Alexanderson of the University of Santa Clara was elected Editor of Mathematics Magazine for the five-year period 1986-1990.

The following were presented Certificates for Meritorious Service. Professor Gerhard N. Wollan of Purdue University, who was nominated by the Indiana Section; Professor Paul K. Rees of Louisiana State University, who was nominated by the Louisiana-Mississippi Section; Professor Henry M. Cox of the University of Nebraska, who was nominated by the Nebraska Section; Professor S.W. Hahn of Wittenberg College, who was nominated by the Ohio Section; and Professor John D. Neff of Georgia Institute of Technology, who was nominated by the Southeastern Section.

The Carl B. Allendoerfer Award for expository articles in Mathematics Magazine was presented to:

Judith V. Grabiner, California State University, Dominguez Hills, "The Changing Concept of Change: The Derivative from Fermat to Weierstrass."

The Lester R. Ford Awards for expository articles in the American Mathematical Monthly were presented to:

Judith V. Grabiner, California State University, Dominguez Hills, "Who Gave You the Epsilon? Cauchy and the Origins of Rigorous Calculus."

Roger E. Howe, Yale University, "Basic Lie Theory."

John W. Milnor, Institute for Advanced Study, "On the Geometry of the Kepler Problem."

Joel H. Spencer, SUNY at Stony Brook, "Large Numbers and Unprovable Theorems."

William C. Waterhouse, Pennyslvania State University, "Do Symmetric Problems Have Symmetric Solutions?"

The George Pólya Awards for expository articles in the College Mathematics Journal were presented to:

Ruma Falk and Maya Bar-Hillel, Hebrew University, "Probabilistic Dependence Between Events."

Richard J. Trudeau, Stonehill College, "How Big is a Point?"

Election of Members

At its meeting on August 15, 1984, the Board elected 1007 applicants for individual membership and 5 institutional members, namely: Anoka-Ramsey Community College, Kennesaw College, Salisbury State College, University of Tennessee at Martin, and the University of Calgary.

Respectfully submitted,

Kenneth A. Ross
Secretary

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polynomial of degree $\leqslant n$ in the a's. And because of (5), R must be of degree $\leqslant m$ in the b's. Now write

$$p(x) = \prod_{i=1}^{m}\left(1 - \frac{x}{\alpha_i}\right), \quad q(x) = \prod_{j=1}^{n}\left(1 - \frac{x}{\beta_j}\right),$$

where the α's and β's are the zeros (possibly in an appropriate extension of the original field). Apart from the sign, the a's and b's are the elementary symmetric polynomials of $\alpha_1^{-1}, \ldots, \alpha_m^{-1}$ and $\beta_1^{-1}, \ldots, \beta_n^{-1}$, respectively. Thus R can be regarded as a polynomial in these variables. Since $R = 0$ if $\alpha_i = \beta_j$, it follows that R is divisible by $(\beta_j^{-1} - \alpha_i^{-1})$ for all i and j. Thus R is divisible by

$$S = \prod_{i=1}^{m}\prod_{j=1}^{n}\left(\beta_j^{-1} - \alpha_i^{-1}\right) = a_m^n b_n^m \prod_{i=1}^{m}\prod_{j=1}^{n}\left(\alpha_i - \beta_j\right).$$

Since

$$S = \prod_{i=1}^{m}\prod_{j=1}^{n}\left(-\alpha_i^{-1}\right)\left(1 - \frac{\alpha_i}{\beta_j}\right) = a_m^n \prod_{i=1}^{m} q(\alpha_i),$$

it follows that S is a polynomial of degree m in the b's, and (because of the symmetry) of degree n in the a's. Thus $R = fS$, where f is a constant factor. The coefficients of a_m^n in R and S are both one, so $f = 1$ and $R = S$. On the other hand, it is well known that the traditional definition of the resultant is equivalent to S. Hence, the two definitions of the resultant are equivalent.

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THE TEACHING OF MATHEMATICS

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AN ELEMENTARY APPROACH FOR EXTENDING POISSON PROCESSES TO THE NONSTATIONARY CASE

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This note describes a way of extending coverage of stationary Poisson processes to include nonstationary applications. Our method has strong intuitive appeal, is mathematically rigorous, is suitable for presentation to upper division courses on probability or mathematical statistics, and can be accomplished in one lecture.

First let us consider a stationary Poisson process defined intuitively to be a process that continues over time where certain rare "happenings" occur randomly in time at a fixed mean rate λ per unit time. Associated with the time window $[0, t]$ of t units duration, the counting random variable $N(t)$ is defined to be the number of happenings that will occur in the interval $[0, t]$. Mathematical theory of stationary Poisson processes is presented in most upper division courses on probability and/or mathematical statistics, establishing the Poisson distribution with parameter λt as the probability distribution of $N(s + t) - N(s)$. However, discussion of the nonsta-

tionary case usually is deferred to a first course on stochastic processes. If so, the thoughtful student of probability who does not go on to take a stochastic processes course is left troubled by the fixed-rate Poisson model, because for many (if not most) applications the mean rate will experience some variation over the time period of interest. Some common examples are: arrivals at service counters, telephone calls, traffic accidents, breakdowns of equipment, errors, etc.

Here we propose that, following presentation of the mathematical theory of stationary Poisson processes (see [5] for a novel approach), one additional lecture be devoted to Poisson processes for coverage of the nonstationary case. In the nonstationary case the mean rate $\lambda(t)$ per unit time is allowed to depend on time t, instead of being a fixed constant. Sufficient modeling flexibility is obtained by insisting that the rate function $\lambda(t)$ be nonnegative-valued and continuous, except possibly for finitely many jump discontinuities in any finite subinterval of $[0, \infty)$. In order to derive the distribution of $N(s + t) - N(s)$ it is necessary, just as in the stationary case, to specify appropriate postulates and formally define a nonstationary Poisson process to be any process that continues over time and satisfies these postulates. Our formulation of the nonstationary Poisson postulates employs the concept of stochastic ordering (see definition below) and builds on the student's intuitive understanding of the fixed-rate Poisson model, avoiding the usual analytic expressions

$$P\{N(t+h) - N(t) = 1\} = \lambda(t)h + o(h) \text{ and } P\{N(t+h) - N(t) \geq 2\} = o(h).$$

DEFINITION. Random variable Y is said to be stochastically larger than random variable X provided

$$P(Y > r) \geq P(X > r)$$

for every real number r.

Nonstationary Poisson Postulates

1. $N(0) \equiv 0$.
2. Events involving the number of happenings in disjoint time intervals are independent.
3. For every pair of nonnegative real numbers $0 \leq a < b$, the random variable $N(b) - N(a)$ is stochastically larger than the Poisson distribution with parameter $(b - a)\lambda_1$ and stochastically smaller than the Poisson distribution with parameter $(b - a)\lambda_2$, where λ_1 and λ_2 are respectively the greatest lower bound and least upper bound of the set $\{\lambda(t): a \leq t \leq b\}$.

For postulate 3 it is necessary to include the degenerate Poisson distribution with rate parameter $\lambda = 0$ in the family of Poisson distributions, because we have allowed $\lambda(t)$ to assume the value zero. To establish the probability distribution of $N(s + t) - N(s)$ from these postulates we will use the fact that the sum of independent Poisson random variables has a Poisson distribution. Also, we will need the following two lemmas on stochastic ordering.

LEMMA 1. *If Y is stochastically larger (smaller) than S_n for every $n = 1, 2, 3, \ldots$, and if S_n converges in distribution to X, then Y is stochastically larger (smaller) than X.*

LEMMA 2. *If X_1, X_2 are independent random variables and Y_1, Y_2 are independent random variables such that Y_i is stochastically larger than X_i for $i = 1, 2$, then $Y_1 + Y_2$ is stochastically larger than $X_1 + X_2$.*

The proofs of these lemmas can be assigned as homework problems, perhaps with the hints for Lemma 2 to first observe that X_1, X_2, Y_1, Y_2 may be assumed to be mutually independent without loss of generality and to then begin by showing that $X_1 + Y_2$ is stochastically larger than $X_1 + X_2$.

Our main objective in lecture is to discuss and prove the following well-known theorem about nonstationary Poisson processes.

THEOREM. *If the nonstationary Poisson postulates are satisfied, then for every* $s \geq 0$ *and* $t > 0$ *the random variable* $N(s+t) - N(s)$ *has a Poisson distribution with parameter equal to* $\int_s^{s+t} \lambda(u)\,du$.

From this theorem students will see that the Poisson distribution with parameter λt remains appropriate for the distribution of $N(s+t) - N(s)$ even in nonstationary examples provided $\lambda = (1/t)\int_s^{s+t} \lambda(u)\,du$, which is the rate function averaged over the time interval of interest. Thus, for modeling the probability distribution of $N(s+t) - N(s)$ it is sufficient to have a good estimate of the average rate over the time period $[s, s+t]$; a detailed knowledge of how $\lambda(u)$ varies over this time interval is not required.

Proof of Theorem. Fix $s \geq 0$ and $t > 0$ arbitrarily. Divide the time interval $[s, s+t]$ into n subintervals of equal length and label the subinterval end-points as follows:

$$t_0 = s, \qquad t_1 = s + \frac{t}{n}, \qquad t_2 = s + \frac{2t}{n}, \ldots, t_n = s + t.$$

Let $\lambda_i = \inf\{\lambda(u)\colon t_{i-1} \leq u \leq t_i\}$, $i = 1, 2, \ldots, n$, and let $X_1, X_2, \ldots, X_n$ be independent random variables such that X_i has a Poisson distribution with parameter $\lambda_i\, t/n$ for $i = 1, 2, \ldots, n$. Now we use the nonstationary Poisson postulates and Lemma 2 to conclude that

$$N(s+t) - N(s) = \sum_{i=1}^{n} \{N(t_i) - N(t_{i-1})\}$$

is stochastically larger than $S_n = X_1 + X_2 + \cdots + X_n$. From theory we know that the sum S_n must have a Poisson distribution with parameter equal to $\sum_{i=1}^{n} \lambda_i\, t/n$. Since, by lower Riemann sum analysis, we have

$$\lim_{n\to\infty} \sum_{i=1}^{n} \lambda_i\, t/n = \int_s^{s+t} \lambda(u)\,du,$$

it follows that S_n converges in distribution to the Poisson distribution with parameter equal to $\int_s^{s+t} \lambda(u)\,du$. Hence, by Lemma 1, we see that $N(s+t) - N(s)$ is stochastically larger than the Poisson distribution with parameter $\int_s^{s+t} \lambda(u)\,du$. But, from upper Riemann sum analysis it follows similarly that $N(s+t) - N(s)$ is stochastically smaller than the Poisson distribution with parameter $\int_s^{s+t} \lambda(u)\,du$. Therefore, the desired result is established. □

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ANSWER TO PHOTO ON PAGE 633

Donald Ornstein has made deep contributions to ergodic theory. He discovered, in particular, a transformation with no invariant measure, and later he proved that for many transformations entropy constitutes a complete set of invariants.

PROBLEMS AND SOLUTIONS

EDITED BY G. L. ALEXANDERSON, DAVID BORWEIN (ADVANCED PROBLEMS),
H. M. W. EDGAR (ELEMENTARY PROBLEMS), AND D. H. MUGLER

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*Send all **proposed** problems, typed and in duplicate if possible, to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053. Please include solutions, relevant references, etc.*

An asterisk () indicates that neither the proposer nor the editors supplied a solution.*

***Solutions** should be sent to the address given at the head of each problem set.*

A publishable solution must, above all, be correct. Given correctness, elegance and conciseness are preferred. The answer to the problem should appear right at the beginning. If your method yields a more general result, so much the better. If you discover that a MONTHLY *problem has already been solved in the literature, you should of course tell the editors; include a copy of the solution if you can.*

ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by April 30, 1985. Please place the solver's name and mailing address on each (double-spaced) sheet. Include a self-addressed card or label (for acknowledgement).

E 3064. *Proposed by Ion Cucurezeanu, Constanta, Romania.*

Find all integers $m, m > 1$, such that m^3 is a sum of m squares of consecutive integers.

E 3065. *Proposed by Armel Mercier, Université du Québec à Chicoutimi, Canada.*

Let $n \geqslant 0$ be any integer and let k be any integer such that $k \geqslant n + 1$. Then find a closed formula for

$$\sum_{j=0}^{n} \frac{(-1)^j \binom{k}{j} \binom{k-1-j}{n-j}}{j+1},$$

where $\binom{m}{i} = \dfrac{m!}{i!(m-i)!}$.

E 3066. *Proposed by Bruce Richter, Utah State University.*

Let A be a subset of $\mathbb{R}^n$ (n-dimensional Euclidean space) that is homeomorphic to $(0,1)$. Let $\bar{A}$ be the closure of A in $\mathbb{R}^n$ and suppose $\bar{A} - A$ contains no homeomorph of $(0,1)$. Show that $\bar{A}$ is homeomorphic to one of $(0,1)$, $[0,1)$, $[0,1]$ and the unit circle.

E 3067. *Proposed by Colin Bennett, University of South Carolina.*

In this MONTHLY, 88(7) (1981), 526–527, D. J. Newman presented the following result: Let $f(x)$ and $g(x)$ be nonnegative C^1-functions, both of which increase without bound as $x \to +\infty$. Write $f(x) \sim g(x)$ to mean that $f(x)/g(x) \to 1$ as $x \to +\infty$. If f is convex and $1/g$ is convex, then

$$f \sim g \Rightarrow f' \sim g'. \tag{1}$$

By considering the equivalent conditions on the reciprocal functions, one might expect to have the following result: If $f(x)$ is concave and $xg(x)$ is convex, then (1) holds.

Show by example that this fails but that the result is true under the additional mild growth condition on g:

$$g(x) \leqslant cg(2x) \qquad (x \geqslant 0), \tag{2}$$

for some constant $c < 1$.

E 3068. *Proposed by George Tsintsifas, Thessaloniki, Greece.*

Let $T = \Delta ABC$ be a triangle with inradius r and circumradius R. We consider a circular disc C with radius $d, r \leqslant d \leqslant R$, in a position such that Area $(T \cap C)$ is a maximum. Prove that as d varies continuously in the closed interval $[r, R]$, the center of the disc C (in the maximum position) moves on a conic τ passing through the incenter, circumcenter and the Lemoine point of ΔABC. Also ΔABC is self-polar with respect to the conic τ.

SOLUTIONS OF ELEMENTARY PROBLEMS

Product of all Positive Divisors of n

E 2946 [1982, 333]. *Proposed by Rodica Simion and Frank W. Schmidt, University of Pennsylvania.*

Let $f(n)$ be the product of all positive divisors of n; e.g., $f(3) = 3$, $f(4) = 8$. Does $f(m) = f(n)$ imply $m = n$?

Solution by Ivan Niven, University of Oregon. Yes. Assuming $f(m) = f(n)$ we prove that $m = n$. For any prime divisor p of m, let p^a and p^b be the highest powers of p dividing m and n, respectively. Write $m = p^a M$. If p^k is the highest power of p dividing $f(m)$, then by a direct calculation $k = a(a+1)d(M)/2$, where $d(M)$ denotes the number of positive divisors of M. Also $d(m) = (a+1)d(M)$, so that $k = ad(m)/2$. Since $f(m) = f(n)$, it follows that $k = bd(n)/2$ and $a/b = d(n)/d(m)$. Thus the ratio a/b is the same for every prime factor p of m. Now $a/b < 1$ implies that m is a proper divisor of n, and $f(m) < f(n)$. Since $a/b > 1$ is likewise ruled out, we conclude that $a/b = 1$, and hence $m = n$.

Also, solutions have been received from eighty-six others.

Remark. The problem is not new. M. S. Klamkin (University of Alberta) points out that he set the problem previously as Problem 518 in Math. Mag., 36 (1963), 37 (1964), and the published solution by Leo Moser gives a generalization.

$p^e \| (x^{p-1} - y^{p-1}),\ 2 \nmid pe,\ p$ Prime Occurs Frequently

E 2948 [1982, 334]. *Proposed by Barry J. Powell, Kirkland, Washington.*

Show that for any fixed pair of coprime positive integers x and y with $x \neq y, xy > 1$, there are infinitely many primes p for which the exponent of p in $x^{p-1} - y^{p-1}$ is odd. [The exponent of p in m is the largest integer e such that $p^e | m$.]

Solution by L. E. Mattics, University of South Alabama. We assume by way of contradiction that there are only finitely many (or no) odd primes $q_1, q_2, \ldots, q_n$ such that q_i has an odd exponent in $x^{q_i-1} - y^{q_i-1}$ and we set

$$s = (q_1 - 1)^2 (q_2 - 1)^2 \cdots (q_n - 1)^2$$

(set $s = 1$ if there are no such primes). $q_1, q_2, \ldots, q_n$ all divide $x^{4s} - y^{4s}$, so none of them divides $x^{4s} + y^{4s}$. If p divides $x^{4s} + y^{4s}$ for p an odd prime, then p has an even exponent in $x^{p-1} - y^{p-1}$, so p has an even exponent in $x^{8s} - y^{8s}$ (and in $x^{4s} + y^{4s}$) because s is a square.

Hence $x^{4s} + y^{4s} = A^2$ or $x^{4s} + y^{4s} = 2A^2$, which is a contradiction since these equations have no nontrivial solutions.

Also solved by the proposer.

Restricted Subadditive Functions

E 2954 [1982, 425]. *Proposed by A. Meir, University of Alberta, Edmonton, Canada.*

Let $f(n)$ be a nonnegative real valued function defined for every natural number n. Suppose $f(a + b) \leqslant f(a) + f(b)$ whenever $|b - a| \leqslant \kappa$ (κ fixed). Does $\lim_n f(n)/n$ exist? (See E 2841 [1980, 577].)

Solution by Vladimir Grinberg, Columbia University, New York. The limit need not exist. Let T be the set of powers of 2: $T = \{2^m\colon m = 0, 1, 2, 3, \ldots, \}$. For each fixed $\kappa \geqslant 0$ define

$$f_\kappa(n) = \begin{cases} n & \text{if distance}\,(n, T) \leqslant \kappa, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $\liminf f_\kappa(n)/n = 0$ and $\limsup_\kappa f_\kappa(n)/n = 1$. Assume now that for some a, b we have

$$(*) \qquad |a - b| \leqslant \kappa \text{ and } f_\kappa(a + b) > f_\kappa(a) + f_\kappa(b).$$

This means that either $f_\kappa(a) = 0$ or $f_\kappa(b) = 0$, say $f_\kappa(a) = 0$. Thus, for some positive integer m,

$$2^{m-1} + \kappa < a < 2^m - \kappa,$$

which implies

$$2^m + \kappa < a + (a - \kappa) \leqslant a + b \leqslant a + (a + \kappa) < 2^{m+1} - \kappa$$

forcing $f_\kappa(a + b) = 0$. Thus $(*)$ is impossible.

Also solved by M. Bencze (Romania), W. Janous (Austria), O. Jørsboe (Denmark), B. Liszkay (Hungary), O. P. Lossers (The Netherlands), V. D. Mascioni (Switzerland), W. Oettli (W. Germany), J. Suck (W. Germany), and the proposer.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be mailed in duplicate to Professor D. H. Mugler, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053, by April 30, 1985. The solver's full post-office address should be on each sheet.

6479. *Proposed by D. W. Masser, The University of Michigan, Ann Arbor.*

Let a, b, c, d be integers such that the polynomial $P(x) = x^4 + ax^3 + bx^2 + cx + d$ is not a perfect square. A well-known theorem of Siegel implies that the diophantine equation $y^2 = P(x)$ has only a finite number of solutions in integers x, y. For $H \geqslant 1$ let $X(H)$ be the maximum of $|x|$ taken over all such solutions of all such equations with $\max(|a|, |b|, |c|, |d|) \leqslant H$. Show that there are absolute constants $k > 0, K$ such that $kH^3 \leqslant X(H) \leqslant KH^3$ for all H.

6480. *Proposed by Robert E. Shafer, Berkeley, CA.*

An expository article in the October 1983 issue of this MONTHLY deals with the result

$$\sum_{m=1}^{\infty} \binom{-1/2}{m} \frac{(-1)^m}{m} = \log 4.$$

Prove the following extension:

$$\sum_{m=1}^{\infty} \binom{-1/2}{m+n} \frac{(-1)^m}{m} = \binom{-1/2}{n} \left(\log 4 + \sum_{m=1}^{n} \frac{1}{m} \right).$$

6481. *Proposed by I. J. Schoenberg, University of Wisconsin-Madison.*

Let

$$f_n(z) = 1 + \frac{z}{4} + \frac{z^2}{4^2} + \cdots + \frac{z^n}{4^n} \qquad (n = 1,2,\ldots).$$

As a very special case of a theorem of Szegő (Theorem II in Collected Papers, vol. 2, page 171) we know that the image $f_n(U)$ of the unit circle $U = \{|z| \leqslant 1\}$ is schlicht and convex. Show that in fact

$$|f_n(z_1) - f_n(z_2)| > |z_1 - z_2|/18$$

whenever $z_1, z_2 \in U$ and $z_1 \neq z_2$.

6482*. *Proposed by G. Vrancken, Utrecht, Holland.*

Let $R(a,p) = \int_0^\infty e^{-ax}(1+x)^p\,dx$. Prove or disprove that

$$\frac{1}{R(a,p)} + \frac{1}{R(b,q)} \geqslant \frac{1}{R(a+b,p+q)}$$

for $a, b > 0$ and $p, q \geqslant 0$.

6483. *Proposed by J. Arias de Reyna, University of Seville, Spain.*

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a differentiable function with a unique critical point at which it has a local minimum. For which values of n is the minimum necessarily an absolute minimum?

SOLUTIONS OF ADVANCED PROBLEMS

The Irrationality of $\sum_{n=1}^{\infty} \frac{1}{n(nm+k)}$

6424 [1983, 289]. *Proposed by Michael J. Dixon, California State University, Chico.*

Let m and k be positive integers. Show that

$$S(m,k) = \sum_{n=1}^{\infty} \frac{1}{n(nm+k)}$$

is rational if and only if $k \equiv 0 \pmod{m}$.

Composite solution by S. Uchiyama, University of Tsukuba, Japan; S. K. Venkatesan, student, Indian Institute of Technology, India; and the proposer. When k/m is an integer, a simple partial fraction decomposition leads to

$$S(m,k) = \frac{1}{m} + \frac{1}{2m} + \cdots + \frac{1}{k},$$

which is, of course, rational.

Suppose, therefore, that k/m is not an integer. Logarithmic differentiation of the infinite product expansion of $\Gamma(x)$ yields the well-known formula [see [2], p. 522]

$$\psi(x) = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x}{n(n+x)}, \qquad x \neq -1, -2, \ldots,$$

where γ is Euler's constant and $\psi(x) = \Gamma'(x)/\Gamma(x)$. Hence

$$kS(m,k) = \psi\left(\frac{k}{m}\right) + \gamma + \frac{m}{k},$$

and so, by a formula of Gauss [see [2], p. 523],

$$kS(m,k) = \frac{m}{k} - \log m - \frac{\pi}{2}\cot\frac{k\pi}{m} + \sum_{r=1}^{m-1} \cos\frac{2\pi rk}{m} \cdot \log 2\sin\frac{r\pi}{m}.$$

Thus, $S(m,k)$ is a nonzero linear combination with algebraic coefficients of logarithms of algebraic numbers, and it follows from a celebrated theorem of Alan Baker [1, Theorem 2.2] that $S(m,k)$ is transcendental.

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Complex Functions with Dense Range

6425 [1983, 289]. *Proposed by J. Miles and L. Rubel, University of Illinois.*

For the Koebe function $K(z) = z/(1-z)^2$, show that there exists a bounded analytic function $B(z)$ in the unit disc $\mathbb{D}$ such that $F(z) = K(z) + B(z)$ maps $\mathbb{D}$ onto the whole complex plane $\mathbb{C}$. Is the same still true for an arbitrary analytic function $K(z)$ in $\mathbb{D}$ that maps $\mathbb{D}$ onto a dense subset of $\mathbb{C}$?

Solution to first part by the proposers. It is well known that $K(z)$ maps $\mathbb{D}$ one-one onto $\mathbb{C} \setminus (-\infty, -1/4]$. We shall show that $F(z) = K(z) - z$ maps $\mathbb{D}$ onto $\mathbb{C}$. Consider the closed curve Γ on $\hat{\mathbb{C}}$ (the extended complex plane) given by

$$F(e^{i\theta}) = -(2\sin(\theta/2))^{-2} - \cos\theta - i\sin\theta, 0 < \theta < 2\pi.$$

Clearly Γ is a simple closed curve in $\hat{\mathbb{C}}$, symmetric about the real axis. By the argument principle, the number of solutions of $F(z) = w$ in $\mathbb{D}$ is constant in each of the two components of $\hat{\mathbb{C}} \setminus \Gamma$. One of the components contains the interval $(-\infty, 3/4)$ and, since

$$F(z) = z^2(2-z)(1-z)^{-2}$$

has two zeros in $\mathbb{D}$, it follows that, for w in this component, $F(z) = w$ has two solutions in $\mathbb{D}$. Next, for $M > 3/4$, the equation $F(z) = M$ has three real solutions z_1, z_2, z_3 with $0 < z_1 < 1, 1 < z_2 < 2$ and $z_3 < -1$. Thus there is one solution of $F(z) = w$ in $\mathbb{D}$ for every point w outside Γ.

Finally consider a finite $w \in \Gamma$. Since

$$F'(z) = z(z^2 - 3z + 4)(1-z)^{-3} \neq 0$$

when $|z| = 1$, F is locally one-one at all points of $\partial\mathbb{D} - \{1\}$. Further, $F(z) = w$ has exactly three distinct solutions, one and only one of which is in $\partial\mathbb{D}$. The other two cannot both be in $\mathbb{D}$ (or in $\mathbb{C} \setminus \overline{\mathbb{D}}$), for then every point in a neighborhood of w would be the image under F of two points in $\mathbb{D}$ (or in $\mathbb{C} \setminus \mathbb{D}$). Thus one of the three solutions lies in $\mathbb{D}$ and another in $\mathbb{C} \setminus \mathbb{D}$. This completes the proof that F maps $\mathbb{D}$ onto $\mathbb{C}$.

Also partially solved by Thomas L. McCoy. The second part of the problem remains open.

Condition for a Semigroup to be a Group

6427 [1983, 338]. *Proposed by Fred Galvin, University of Kansas.*

Suppose that S is a semigroup, and there are elements a and b in S such that (1) for each x in S there is a unique y in S such that $xy = a$, and (2) for each x in S there is a unique y in S such that $yx = b$. Show that S is a group.

Solution by Phillip Barry, Idaho Falls, Idaho. By (1) there exists a unique w such that $aw = a$. Similarly, there is a unique v such that $vb = b$. For every t in S, using (1) with $x = at$, there is a unique t_R such that $(at)t_R = a$. Reassociating gives $a(tt_R) = a$. By the uniqueness of w, $tt_R = w$. By a similar argument there exists a unique t_L such that $t_L t = v$.

Next $(vb)b_R = bb_R$ implies $vw = w$, and $a_L(aw) = a_L a$ implies $vw = v$; hence $v = w$ and $t_L t = w$. Now

$$(at_L)t = a(t_L t) = aw = a,$$

so t is the unique solution to (1) with $x = at_L$. But $((at_L)t)w = aw = a$. Reassociating gives $(at_L)(tw) = a$; therefore $tw = t$. We now have a right identity and right inverses, so the result follows.

Also solved by the proposer and sixty others.

REVIEWS

EDITED BY ALLAN L. EDMONDS AND JOHN H. EWING
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER

Zermelo's Axiom of Choice: Its Origins, Development, and Influence. By Gregory H. Moore. Studies in the History of Mathematics and Physical Sciences, Number 8. Springer-Verlag, New York, 1982. xiv + 410 pp.

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This fine book provides not only a history of the axiom of choice beginning with its earliest implicit uses, covering both its mathematical and philosophical aspects; it contains a good deal of material on the conceptual development and foundations of set theory. There is a full account of the various controversies and the investigations of the axiom's applications in set theory, algebra, analysis, topology, and logic up to the proof of its relative consistency by Gödel. A brief Epilogue on developments after Gödel's work is also included, as well as an appendix containing a translation of the five letters by Borel, Baire, Lebesgue, and Hadamard dating from the early days of the controversy over the axiom. The work is an excellent example of historical scholarship in the area of foundations of mathematics: it is interesting, comprehensive, and exact. While the ideas on foundations and mathematical contributions of many mathematicians and logicians are described, it is above all the work of Zermelo which is featured, and there is an account of all facets of his investigations in both the early period and the later around 1930. Since Zermelo is also one of my own great heroes of the intellectual realm, I shall deal most fully with what relates to him.

The axiom of choice was first stated as a basic general principle of mathematics in 1904 by Ernst Zermelo. In his formulation, the axiom asserted that for any set A of non-empty sets there is a function f such that for each M in A, $f(M)$ is an element of M. In the more psychological terms which were commonly used, one speaks of postulating the possibility of infinitely many arbitrary choices. Finitely many arbitrary choices require no special assumption, and the axiom restricted to finite sets of sets is demonstrable by means of the principles of logic and mathematical induction. The first non-trivial case arises for denumerable sets of non-empty sets, and the axiom restricted to such sets is known as the denumerable axiom of choice. Of course when a rule can be given for making the choices, the axiom is not required. In 1904, independently of Zermelo,

Bertrand Russell recognized a proposition which turned out to be equivalent to Zermelo's axiom, as an assumption involved in arguments he and others had been giving. This principle, which he called the multiplicative axiom, stated that for any set A of mutually exclusive non-empty sets M there is a set S which contains exactly one element from each member M of the set A. But, unlike Zermelo, Russell regarded this proposition and the axiom of choice with skepticism and used it only as a hypothesis. In a moment we shall see why.

Long before Zermelo explicitly formulated the axiom of choice, many mathematicians had given arguments involving infinitely many choices without stating a rule by which they could be made. This occurred in algebra and analysis as well as in set theory. These arguments were given by mathematicians who were unaware that the validity of their reasoning depended on a basic principle not previously recognized, and indeed, one which would be frequently rejected as soon as it was made explicit—rejected by many who had often implicitly used it. In fact, some of the axiom's most persistent critics continued to lapse into arguments in which the axiom was implicitly used (p. 103).

Moore specifies four stages of involvement with choices: (1) choice of an unspecified element of a set, which represents a basic logical method present already in ancient mathematics, or choices from finitely many sets. Infinitely many choices (2) made according to a given rule, or (3) made without a stated rule—an example of this stage can be found in an argument by Cauchy as early as 1821 (p. 12). (4) Infinitely many arbitrary choices for which it is impossible to provide a rule. Perhaps the earliest example of this occurs in 1871 in an argument due to Cantor but published by E. Heine—an argument for a proposition from analysis asserting that two definitions of the continuity of functions are equivalent. Here, however, only the denumerable axiom of choice is needed. In 1877, in an argument for a proposition of algebraic number theory, Dedekind utilized nondenumerably many choices from congruence classes which could not be made according to a rule. Of course, at the time Cantor and Dedekind were unaware of the novelty of their methods, which, moreover, attracted little attention (p. 16). The examples cited are notable for their early date, but they are only three among numerous pre-1904 cases of at least stage three described by Moore—instances are given from Cantor, Dedekind, Jordan, Lebesgue, Borel, Russell, Hardy, and others.

Although the axiom of choice slipped into mathematics unnoticed, as soon as mathematicians were confronted with it and its deductive power was realized, it became the center of a considerable amount of contention among those interested in set theory and the foundations of mathematics. Indeed its deductive power—in particular its implying the well-ordering theorem: every set can be well-ordered—seems to have been a main source of the disputes concerning the axiom which began at once with the publication of Zermelo's paper in which that implication was established. Especially during the next four years, objections and reservations were registered in many European countries and in the U.S. In France, England, and Germany, series of articles appeared in which the axiom and Zermelo's proof were critically discussed, often in association with the newly discovered antinomies of set theory. "Rarely have the practitioners of mathematics, a discipline known for the certainty of its conclusions, differed so vehemently over one of its central premises as they have done over the Axiom of Choice. Yet without the Axiom, mathematics today would be quite different. The very nature of modern mathematics would be altered and, if the Axiom's most severe constructivist critics prevailed, mathematics would be reduced to a collection of algorithms. Indeed, the Axiom epitomizes the fundamental changes—mathematical, philosophical, and psychological—that took place when mathematicians seriously began to study infinite collections of sets" (p. 1).

The Axiom of Choice and Well-Ordering. Perhaps the most important problems of early set theory were those of the comparability of sets (or trichotomy of cardinals), of well-ordering, and of the cardinal number of the continuum. If Cantor's transfinite "powers" did not exhibit

trichotomy, then they would not deserve the designation "cardinal numbers." Trichotomy for alephs or powers of well-ordered sets could be shown, so if it could be proved that every set could be well-ordered, it would follow that the power of every set is an aleph and consequently that trichotomy applies for all transfinite powers. It would also follow in particular that the power of the continuum is an aleph; *which* aleph it is constituted the continuum problem, and Cantor's continuum hypothesis asserted that is was $\aleph_1$.

Cantor originally stated the proposition that every set can be well-ordered as a "law of thought," but it was not accepted as such by others. In the same paper of 1883 in which Cantor proposed the well-ordering principle as a law of thought, he formulated the continuum hypothesis in a way (not the original formulation) which implied that the continuum can be well-ordered. "Cantor's correspondence suggests that it was this form of the Continuum Hypothesis which led him to the Well-Ordering Principle..." (p. 83). In this final presentation of the theory of the transfinite (1895, 1897), Cantor clearly recognized the need for proof of such propositions as that every power is an aleph, that every set can be well-ordered, and that for unequal powers one is greater than the other. He even formulated an argument for the aleph theorem (from which the others follow) in letters to Dedekind (1899) and others which makes use of what were later called paradoxes of set theory. It is plausibly suggested by Moore that the source of Cantor's proof of the aleph theorem was the idea of his proof that every infinite set contains a denumerable subset. If a set A is not finite, then it has proper subsets of every finite number. Cantor reasoned, implicitly using infinitely many choices, that A must contain a denumerable sequence of elements: let the first term be any element of A, let the second be any remaining element, etc. (p. 29). Now he could have thought along similar lines regarding the possibility of sets whose power is not an aleph, supposing there are such. They must have proper subsets of every aleph, and a sequence of elements in which all ordinal numbers are used as indices. Whether or not Cantor thought along precisely these lines, G. H. Hardy did in a paper of 1903 in which he argued that every infinite power is either an aleph or is greater than every aleph (p. 60). Cantor was able to proceed further than Hardy through the use of his discovery of the need to distinguish between consistent and inconsistent multiplicities; the former can be conceived as unities (which he called sets) while the supposition that the latter (which he referred to as absolutely infinite) are unities leads to a contradiction. This is the case for the multiplicities of all ordinal numbers, all sets, all things, all alephs—as Cantor showed in his correspondence. Invoking the principle that a multiplicity containing a submultiplicity equipollent to an inconsistent multiplicity is also inconsistent, Cantor could argue that there cannot be an infinite set which does not have an aleph as its power. A multiplicity without an aleph as its power must contain a sequence of elements isomorphic to the inconsistent multiplicity of all ordinals.

Cantor seems to have had reservations about this argument and did not care to publish it. He seemed uncertain regarding the distinction between consistent and inconsistent multiplicities, and asked Dedekind for his opinion of it. Dedekind replied that the meaning of part of what Cantor had said on this matter was not clear to him [**4**]. Later Philip Jourdain sent to Cantor an argument for the aleph theorem only to find that Cantor knew all about it [**3**]. While encouraging Jourdain to publish his own proof, Cantor would not allow a letter containing his proof to be published by Jourdain (p. 61).

In his edition of Cantor's works, Zermelo explained his objections to the type of argument given by Cantor and others. He criticized the transfinitely many *successive* choices that seem to be implicit in Cantor's argument on the ground that "the intuition of time is applied here to a process that surpasses every intuition; a being is simulated who could carry out *successive* arbitrary choices...". What Cantor needed to employ was the axiom of choice, "which postulates the possibility of a *simultaneous* choice..." (tr. Moore p. 53). Yet one might also entertain doubts about the being who would make the simultaneous choices, etc. The fact is, however, that the formulation in terms of the possibility of simultaneous choices is not the ultimate one; for as

Zermelo clearly puts it himself in his 1908 paper on well-ordering, "the principle taken as fundamental still appears to be somewhat tainted with subjectivity and liable to misinterpretation" [**16**, p. 186]. The equivalent principle which he then states and describes as "purely objective" is the multiplicative axiom, which says nothing whatever about the possibility of choices, but only postulates the existence of a set. Zermelo also objected to the use of inconsistent multiplicities. In his paper of 1904 he proved the well-ordering theorem while avoiding both successive arbitrary choices and inconsistent multiplicities. But in doing so he certainly did not succeed in demonstrating the principle of well-ordering to everyone's satisfaction; on the contrary few were satisfied with his proof or with the new axiom on which it was based. Indeed, it was with the publication of this paper that the great controversy over the axiom of choice began.

Objections to the Axiom of Choice. The first objections to the use of infinitely many choices not made in accordance with a rule occurred much earlier than 1904. In a paper of 1890, Peano stated such to be inadmissible without elaborating the reasons for this stricture. (He did in 1906 (p. 134), and was answered by Zermelo (p. 145).) Two years later R. Bettazzi did provide a reason: "one cannot regard as determinate an *infinite* number of objects all chosen arbitrarily in given classes. This follows clearly when one notes that giving them arbitrarily is equivalent to defining them separately one at a time" (tr. Moore p. 77). Later, in 1896, Bettazzi identified infinitely many arbitrary choices in Dedekind's proof of the equivalence of two concepts of finite sets (p. 26). Bettazzi might well have initiated a discussion of the matter of arbitrary choices at this time if he had not become convinced of the truth of Dedekind's theorem due to a work by Burali-Forti which was based on an erroneous postulate. "In this way the only mathematician to suspect at an early stage that these two notions of finite set need not be equivalent joined those who employed the two interchangeably. As a result, neither Bettazzi nor anyone else scrutinized the use of arbitrary choices until Bepo Levi did so, in a different context, five years later" (pp. 28–29).

Some of the first critics of the axiom of choice after Zermelo's proof that it implied the well-ordering theorem were the French mathematicians Borel, Baire, and Lebesgue. It was their contention that mathematical existence was dependent upon definitions; nonconstructive methods were considered to be "outside mathematics." Baire even based his rejection of Zermelo's argument ultimately on the denial that the set of all subsets of an infinite set is given with that set, the reason presumably being that not all the subsets are specifiable (see [**8**, p. 95] for a suitable response to such criticism of the power set concept). In opposition to these views, Hadamard, who accepted both the axiom and Zermelo's proof, was inclined to think that questions of what we can do (choose, define) are subjective and belong to psychology, rather than to mathematics. It seemed to him that history showed that "the essential progress in mathematics has resulted from successively annexing notions which, for the Greeks or the Renaissance geometers or the predecessors of Riemann, were 'outside mathematics' because it was impossible to define them" (tr. Moore p. 96). Actually no ideological claims about what is "outside mathematics" hold up very well. Mathematics has, as indicated by Hadamard, constantly changed and expanded throughout its history, and no one is in a position to pontificate on what the true nature of mathematics is or ought to be. Of course there are different individual preferences as Hadamard well recognized: "there are two conceptions of mathematics, two mentalities, in evidence. After all that has been said up to this point, I do not see any reason for changing mine" (tr. Moore p. 318). His conception seems to have become the predominate one, for, as Moore says: "The constructivists' restriction of mathematics (perhaps because it became an anomaly) acquired a name, whereas the viewpoint espoused by Hadamard and Zermelo became the established position and so found only the name: modern mathematics" (p. 310).

Debate about the axiom of choice and Zermelo's proof was also vigorously carried on in England by Hobson, Russell, Hardy, and Jourdain. In 1904, while examining an attempted proof of the multiplicative axiom by Whitehead, Russell had found that the argument was circular. "At first I thought probably a proof could easily be found; but gradually I saw, if there is a proof, it

must be very recondite" (p. 122). Russell must have thought the axiom true initially, since he expected to prove it; but failing to find a proof, he became skeptical. Theorems depending on the axiom, such as the equivalence of Dedekind's definition of "finite" with the usual definition and that every infinite set has a denumerable subset, were now regarded by Russell as only conditionally demonstrated. On hearing this, Louis Couturat, who was in correspondence with Russell at the time, objected to such skepticism. He didn't seem surprised that Russell was unable to prove the multiplicative axiom, but he was astonished at Russell's doubting the truth of the axiom which to Couturat was self-evident, a proposition of common sense. But the antinomies had made Russell very wary. Common sense had been shown to be unreliable, and he apparently regarded the multiplicative axiom as possibly a new source of contradictions (p. 125). Yet a short time later, Russell granted to G. H. Hardy, who maintained that the denial of the axiom "seems paradoxical" and "seems to make hay of a lot of the most interesting mathematics," that "if it really strikes anyone as obvious, I know no reason for refusing to admit it" (p. 127). There was, however, a basic reason which prevented the axiom from seeming obvious to Russell—the concept of set which he preferred.

There are many different concepts of set, not always clearly distinguished; most of them are unsuitable for Cantorian set theory, including the concept often used (no doubt without considering its implications) by Cantor. Whether or not the axiom of choice or the multiplicative axiom is evident depends upon the conception of set one has in mind. Thus, the objections which Russell and some others made against the axiom of choice could be entirely justified given the concept of set accepted by the critic. Let us briefly consider two of the relevant concepts. According to the intensional concept, (the existence of) a set is determined by a property common and peculiar to its elements. Some would prefer to replace properties (considered as abstract entities) by laws, rules, or formulas composed of a finite number of words; this concept, which makes the existence of a set depend on the availability of a definition determining it, is the one accepted by the French constructivists discussed above. In sharp contrast with any form of intensional concept, there is the purely extensional concept of set as an abstract object determined or constituted by its elements and marked off from other sets by its elements; this concept of set as a "mere plurality" of things makes no mention of properties, rules, or definitions. For reasons that cannot be discussed here (see [**5**, p. 141]), but which do not seem sound, Russell preferred the intensional concept. Given this conception of sets, the multiplicative axiom means that for any class of nonempty disjoint sets, there is a property which is possessed in common by exactly one element from each of these sets and by only those elements. Now this was the proposition which Russell thought "is not at all self-evident." Thus he was "led to the conclusion that the axiom ceases to be self-evident as soon as we grasp what it means" (p. 176). But as Gödel says, nothing can better express what is intended by the extensional concept than the multiplicative axiom [**5**, p. 151], which is on this concept "just as evident as the other set-theoretical axioms" [**6**, p. 516, fn. 2].

Zermelo also maintained that the axiom of choice, or rather the equivalent form the multiplicative axiom, was a self-evident proposition, as indeed it is *provided* a suitable concept of set is selected. Yet the most effective way to defend the axiom may have been to indicate the wealth of interesting consequences which follow from it—this probably had impressed Hardy most about the axiom. Moore traces the development of the investigation of the consequences of the axiom, which were due more than anyone else to W. Sierpinski, beginning around 1916, and the Warsaw school. One may well agree with Moore's comment: "As Zermelo had foreseen, the Axiom's usefulness finally won it a secure place within the mainstream of mathematical research" (p. 197).

With the notable exception of P. Jourdain, the axiom of choice was granted from the days of its first appearance to be indemonstrable. But some of its critics, especially those like Russell who were inclined to think that not every class can be well-ordered, thought it possible that the axiom of choice might lead to a contradiction. In a lecture delivered to the Société Mathématique de France in 1911, Russell even encouraged his listeners to look for contradictions (p. 176). But in

1938 Gödel put an end to the hopes of the opponents of the axiom of choice by proving that if the rest of the axioms of set theory are consistent, those axioms together with the axiom of choice are also consistent.

Zermelo's Axioms for Set Theory. In the summer of 1907 Zermelo took stock of the objections that had been made against the axiom of choice and his proof of the well-ordering theorem. In 1908 he published two papers which Moore regards as very closely connected: the first containing a new proof of well-ordering followed by a refutation of his critics, the second containing his system of axioms. The usual interpretation of Zermelo's motivation in constructing his system of axioms is that he intended to provide a foundation for set theory in view of the antinomies. Pointing to other sources of motivation for Zermelo's axiomatization, Moore minimizes the role of the antinomies. He emphasizes the differing attitudes towards the antinomies of Zermelo, Russell, and Hausdorff, all of whom published major papers relevant to set theory in 1908. Hausdorff was the least concerned of the three and apparently felt that too much fuss was being raised over the antinomies; he was content merely to avoid the sets known to lead to contradictions (p. 158). Regarding the difference between Russell and Zermelo, two facts are particularly notable. First, both independently discovered the antinomy, later known as Russell's paradox, of the set of all sets which are not members of themselves (see [**12**]). Zermelo communicated his discovery to Hilbert and others, but that was all; in general he devoted a minimum of space to the discussion of antinomies in his publications. Russell wrote to Frege, who regarded arithmetic as shaken by the news, and discussed it extensively in his book *The Principles of Mathematics*. Secondly, Russell was occupied for years with schemes for "solving" the antinomies and was prepared (reluctantly) to sacrifice portions of Cantorian set theory to what he regarded as a satisfactory solution. In contrast, set theory came first for Zermelo, who certainly would not have been willing to spend his time dwelling on the antinomies after the fashion of Russell. Also, Zermelo did not seem to have been so impressed by the seriousness of the antinomies as Russell was (but see the discussion and quotations in [**9**, p. 598]). Moore interprets Zermelo as seeing "the threat of the paradoxes, from a mathematical standpoint, as more apparent than real" (p. 158). It is true that the impression given by the papers of 1908 is that too much scepticism has been produced by the antinomies and that everything can be straightened out by "a suitable restriction of the concept of set." Moreover, Zermelo does not seem to have thought it necessary to go to any great lengths (as Hilbert did) to uphold the distinctive certainty traditionally attributed to mathematics. What he did intend was to show that the full extent of set theory as developed by Cantor and Dedekind follows from his principles. Moreover, one can see that "the antinomies discovered so far vanish one and all" when set theory is based on his axioms. Of course, from these same principles, Zermelo's own much contended well-ordering theorem also follows.

While granting that the antinomies had something to do with the axiomatization, Moore is inclined to think that Zermelo's "axiomatization was primarily motivated by a desire to secure his demonstration of the Well-Ordering Theorem, and in particular, to save his Axiom of Choice" (p. 159). It does seem that opposition to his work was the occasioning cause of the papers of 1908, though Zermelo had been thinking of restricting the concept of set—presumably by means of axioms—before the controversy started, and Moore provides information on some earlier axioms used by Zermelo (p. 155). But, in what way could an axiomatization have figured in a defense of Zermelo's proof? How was presentation of the system of axioms to aid in combating his critics?

The antinomies may have motivated Zermelo's axiomatization as much indirectly as directly insofar as they influenced the reception of his proof of the well-ordering theorem. Some of the objections to Zermelo's proof were directed not against the axiom of choice but were inspired by the suspicion that Zermelo's method was endangered by Burali-Forti's antinomy involving the set of all ordinal numbers (see [**12**] on "Burali-Forti's" antinomy); this was especially the line of attack of his German critics, but Hardy's reservations also resulted from thinking that Zermelo's arguments depended on that antinomy (p. 128). It would appear that such were the only critics

whom Zermelo could have hoped to rebut by an axiomatization. So even if Zermelo wrote the paper on the axioms primarily in defense of his proof of well-ordering, it still would have had to function as a convincing way of avoiding the antinomies.

The axiom system seems to have satisfied Gerhard Hessenberg (p. 114), and a few others, but on the whole Zermelo's system was poorly received at first. Bernstein, who seems to have been satisfied in other respects, held it against the system that it implied the well-ordering theorem (p. 166). Many objected to his axiom of separation: the concept of definite property or definite propositional function, invoked on account of the paradoxes of finite definability (see [**16**, p. 192, fn. 10; p. 202]) was considered too vague. Furthermore, the principle for avoiding the antinomies was not clear to all. In fact, Zermelo's system is based on the theory of limitation of size—the idea, going back to Cantor, that the "inconsistent multiplicities" are too big or too comprehensive or, in Cantor's terminology, "absolutely infinite." Zermelo, following Hessenberg, referred to the antinomies as "ultrafinite paradoxes" [**16**, p. 202]. To those who were concerned about Burali-Forti's antinomy spoiling his proof of well-ordering, Zermelo could reply that his axioms did not imply the existence of such comprehensive sets as the set of all ordinals or the set of all sets. On the contrary, Zermelo could prove, for example, that every set must have at least one subset which does not belong to it; consequently there can be no set of all sets. Russell very suggestively spoke of "*self-reproductive* processes and classes" which would correspond to "properties such that, given any class of terms all having such a property, we can always define a new term also having the property in question. Hence we can never collect *all* the terms having the said property into a whole; because, whenever we hope we have them all, the collection which we have immediately proceeds to generate a new term also having the said property" [**15**].

One can circumscribe the sets which are too big in terms of cardinality as von Neumann did: sets are too big to be elements of sets if and only if they contain a subset equipollent to the set of all sets which are elements; but the sets too big to be elements are not too big to exist—they are not Cantor's inconsistent multiplicities. Cantor had advanced the proposition that a multiplicity equipollent to an inconsistent multiplicity is also inconsistent, and Jourdain had asserted that any multiplicity containing a submultiplicity equipollent to the multiplicity of all ordinals is inconsistent; but neither Cantor nor Jourdain developed his ideas in a logically satisfactory way, as von Neumann did. (Recall that Zermelo wanted nothing to do with inconsistent multiplicities.) One can think of von Neumann's distinction between sets which can and cannot be elements as a way of putting an end to Russell's "self-reproductive processes and classes."

Although the theory of limitation of size can easily be embodied in an axiom system like Zermelo's and the sets which are too big can be specified in von Neumann's way, when defending the theory of limitation of size one would like to be able to point to something wrong with being too comprehensive—to some conceptual problem associated with the sets which are too large. Can one suggest anything at all that is dubious about the 'sets' involved in the antinomies other than that they lead to contradictions (are inconsistent multiplicities)? Leaving aside the case of sets of ordinals or cardinals not conceived of as sets, the matter of self-membership seems the only possibility—and one that was advanced quite early. In 1904, Hausdorff was willing to take it as an axiom that "a well-defined class of objects... is never itself an object of this class, but always represents a new object" [**7**, p. 123]. As Gödel has remarked, it seems that the extensional concept of set implies that a set depends on its elements in a way that excludes sets which are not well-founded [**5**, p. 140]. Around 1905, Zermelo included an axiom ruling out self-membership in a preliminary version of his system (p. 155) though it did not appear in his 1908 paper. But it was not until 1930 that Zermelo included the axiom of foundation in his system [**17**]; this axiom excludes sets which are members of themselves or members of members of themselves, etc., or which contain infinite descending element chains. The sets satisfying this axiom, the well-founded sets, form an absolutely endless cumulative hierarchy whose foundation is a totality of non-sets, e.g., concrete individuals. First there are sets of individuals, then sets containing individuals and

sets of individuals, and so on. This structure of well-founded sets can be better described using the concepts of set theory:

$$R_0 = I$$
$$R_{\alpha+1} = P(R_\alpha) \cup R_\alpha$$
$$R_\alpha = \bigcup_{\beta<\alpha} R_\beta, \text{ for limit ordinals } \alpha.$$

This may be simplified for mathematical purposes by letting the set of individuals I be empty. The axioms of set theory are satisfied in the universe of well-founded sets as well as in suitable segments of it—[**17**] investigates these segments.

According to Moore, and many others, "Zermelo had given no explicit rationale for his choice of axioms, except that they yielded the main theorems of Cantorian set theory. Such a rationale had to await the cumulative type hierarchy which Zermelo introduced two decades later" (p. 167, cp. 270). This is debatable, at least to some extent. The theory of limitation of size was what guided the restriction of the concept of set represented by Zermelo's axioms. Furthermore, Zermelo does not seem to present the cumulative hierarchy as a rationale for his axioms; he regards it as "a suitable clarification of the 'ultrafinite antinomies,'" but there is no indication that he regards it as a conceptual beginning motivating the choice of axioms. It seems that what Zermelo says in [**17**] is in accord with considering the cumulative hierarchy as a supplement to the theory of limitation of size—it is of course a most valuable supplement! If one prefers to take the concept of *set of x's* (the so-called iterative concept of set) as generating the cumulative hierarchy, one will at least have to understand *set of* in the sense of the extensional concept of set, in order that the concept imply the axiom of choice (or, rather the multiplicative axiom)—it is made clear that this is intended in [**6**]. But there will be some doubts about the justification of the axiom of replacement, which seems better accounted for in terms of the theory of limitation of size, and also about the amount of set theory needed to explain the iterations (the latter objection is raised in [**2**]; for the former, as well as general discussion see [**1**] and [**13**]). It seems that the cumulative hierarchy enters set theory as the result of reflection both on the axioms formulated in accordance with the theory of limitation of size and on the fact that the big sets are the ones which are not well-founded—one might say that they are too comprehensive in the sense that they would have to contain themselves as elements, or contain sets of which they were elements, etc. In any case, the three main ideas involved in the conceptual foundations of a reconstructed set theory of Cantorian extent are the extensional concept of set, the theory of limitation of size, and the cumulative hierarchy of sets. The reason one cannot say *simply* that the axiom of foundation and the cumulative hierarchy follow from the extensional concept of set is that it *also seems* to follow from the concept of set as a mere plurality that any objects should together constitute a set—that the existence of many objects is sufficient for the existence of the set of them all. Though they are incompatible, both this general sufficient condition for existence and the axiom of foundation seem to follow from the extensional concept of set; but things work out quite well, if the former is restricted in accordance with the latter and the theory of limitation of size is employed: one can never establish the existence of sets which are too comprehensive by beginning from a small infinite set and employing axioms of power set, union, and replacement.

Perhaps it would not be amiss in ending to consider briefly the question of the reality of the universe of set theory. Some are inclined to the view—Gödel seems the foremost among these (see [**5**], [**6**], and for discussion [**10**])—that abstract objects like sets are just as real as concrete objects and in the same way exist independently of thought. Zermelo, at least in his later writings, does not seem to have held this view; on the other hand, he was most certainly not a formalist, considering the language of set theory just marks to be shifted about according to formal rules. As he said in a lecture given in 1931, "the true subject matter of mathematics is *not*, as many would have it, 'combinations of signs' but *conceptually-ideal relations* between the elements of a

conceptually determined *infinite manifold*" (tr. Moore in [**11**], p. 126). It seems to matter little whether the objects of mathematics are really there or whether the universe of sets is so-to-speak superimposed on reality, constituting a universe in which nevertheless real discoveries can be made. In either case, set theory really has an intelligible content, and we may admire with Hilbert "Cantor's magnificent conceptions," a "majestic world of ideas" of which Zermelo was a most able champion.

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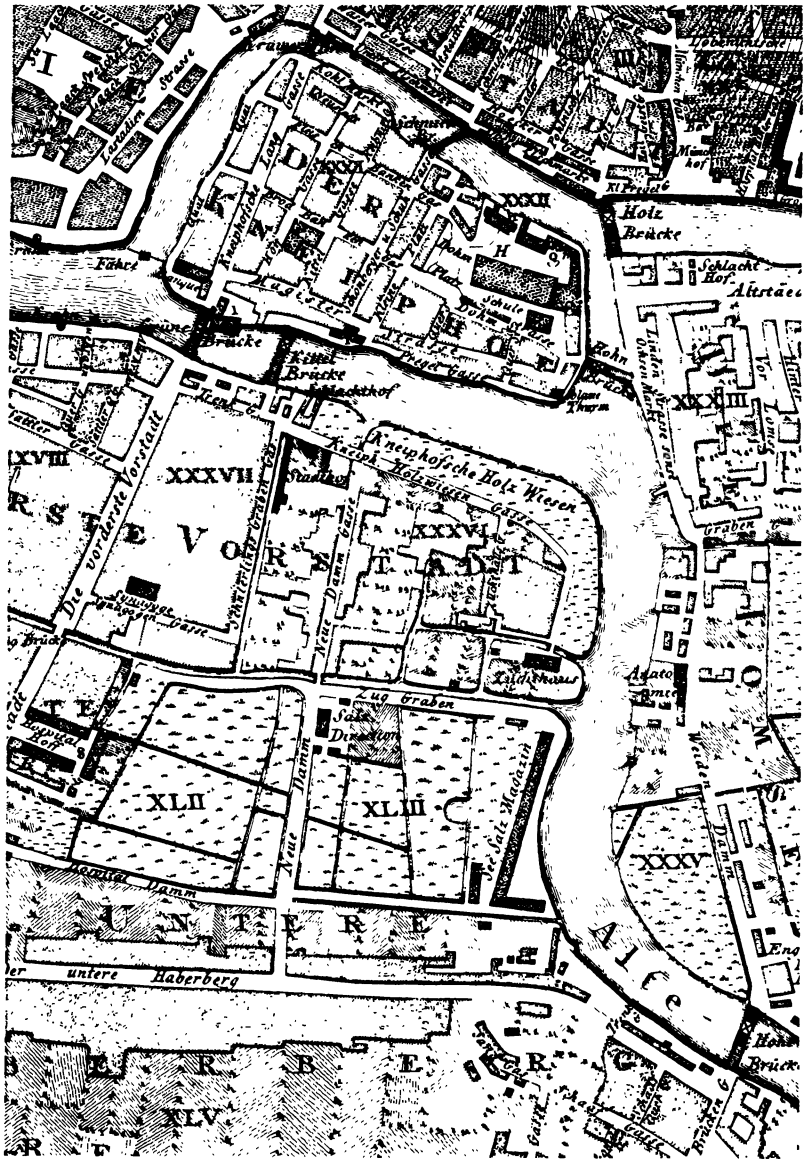
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LETTERS TO THE EDITOR

Material for this department should be prepared exactly the same way as submitted manuscripts (see the inside front cover) and sent to Professor P. R. Halmos, Department of Mathematics, Indiana University, Bloomington, IN 47405 (after December 15, 1984: Department of Mathematics, University of Santa Clara, Santa Clara, CA 95053).

Editor:

On p. 123 of his book *Why the Professor Can't Teach* (New York: St. Martin's Press, 1977) Professor Kline asks, "What is the major problem facing this nation today? Is it inflation? Unemployment? The absorption of minorities? Women's rights? Retaining the respect of other nations? If one were to judge by the contents of the liberal arts courses, it is the Koenigsberg bridge problem." Although I wholeheartedly concur with Professor Kline's withering scorn, I



nevertheless acquiesced when requested to talk about the problem at a local high school. To spice up the old chestnut I decided to take along an actual map of old Königsberg. The best one I found was from a collection entitled "Karte von Ost-Preußen nebst Preußisch-Litthauen des Kgl. Preuß. Staatsministers Frey Herrn von Schroetter in den Jahren 1796–1802" reproduced in the *Historisch-geographischer Atlas des Preußenlandes*, Lieferung 6 (Wiesbaden: Franz Steiner Verlag GmbH, 1978). A section of this map accompanies this letter. To my surprise I found, in addition to the famous seven bridges, a ferry from the Kneiphof to the mainland, thus providing the possibility of a unicursal tracing of the bridges.

It is not certain whether Euler had actually visited Königsberg by 1735, when he solved the problem. If he had, he might have used the ferry to solve the problem affirmatively, with who knows what consequences for the development of topology. Possibly too the wily Prussians,

having read Euler's solution ("Solutio problematis ad geometriam situs pertinentis" *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, T. 8, p. 128) installed the ferry out of spite, hoping to confound the speculations of the great Swiss thinker.

Having dug up the old fossil for (I hope) the last time, I feel I should give a suitably Eulerian benediction as it is reinterred: Requiescat in pace problema pontium Regiomontanae.

Roger Cooke
Department of Mathematics
University of Vermont
Burlington, VT 05405

Editor:

A CUPM panel recently presented in this MONTHLY (vol. 89, pp. 266–272) some 17 topics for Minimal Mathematical Competence (of college graduates). Some of these topics were described in a sprightly fashion, as in topic 14 "Statistics and its dangers". To this description, Danny W. Turner (this MONTHLY, vol. 90, p. 144) objected. He is correct in observing that one might also speak of "Mathematics and its dangers". (Just think of the nonsense which has been written on pseudo applications of catastrophe theory or of fuzzy sets.) But he claims that "statistics is no more dangerous than percent (topic 9)".

False. For some years I had the painful task of guiding the review of reports for the National Research Council. In these reports there were many bogus uses of statistics—especially the use of regression. The most flagrant case was an attempt to determine a shadow price for clean air by regressing the cost of houses in greater Boston on some 14 different variables (see Harny and Rubinfeld, Hedonic prices and the demand for Clean Air, Journal for Environmental Economics and Management, vol. 5 (1978) pp. 81–102). No regression on that many variables can have a dependable meaning. For this and many other reasons, I find that the stated topic 14 is well put. Now that programs for doing regression and machines to carry out these programs are readily available it can be easy to program and produce nonsense. Such dangers may not be well understood by non-mathematicians—so it is good that mathematical competence for undergraduates should include some attention (even sprightly attention) to all the dangers.

Saunders Mac Lane
Department of Mathematics
University of Chicago
Chicago, IL 60637

136. MISCELLANEA

...Even the most factual writing may involve feeling. Even the coldest biological monograph on the habits of flatworms, or the most detached piece of historical research into the price of eggs under Edward I, may be written so lucidly, argued so neatly, as to stir pleasure and admiration. Even mathematical solutions (though here I speak with trembling) can have aesthetic beauty.

—F. L. Lucas, *Style*,
Cassell, London, 1955.

REVIEWS

Names of authors are in ordinary type; those of reviewers in CAPITALS.

Askey Richard (ed.) *Gabor Szegő Collected Papers Volume I 1915–1927 Volume II 1927–1943 Volume III 1945–1972* MARK KAC 591–592

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van Lint JH *Introduction to Coding Theory* HN WARD 522–525

ERRATA AND ADDENDA

Professor David E. Dobbs, University of Tennessee, has informed us that his paper, titled "Discs and shells," and published in the *MATYC Journal*, 11 (1977) 35–37, contains the results appearing in "The disk and shell method", this MONTHLY, 91 (1984) 139, by Professor Charles A. Cable.

Dr. Ruma Falk wishes to correct her affiliation in the paper "Multiplicative analogues of some statistics", this MONTHLY, 91 (1984) 198–202. She is in the Department of Psychology, The Hebrew University, Jerusalem 91–905, Israel.

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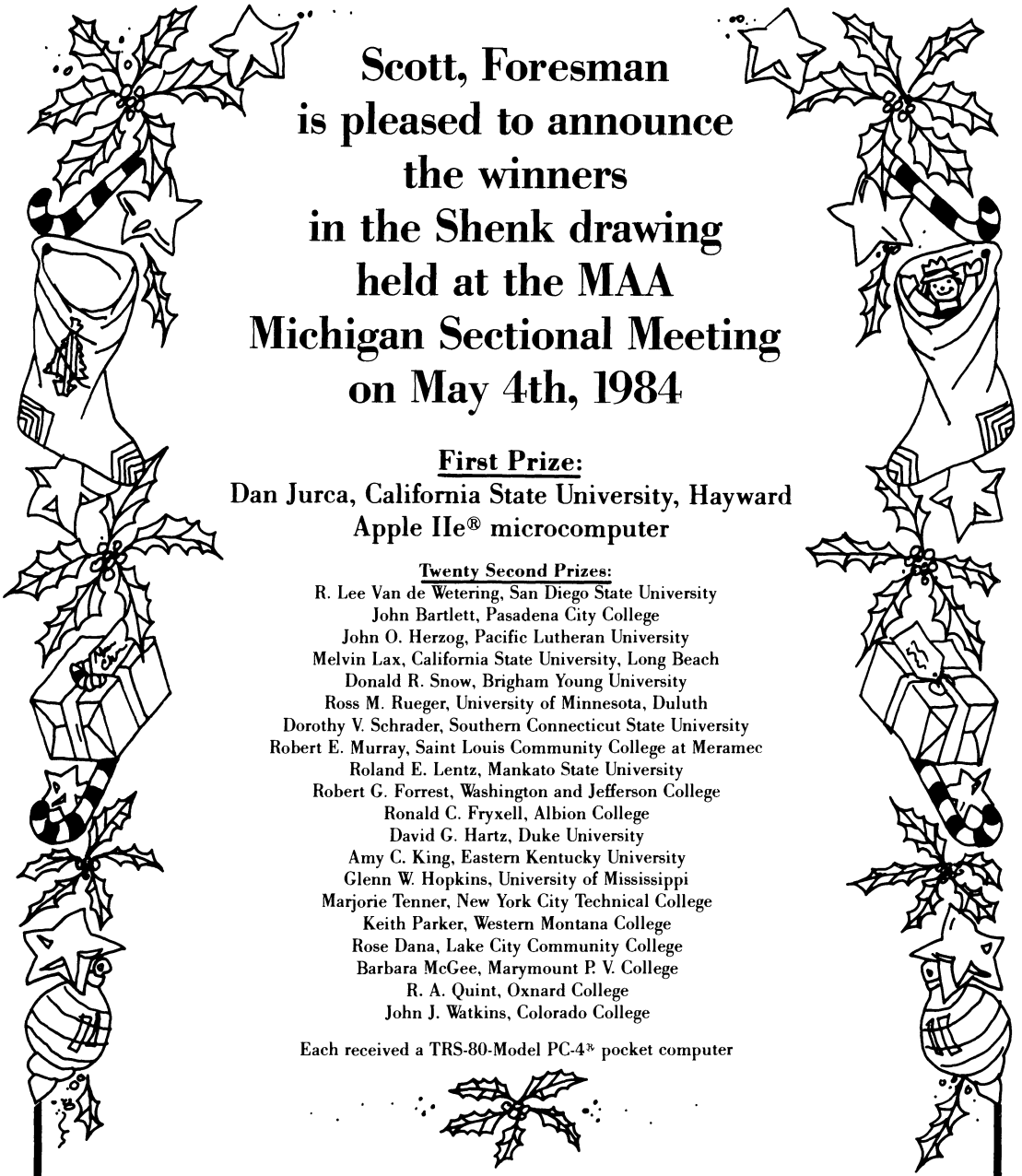
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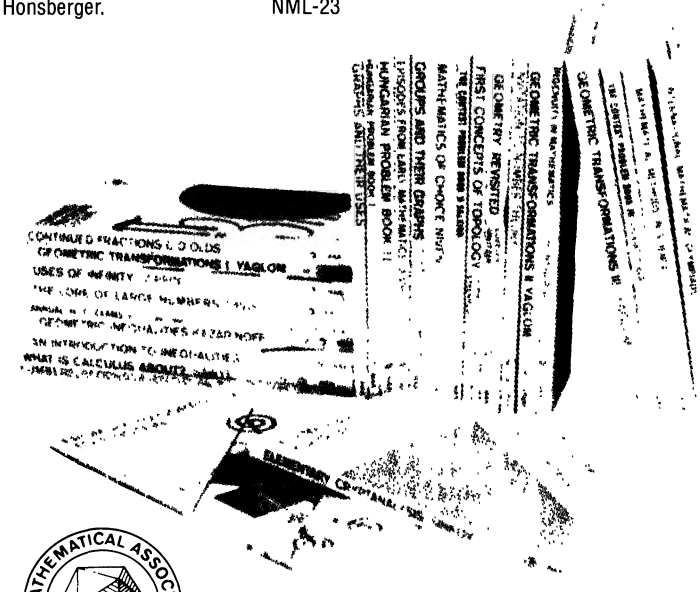
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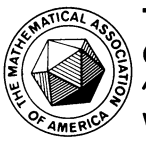
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